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On asymptotic and Hyers-Ulam stability of Hilfer fractional initial value problem involving a $(p_1, p_2, \ldots, p_n)$-Laplacian operator

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Abstract

In this paper, we investigate the existence, asymptotic, Hyers-Ulam, and semi-Hyers-Ulam-Rassias stability results for the Hilfer fractional initial value problem involving the (p_1,p_2...p_n)-Laplacian operator by using the fixed point arguments.

Keywords: Hilfer fractional IVP, Hyers-Ulam stability, fixed point
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1 Introduction

In this paper, we study the following Hilfer fractional initial value problem

$$\begin{cases} -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.u\right)'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}\left(\delta.u\right)(t) + R\left(t,u\left(t\right)\right) = 0,\ t > 0, \\ u(0) = 0, \end{cases} \tag{1.1}$$

where $\phi_{p_i}(x) = |x|^{p_i-2}.x$, $p_i > 1$, for $i \in \{1, .., N\}$,

$$R\left(t,u\left(t\right)\right) = p(t,u\left(t\right)) + q\left(t\right) f(t,u(t)), \quad p\left(t,u\right) = \sum_{n=1}^{n=m} \eta_n\left(t\right) u^n,\ N, m \in \mathbb{N}^*,$$

with $\eta_n : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive function, $D_{0^+}^{\mu,\omega,\sigma}$is the $\sigma-$Hilfer fractional derivative of order $\mu \in \{\alpha, \beta\}$ and type $0 \le \omega \le 1$ with $0 < \beta < \alpha < 1$.

Fractional differential equations with different initial conditions and boundary conditions were studied by many authors [2, 8, 9, 15] and extended to p-Laplacian fractional differential equations, see, [5, 6, 7, 17, 21].

Besides, the subject of stability is a very important notion in physics, and for the sake of such importance and applicability, one can observe a lot of work in the numerous publications (for example, refer to the references [14, 23, 24, 26]). The study of Ulam and Hyers-Ulam stability for various equations originated from a famous talk of Ulam [28]. In 1940, Ulam posed a problem concerning the stability of functional equations: "Give Conditions in order for a

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linear function near an approximately linear function to exist" . Since then, this question has attracted the attention of many researchers, see, [3, 4, 11, 16, 18, 20, 21, 25, 27, 28], and references therein.

In [16], the authors worked on the existence of positive solution and Hyers–Ulam stability results, for p-Laplacian fractional differential system :

$$\begin{cases} D^{\beta}\left[\phi_p\left[D^{\nu_0}u(t)-y_2(t,u(t))\right]\right]=-A(t)\,y_1(t,u(t-\tau)), \\ \phi_p\left[D^{\nu_0}\left(u(.)-y_2(.,u)\right)\right](0)=0=\left(\phi_p\left[D^{\nu_0}u(.)-y_2(.,u)\right]\right)'(0), \\ u(0)=0=u'(1),\ I^{2-\nu_0}\left[u(.)-y_2(.,u(.))\right](0)=0, \end{cases}$$

where D^{β}, D^{ν_0} are the Caputo fractional derivatives of orders $\beta\in(1,2]$ and $\nu_0\in(2,3]$ respectively and $\phi_p(x)=|x|^{p-2}.x$ is the p-Laplacian operator.
The authors in [5] investigated the existence of solutions and Hyers-Ulam stability for the following φ-Hilfer fractional order differential equation involving a p-Laplacian operator

$$\begin{cases} D_{a^+}^{\alpha_1,\beta_1,\varphi}\psi_p\left(D_{a^+}^{\alpha_2,\beta_2,\varphi}u\right)(t)=h\left(t,u(t),{}^{RL}D_{a^+}^{\mu,\varphi}u(t)\right),\ t\in(a,b] \\ u(a)=u(b)=\sum\limits_{i=1}^{i=n}\lambda_i u(\zeta_i), \\ \psi_p\left(D_{a^+}^{\alpha_2,\beta_2,\varphi}u\right)(a)=0, \\ \psi_p\left(D_{a^+}^{\alpha_2,\beta_2,\varphi}u\right)(b)=I_{a^+}^{\rho,\varphi}u(\zeta),\ a<\zeta,\ \zeta_i<b, \end{cases}$$

where $D_{a^+}^{\alpha_1,\beta_1,\varphi}$, $D_{a^+}^{\alpha_2,\beta_2,\varphi}$ are the $\varphi-$Hilfer fractional derivative of orders $\alpha_1,\alpha_2\in(1,2)$ with parameters $\beta_1,\beta_2\in[0,1]$, ${}^{RL}D_{a^+}^{\mu,\varphi}$ the $\varphi-$-Riemann-Liouville fractional derivative of order $\mu<\alpha_2$ and $I_{a^+}^{\rho,\varphi}$ the left-sided $\varphi-$Riemann Liouville fractional integral of order $\rho>0$ and $\psi_p(x)=|x|^{p-2}.x$ is the p-Laplacian operator.

Motivated by the cited papers, in the present article, we discuss the existence, asymptotic, Hyers-Ulam and semi-Hyers-Ulam-Rassias stability for Hilfer fractional equation involving the (p_1,p_2...p_n)-Laplacian operator (1.1).

Throughout the article, we assume that $\sigma\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ is increasing with $\sigma(0)=0$ and $\sigma'(t)\neq 0$ for all $t\geq 0$, $p:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a two variable polynomial function and $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a continuous function, and there exist $k\in\mathbb{N}^*$, $\lambda,\epsilon>0$ and $r>0$ such that for all $x\in[0,r]$

$$0<f(t,e^{kt}x)\leq\lambda.x+\epsilon,t>0. \tag{1.2}$$

The functions $g,h,\delta,\eta_n,q:\mathbb{R}^+\to\mathbb{R}^+$ are continuous, where g is increasing such that $g(0)>0$, h,δ and q do not vanish identically on any subinterval of $\mathbb{R}^+$. We consider the following conditions

$$\begin{cases} \frac{1}{h}\in L^1_{loc}(\mathbb{R}^+),\ \frac{e^{-kt}}{g(t).h(t)}\in L^1(\mathbb{R}^+), \\ \lim\limits_{x\to+\infty}\frac{e^{-kx}}{g(x)}\int_0^x\frac{ds}{h(s)}=0,\ \hat{\delta}_k\in L_{\sigma}^{\alpha-\beta}(\mathbb{R}^+,\mathbb{R}^+)\ \text{and}\ \bar{\eta}_{n,k},q\in L_{\sigma}^{\alpha}(\mathbb{R}^+,\mathbb{R}^+) \end{cases} \tag{1.3}$$

where $\bar{\eta}_{n,k}(s)=e^{nks}\eta_n(s)$ and $\hat{\delta}_k(s)=e^{ks}\delta(s)$, and for $\mu>0$

$$L_{\sigma}^{\mu}(\mathbb{R}^+,\mathbb{R}^+)=\left\{u:\mathbb{R}^+\to\mathbb{R}^+,\sup_{x\geq 0}\int_0^x\sigma'(s)(\sigma(x)-\sigma(s))^{\mu-1}u(s)\,ds<\infty\right\}.$$

The rest of the paper is organized as follows. In Section 2, some preliminary materials to be used later are stated. In Section 3, we present and prove our main results consisting of the existence of nontrivial positive solution, asymptotic, Hyers-Ulam and semi-Hyers-Ulam-Rassias stability results of the initial value problem (1.1) by using the fixed point arguments. Finally, example is given to illustrate our results.

2 Preliminaries

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm denoted $\|.\|$. A nonempty closed convex subset P of E is said to be a cone if $P\cap(-P)=0$ and $(tP)\subset P$ for all $t\geq 0$. It is well known that a cone P induces a partial order in the Banach space E. We write for all $x;y$:$\in E$; $x\leq y$ if $y-x\in P$.

The mapping $L:E\to E$ is said to be positive in P if $L(P)\subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E.

Definition 2.1. [29] Let $a \in \mathbb{R}^+$ and $\alpha > 0$. Also, let $\sigma(x)$ be an increasing and positive function having a continuous derivative $\sigma'(x)$ on $(a, +\infty)$. Then the left-sided fractional integral of a function u with respect to another function σ on $\mathbb{R}^+$ is defined by

$$I_{a^+}^{\alpha,\sigma} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \sigma'(t) \left(\sigma(x) - \sigma(t)\right)^{\alpha-1} u(t) dt.$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator $I_{a^+}^{0,\sigma} u = u$.

Definition 2.2. [29] Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, and $u, \sigma \in C^n(\mathbb{R}^+, \mathbb{R})$ two functions such that σ is increasing and $\sigma'(t) \neq 0$, for all $t \in \mathbb{R}^+$. The σ-Hilfer fractional derivative $D_{a^+}^{\alpha,\omega,\sigma}$ of u of order $n-1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$$D_{a^+}^{\alpha,\omega,\sigma} u(x) = I_{a^+}^{\omega(n-\alpha),\sigma} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x}\right)^n I_{a^+}^{(1-\omega)(n-\alpha),\sigma} u(x).$$

Let's also recall the following important result [29]:

Theorem 2.3. If $u \in C^n(\mathbb{R}^+)$, $n-1 < \beta < \alpha < n$, $0 \leq \omega \leq 1$ and $\xi = \alpha + \omega(n-\alpha)$, then

$$I_{a^+}^{\alpha,\sigma} . D_{a^+}^{\alpha,\omega,\sigma} u(x) = u(x) - \sum_{k=1}^{n} \frac{(\sigma(x) - \sigma(a))^{\xi-k}}{\Gamma(\xi - k + 1)} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x}\right)^{n-k} I_{a^+}^{(1-\omega)(n-\alpha),\sigma} u(a).$$

Moreover, $I_{a^+}^{\beta,\sigma} I_{a^+}^{\alpha-\beta,\sigma}(u) = I_{a^+}^{\alpha-\beta,\sigma} I_{a^+}^{\beta,\sigma}(u) = I_{a^+}^{\alpha,\sigma}(u)$ and ${}^H D_{a^+}^{\alpha,\omega,\sigma} I_{a^+}^{\alpha,\sigma}(u) = u$.

Remark 2.4. In this paper, we assume that $\sigma(x)$ is increasing and positive with $\sigma(0) = 0$, having a continuous derivative $\sigma'(x)$ on $\mathbb{R}^+$ and $\sigma'(x) \neq 0$, for all $x \in \mathbb{R}^+$. If $\alpha \in (0,1)$, then $n = 1$ and for $x > 0$

$$I_{0^+}^{\alpha,\sigma} . D_{0^+}^{\alpha,\omega,\sigma} u(x) = u(x) - \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma} u\right)(0).$$

Moreover, if $u : \mathbb{R}^+ \to \mathbb{R}$ is continuous, then

$$\lim_{x \to 0^+} \left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma} u\right)(x) = 0$$

and so $I_{0^+}^{\alpha,\sigma} . {}^H D_{0^+}^{\alpha,\omega,\sigma} u(x) = u(x)$.

In what follows, we use of the following Schauder's fixed-point theorem [13] :

Theorem 2.5. [13] Let E be a Banach space, C be a nonempty bounded convex and closed subset of E, and $T : C \to C$ be a compact and continuous map. Then T has at least one fixed point in C.

We will use the following lemma concerning existence of fixed point for a compact map $T : P \cap \bar{B}(0, r) \to P$, where $r > 0$ and P is a cone in a Banach space F.

Lemma 2.6. [10] If $\|Tu\| < \|u\|$ for all $u \in P \cap \partial B(0, r)$, then T has a fixed point u in $P \cap \bar{B}(0, r)$.

Definition 2.7. [10] Solutions of IVP (1.1) are locally asymptotically stable in a cone K of a Banach space E if there exits a nonempty bounded convex and open subset Ω of E such that, for any solutions $u, v \in K \cap \Omega$ of IVP (1.1), we can write

$$\lim_{x \to +\infty} (u(x) - v(x)) = 0 \tag{2.1}$$

uniformly with respect to $K \cap \Omega$. Moreover, if (2.1) is verified for all solutions $u, v \in K$, (1.1) is said to be asymptotically stable.

Definition 2.8. [18] We say that IVP (1.1) has the Hyers-Ulam stability in a cone K of a Banach space E if there exits a constant $M>0$ such that for every $\epsilon>0$, $v\in K$, if

$$\left|-\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.v\right)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}\left(\delta.v\right)(t)+p(t,v\left(t\right))+q\left(t\right)f(t,v(t))\right|\leq\epsilon, \tag{2.2}$$

then there exists a solution $u\in K$ of IVP (1.1), such that

$$|u(x)-v(x)|\leq M.\epsilon. \tag{2.3}$$

We call such M a Hyers-Ulam stability constant for (1.1).

Definition 2.9. [11]We say that IVP (1.1) has the semi-Hyers-Ulam-Rassias stability in a cone K of a Banach space E if it has the following properties:
For every $\epsilon>0$, there exists $\omega\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ such that for $v\in K$, if

$$\left|-\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.v\right)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}\left(\delta.v\right)(t)+p(t,v\left(t\right))+q\left(t\right)f(t,v(t))\right|\leq\epsilon, \tag{2.4}$$

then there exists a solution $u\in K$ of IVP (1.1), such that

$$|u(t)-v(t)|\leq\omega\left(t\right),\ t>0. \tag{2.5}$$

Definition 2.10. We say that IVP (1.1) has the generalized semi-Hyers-Ulam-Rassias stability in a cone K of a Banach space E if it has the following properties:
There exists $\Phi\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ such that for every positive function $\varphi\in L_\sigma^\alpha\left(\mathbb{R}^+,\mathbb{R}^+\right)$ and $v\in K$, if

$$\left|-\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.v\right)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}\left(\delta.v\right)(t)+p(t,v\left(t\right))+q\left(t\right)f(t,v(t))\right|\leq\varphi\left(t\right), \tag{2.6}$$

then there exists a solution $u\in K$ of IVP (1.1), such that

$$|u(t)-v(t)|\leq\Phi\left(\varphi\left(t\right)\right),\ t>0. \tag{2.7}$$

Remark 2.11. Assume that the function σ is bounded and IVP (1.1) has the generalized semi-Hyers-Ulam-Rassias stability. Then the constant function $\varphi=\epsilon\in L_\sigma^\alpha\left(\mathbb{R}^+,\mathbb{R}^+\right)$ and so, IVP (1.1) has the semi-Hyers-Ulam-Rassias stability.

Let F be a real Banach space defined by

$$F=\left\{u\in C(\mathbb{R}^+,\mathbb{R}):\sup_{x\geq 0}|u(t)|<\infty\right\}$$

equipped with the sup-norm $\|u\|_0=\sup_{t\in\mathbb{R}^+}\left(|u(t)|\right)$ and P the cone in F defined as

$$P=\left\{u\in F:u\left(0\right)=0 \text{ and } u(t)\geq 0 \text{ for all } t\in\mathbb{R}^+\right\}.$$

For $k\in\mathbb{N}^*$ given in (1.3), let E be a real Banach space defined as

$$F\subset E=\left\{u\in C(\mathbb{R}^+,\mathbb{R}):\lim_{t\to\infty}e^{-kt}u(t)=0\right\}$$

equipped with the norm $\|\cdot\|$, where for $u\in E$, $\|u\|=\sup_{t\in\mathbb{R}^+}\left(e^{-kt}\left|u(t)\right|\right)$, and

$$K=\left\{u\in E:u\left(0\right)=0 \text{ and } u(t)\geq 0 \text{ for all } t\in\mathbb{R}^+\right\}$$

be the cone of E.

Lemma 2.12. [10] A non empty subset M of E is relatively compact if the following conditions hold :

1. M is bounded in E,
2. The set $\left\{e^{-kt}u, u \in M\right\}$ is locally equicontinuous on $[0, +\infty)$, and
3. The set $\left\{e^{-kt}u, u \in M\right\}$ is equiconvergent, that is, for any given $\epsilon > 0$, there exists $A > 0$ such that

$$\left|e^{-kx}u(x) - \lim_{y \to +\infty} e^{-ky}u(y)\right| < \epsilon,$$

for any $x > A$, $u \in M$.

3 Main results

We consider the operator $T : E \to C^1(\mathbb{R}^+)$ defined by

$$Tu(x) = \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(t,u) + q.f(.,u))\right)(t)dt$$

where $\psi = \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is the inverse function of sum of p$_i$-Laplacian operators $\phi = \sum_{i=1}^{i=N} \phi_{p_i}$, with $\phi_{p_i}(x) = |x|^{p_i-2}.x$ and ψ_{p_i} is the inverse function of ϕ_{p_i}.

Remark 3.1. Let $p^- = \min\{p_1, p_2...p_N\}$ and $p^+ = \max\{p_1, p_2...p_N\}$. For all $x \geq 0$, $i \in \{1, 2...N\}$

$$\phi_{p_i}(x) \leq \phi(x) \leq N.\phi^+(x)$$

where

$$\phi^+(x) = \begin{cases} \phi_{p^+}(x) & if\ x \geq 1 \\ \phi_{p^-}(x) & if\ x \leq 1 \end{cases}$$

and so, we conclude that

$$\psi^+\left(\frac{x}{N}\right) \leq \psi(x) \leq \psi_{p_i}(x) \tag{3.1}$$

where

$$\psi^+\left(\frac{x}{N}\right) = \begin{cases} \psi_{p^+}\left(\frac{x}{N}\right) & if\ x \geq 1 \\ \psi_{p^-}\left(\frac{x}{N}\right) & if\ x \leq 1. \end{cases}$$

Moreover, for $x \geq y \geq 0$,

$$\begin{cases} \psi_p(x+y) \leq \psi_p(x) + \psi_p(y), & \text{if } p \geq 2, \\ \psi_p(x+y) \leq (2)^{\frac{2-p}{p-1}}.\left[\psi_p(x) + \psi_p(y)\right], & \text{if } p < 2. \end{cases} \tag{3.2}$$

3.1 Existence and asymptotic stability results

For $p \in \{p_1, p_2...p_N\}$, let

$$\Lambda_r(p) = \sup_{x \geq 0} \psi_p\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \bar{\eta}_{n,k}.r^{n-1} + \left(\lambda + \frac{\epsilon}{r}\right).q\right)\right),$$

where $r > 0$ is the constant given in (1.2). Hypothesis (1.3) gives that that $\Lambda_r < \infty$.

Lemma 3.2. $u \in C^1(\mathbb{R}^+)$ is solution of IVP (1.1) if and only if u is fixed point of T (i.e $Tu = u$).

Proof . Let $u \in E$ be a fixed point of T, then $u \in C^1(\mathbb{R}^+)$, $u(0) = 0$ and

$$\phi(h.u')(t) = I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(t,u) + q.f(.,u))(t)$$

then it follows from Theorem 2.3 that

$$D_{0^+}^{\alpha,\omega,\sigma}\phi(h.u')(t) = D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t) + p(t,u) + q.f(.,u)(t) \tag{3.3}$$

The continuity of the function $t \mapsto \delta(t).u(t)$ gives that

$$\lim_{t\to 0^+}\left(I_{0^+}^{(1-\omega)(1-\beta),\sigma}(\delta.u)\right)(t)=0$$

and so

$$\begin{aligned} D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) &= D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}\left(I_{0^+}^{\beta,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u)+\frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)}\left(I_{0^+}^{(1-\omega)(1-\alpha),\sigma}(\delta.u)\right)(0)\right)\\ &= D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha-\beta,\sigma}\left(I_{0^+}^{\beta,\sigma} D_{0^+}^{\beta,\omega,\sigma}(\delta.u)\right)=D_{0^+}^{\alpha,\omega,\sigma} I_{0^+}^{\alpha,\sigma}\left(D_{0^+}^{\beta,\omega,\sigma}(\delta.u)\right)\\ &= D_{0^+}^{\beta,\omega,\sigma}(\delta.u)\end{aligned}$$

then equation (3.3) means that u is solution of IVP (1.1).
Conversely, it is easily to show, by a direct calculation, that the solution u of the IVP (1.1) satisfies the equation $u = Tu$. This completes the proof. □ □

Lemma 3.3. Assume that Hypothesis (1.2) and (1.3) hold true.
Then the operator $T : K \cap \bar{B}(0, r) \to K$ is compact, where r is the constant given in (1.2).

Proof .
Let $M_r = T(\Omega_r)$, where $\Omega_r = K \cap \bar{B}(0, r)$ and set

$$A_0 u = g(x).Tu = \int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)+I_{0^+}^{\alpha,\sigma}(p(t,u)+q.f(.,u))\right)(t)dt.$$

We show that the set $M_r = T(\Omega_r)$ is a subset of E. Let $u \in \Omega_r$. The continuity of the functions $g, \psi,\ f(.,u)$ and $p(.,u)$ and Hypothesis (1.3) make that

$$\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)+I_{0^+}^{\alpha,\sigma}(p(t,u)+q.f(.,u))\right)(t)\in L^1_{loc}\left(\mathbb{R}^+\right)$$

and so $T\ (\Omega_r) \subset C(\mathbb{R}^+, \mathbb{R})$.
For $x > 0$

$$\begin{aligned} e^{-kx}Tu(x) &= \frac{e^{-kx}}{g(x)}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)+I_{0^+}^{\alpha,\sigma}(p(t,u)+q.f(.,u))\right)(t)dt\\ &\leq \frac{e^{-kx}}{g(x)}\int_0^x \frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}_k\right)+I_{0^+}^{\alpha,\sigma}\left[\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}\left(\tilde{u}_k\right)^n+\lambda q\tilde{u}_k+\epsilon q\right)\right]\right)(t)dt\\ &\leq \psi_{p_i}(r)\,\Lambda_r(p_i)\,.\frac{e^{-kx}}{g(x)}\int_0^x\frac{dt}{h(t)}\end{aligned}$$

and from Hypothesis (1.3) leads

$$\lim_{x\to+\infty} e^{-kx}Tu(x)=0,$$

thus M_r is a subset of.E.

We show that the operator $T : \Omega_r \to E$ is continuous. Let $u \in \Omega_r$ and $(u_p)_p \subset \Omega_r$ a sequence such that

$$\lim_{p\to+\infty} u_p = u.$$

We condider the operator $B : \Omega_r \to E$ defined by

$$B(u)(t) = I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t) + I_{0^+}^{\alpha,\sigma}(p(.,u)+q.f(.,u))(t).$$

For $x > 0$

$$\begin{aligned} e^{-kx}\left|Tu(x)-Tu_p(x)\right| &\leq \frac{e^{-kx}}{g(x)}\int_0^x \frac{1}{h(t)}\Sigma_p(t)\,dt\\ &\leq \int_0^x \frac{e^{-kt}}{g(t).h(t)}\Sigma_p(t)\,dt\end{aligned}$$

where

$$\begin{aligned}\Sigma_p(t) &= |\psi(B(u_p)(t)) - \psi(B(u)(t))| \\ &\leq C = 2\sup_{t\geq 0}\left\{\psi\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.r^n + q.(\lambda r+\epsilon)\right)\right)(t)\right\}\end{aligned}$$

and

$$\lim_{p\to+\infty}\Sigma_p(t) = 0, \text{ for all } t\geq 0.$$

As $\dfrac{e^{-kt}}{g(t).h(t)} \in L^1(\mathbb{R}^+)$, then

$$\|Tu - Tu_p\| \leq \int_0^\infty \frac{e^{-kt}}{g(t).h(t)}\Sigma_p(t)\,dt.$$

with $\dfrac{e^{-kt}}{g(t).h(t)}\Sigma_p(t) \leq \dfrac{C.e^{-kt}}{g(t).h(t)} \in L^1(\mathbb{R}^+)$. Thus, we deduce from the Lebesgue's dominated convergence theorem that

$$\lim_{p\to+\infty} Tu_p = Tu,$$

proving the continuity of T.

Now, we show that M_r is relatively compact.
First, we show that M_r is bounded. Hypothesis (1.2) and inequality (3.1) of Remark 3.1 lead that for all $i\in\{1,2...N\}$

$$\begin{aligned}e^{-kx}Tu(x) &\leq \frac{e^{-kx}}{g(x)}\int_0^x \frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}_k\right) + I_{0^+}^{\alpha,\sigma}\left[\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}(\tilde{u}_k)^n + \lambda q\tilde{u}_k + \epsilon q\right)\right]\right)(t)dt \\ &\leq \psi_{p_i}(r)\Lambda_r(p_i).\frac{e^{-kx}}{g(x)}\int_0^x\frac{dt}{h(t)}\end{aligned}$$

this is for all $x\geq 0$, where $\tilde{u}_k(s) = e^{-ks}u(s)\in[0,r]$. Then

$$\|Tu\| \leq R = \psi_{p_i}(r)\Lambda_r(p_i).\sup_{x\geq 0}\left\{\frac{e^{-kx}}{g(x)}\int_0^x\frac{dt}{h(t)}\right\}.$$

proving the boundeness of M_r.
Let $b_1\leq t_1 < t_2 \leq b_2$, $b_1, b_2\in\mathbb{R}^+$ and set $w(t) = \dfrac{e^{-kt}}{g(t)}$. For all $u\in\Omega_r$ we have

$$\begin{aligned}\left|e^{-kt_2}Tu(t_2) - e^{-kt_1}Tu(t_1)\right| &= |w(t_2)A_0u(t_2) - w(t_1)A_0u(t_1)| \\ &\leq w(t_2)|A_0u(t_2) - A_0u(t_1)| + A_0u(t_1)|w(t_2) - w(t_1)| \\ &\leq w(t_2)|A_0u(t_2) - A_0u(t_1)| + e^{kb_2}R|w(t_2) - w(t_1)|\end{aligned}$$

with

$$\begin{aligned}w(t_2)|A_0u(t_2) - A_0u(t_1)| &\leq w(t_2)\int_{t_1}^{t_2}\frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(t,u) + q.f(.,u))\right)(t)dt \\ &\leq w(t_2)\int_{t_1}^{t_2}\frac{1}{h(t)}\psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}_k\right) + I_{0^+}^{\alpha,\sigma}[(p(t,u) + q.(\lambda\tilde{u}_k + \epsilon))]\right)(t)dt \\ &\leq w(t_2).\psi_{p_i}(r)\Lambda_r(p_i).\int_{t_1}^{t_2}\frac{dt}{h(t)} \\ &\leq w(b).\psi_{p_i}(r)\Lambda_r(p_i).\int_{t_1}^{t_2}\frac{dt}{h(t)}\end{aligned}$$

Because that w and $x\to\int_0^x\frac{dt}{hg(t)}$ are uniformly continuous on compact intervals, the above estimates prove that $\left\{e^{-kt}u, u\in M_r\right\}$ is locally equicontinuous on $[0,+\infty)$.

Now, let $u \in \Omega_r$, $x \in \mathbb{R}^+$. For $y > x$

$$\begin{aligned}\left|e^{-kx}T(u)(x) - e^{-ky}T(u)(y)\right| &\leq w(y)\left|A_0u(x) - A_0u(y)\right| \\ &\quad + A_0u(x)\left|w(y) - w(x)\right| \\ &\leq w(y).\psi_{p_i}(r)\Lambda_r(p_i).\int_x^y \frac{dt}{h(t)} \\ &\quad + w(x).Au(x)\left|\frac{w(y)}{w(x)} - 1\right| \\ &\leq w(y).\psi_{p_i}(r)\Lambda_r(p_i).\int_0^y \frac{dt}{h(t)} \\ &\quad + w(x).\psi_{p_i}(r)\Lambda_r(p_i).\int_0^x \frac{dt}{h(t)}\left|\frac{w(y)}{w(x)} - 1\right|\end{aligned}$$

then

$$\begin{aligned}\left|e^{-kt}T(u)(x) - \lim_{y\to+\infty} e^{-ky}T(u)(y)\right| &\leq \psi_{p_i}(r)\Lambda_r(p_i).\lim_{y\to\infty} w(y).\int_0^y \frac{dt}{h(t)} \\ &\quad + \psi_{p_i}(r)\Lambda_r(p_i).w(x).\int_0^x \frac{dt}{h(t)}\end{aligned}$$

with

$$\lim_{x\to+\infty} w(x).\int_0^x \frac{dt}{h(t)} = 0,$$

so, the equiconvergence of $\left\{e^{-kt}u, u \in M_r\right\}$ holds. By Lemma 2.12, we deduce that M_r is relatively compact.
Finally, we have from hypothesis (1.2) and (1.3) that for $u \in \Omega_r$ the functions $q.f(.,u), \eta.p(u)$ and $\delta.u$ are positive, and so $T\left(K \cap \bar{B}(0,r)\right) \subset K$.
Proving our claim. □

Remark 3.4. For $u \in K$ and $x > 0$

$$\begin{aligned}I_{0^+}^{\alpha,\sigma}(u)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x \sigma'(t)(\sigma(x) - \sigma(t))^{\alpha-1}(u)(t)dt \\ &= \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha).\Gamma(\alpha-\beta)}\int_0^x \sigma'(t)(\sigma(x) - \sigma(t))^{\alpha-\beta-1}(\sigma(x) - \sigma(t))^{\beta}(u)(t)dt \\ &\leq \nu(x).I_{0^+}^{\alpha-\beta,\sigma}(u)(x)\end{aligned}$$

where

$$\nu(x) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}(\sigma(x))^{\beta}$$

and so

$$\begin{aligned}Tu(x) &= \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \int_0^x \frac{1}{h.g(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t)I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt.\end{aligned}$$

□

Consider the condition

$$\begin{cases} \text{there exists } i \in \{1, 2...N\} \text{ such that} \\ \Lambda_r(p_i) \leq r^{\dfrac{p_i-2}{p_i-1}}\left(\sup \frac{1}{g(x)}\int_0^x \frac{dt}{h(t)}\right)^{-1}, \end{cases} \tag{3.4}$$

where $r > 0$ is the constant given in (1.2).

Theorem 3.5. Assume that Hypothesis (1.2), (1.3) and (3.4) hold true.
Then IVP (1.1) admits at least one positive solution.

Proof . Let $u \in K \cap \bar{B}(0,r)$, for $x > 0$

$$\begin{aligned} e^{-kx}Tu(x) &= \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(t,u) + q.f(.,u))\right)(t)dt \\ &\leq \psi_{p_i}(r)\Lambda_r(p_i) . \frac{e^{-kx}}{g(x)} \int_0^x \frac{dt}{h(t)} \leq r \end{aligned}$$

then

$$\|Tu\| \leq \|u\| .$$

We have that the compact operator T maps the closed bounded convex set $K \cap \bar{B}(0,r)$ into itself. So, Schauder's fixed point theorem guarantees existence of a fixed point u of T, which is a positive solution of IVP (1.1). □

We consider the following hypothesis

$$\left\{ \begin{array}{c} \text{There exist } \pi : \mathbb{R}^+ \to \mathbb{R}^+,\ r > 0 \text{ and } i \in \{1, 2...N\} \text{ such that for all } (t,x) \in \mathbb{R}_+^* \times (0,r] \\ p(t, e^{kt}x) + q.f(t, e^{kt}x) \leq \pi(t).x \text{ and for all } i \in \{1, 2...N\} \\ (r)^{\frac{2-p_i}{p_i-1}} \sup\limits_{x\geq 0} \left\{ \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + \nu(t) I_{0^+}^{\alpha-\beta,\sigma}(\pi)\right)(t)dt \right\} \leq 1. \end{array} \right. \tag{3.5}$$

Theorem 3.6. If Hypothesis (1.2), (1.3) and (3.5) hold true, then IVP (1.1) admits at least one positive solution.

Proof . For $u \in K \cap \bar{B}(0,r)$

$$\begin{aligned} Tu(x) &= \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t) I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \end{aligned}$$

and then

$$\begin{aligned} e^{-kx}Tu(x) &\leq \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t) I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(\phi_{p_i}\left(e^{-kt}\right) I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + \nu(t)\phi_{p_i}\left(e^{-kt}\right) I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{u}\right) + \nu(t)\phi_{p_i}\left(e^{-kt}\right) I_{0^+}^{\alpha-\beta,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt \\ &\leq \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right) + \nu(t)\phi_{p_i}\left(e^{-ks}\right) I_{0^+}^{\alpha-\beta,\sigma}(\pi.\tilde{u})\right)(t)dt \\ &\leq (r)^{\frac{1}{p_i-1}} \sup_{x\geq 0} \left\{ \frac{e^{-kx}}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p_i}\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + \nu(t) I_{0^+}^{\alpha-\beta,\sigma}(\pi)\right)(t)dt \right\} \leq r \end{aligned}$$

So, Schauder's fixed point theorem guarantees existence of a fixed point u of T, which is a positive solution of IVP (1.1). □ □

Now, we consider the following hypothesis

$$\left\{ \begin{array}{c} \text{There exists a function } \rho : \mathbb{R}^+ \to \mathbb{R}^+, \text{ such that for all } t > 0, \text{ if } x, y \in [0,r] \text{ then} \\ \left|f(t, e^{kt}x) - f(t, e^{kt}y)\right| \leq \rho(t).|x - y| \text{ and} \\ \lim\limits_{x\to+\infty} \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} A(t) \left[I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\rho + \sum_{n=1}^{n=m} \bar{\eta}_{n,k}.nr^{n-1}\right)\right](t)dt = 0, \end{array} \right. \tag{3.6}$$

where r is the constant given in hypothesis (1.2),

$$A(t)=\frac{1}{\sum_{i=1}^{i=N}(p_i-1)\left(\psi\left(N_i(t)\right)\right)^{(p_i-2)}},$$

with

$$N_i(t)=\begin{cases} I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right)(t)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.r^n+(\lambda.r+\epsilon).q\right)(t), & \text{if } 1<p_i\le 2\\ I_{0^+}^{\alpha,\sigma}(q).\min\left\{f(t,x),(t,x)\in\mathbb{R}^+\times\left[0,e^{kt}r\right]\right\} & \text{if } \quad p_i>2.\end{cases}$$

Let $B:K\to E$ the operator defined as

$$Bu(t)=I_{0^+}^{\alpha-\beta,\sigma}(\delta.u)(t)+I_{0^+}^{\alpha,\sigma}(p(.,u)+q.f(.,u))(t).$$

Theorem 3.7. Assume that Hypothesis (1.2), (1.3) and (3.6) hold true, and one of the conditions (3.4) or (3.5) is satisfied.
Then the positive solutions of problem (1.1) are locally asymptotically stable in K.

Proof . We have from theorems 3.5 and 3.6 that T admits a fixed point in $K\cap\bar{B}(0,r)$, which is a solution of IVP (1.1) in $\bar{B}(0,r)$.

Now, we show that the solutions are locally asymptotically stable in K. We assume that $u,v\in K\cap B(0,r)$ are solutions of IVP (1.1). For $x>0$, we have

$$u(x)-v(x)=Tu(x)-Tv(x)=\frac{1}{g(x)}\int_0^x\frac{1}{h(t)}\left(\psi(Bu)-\psi(Bv)\right)(t)\,dt.$$

Then there exists a function $\chi\in[\min(Bu,Bv),\max(Bu,Bv)]$ such that

$$\begin{aligned}(u-v)(x) &= \int_0^x\frac{1}{h(t)}A(\chi(t))\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.(u-v))\right)(t)dt\\ &+\int_0^x\frac{1}{h(t)}A(\chi(t))\left(I_{0^+}^{\alpha,\sigma}(p(.,u)-p(.,v))\right)(t)dt\\ &+\int_0^x\frac{1}{h(t)}A(\chi(t))\,I_{0^+}^{\alpha,\sigma}(q.[f(.u)-f(.,v)])(t)\ \ dt\end{aligned}$$

where

$$A(\chi(t))=\frac{1}{\sum_{i=1}^{i=N}(p_i-1)\left(\psi(\chi(t))\right)^{(p_i-2)}}.$$

For $w\in\{u,v\}$ and $t\in[0,x]$

$$\begin{aligned}Bw(t) &= I_{0^+}^{\alpha-\beta,\sigma}(\delta.w)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(w)^n+q.f(t,w)\right)\\ &\le I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.\tilde{w}\right)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.(\tilde{w})^n+\lambda q.\tilde{w}+q\epsilon\right)\\ &\le I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.r\right)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.(r)^n+q(\lambda.r+\epsilon)\right)\end{aligned}$$

and

$$Bw(t)\ge I_{0^+}^{\alpha,\sigma}(q.f(t,w))\ge I_{0^+}^{\alpha,\sigma}(q).\min\{f(t,x),t,x\ge 0\},$$

where $\tilde{w}(s)=e^{-ks}w(s)\in[0,r]$.

Then inequality of hypothesis (3.6) gives for $x > 0$

$$\begin{aligned}
|u-v|(x) &\leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(\chi(t))\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|) + I_{0^+}^{\alpha,\sigma}|p(.,u)-p(.,v)|\right)(t)dt \\
&+\frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(\chi(t))\, I_{0^+}^{\alpha,\sigma}(q.(f(.,u)-f(.,v)))(t)\ dt \\
&\leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k.|\tilde{u}-\tilde{v}|\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n n.(r)^{n-1}|u-v|\right)\right)(t)dt \\
&+\frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(t)\, I_{0^+}^{\alpha,\sigma}(\rho.|\tilde{u}-\tilde{v}|)(t)\ dt \\
&\leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k r\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.n.(r)^{n}\right)\right)(t)dt \\
&+\frac{1}{g(x)}\int_0^x \frac{1}{h(t)} A(t)\, I_{0^+}^{\alpha,\sigma}(\rho.r)(t)\,dt \\
&\leq \frac{r}{g(x)}\int_0^x \frac{1}{h(t)} A(t)\left(I_{0^+}^{\alpha-\beta,\sigma}\left(\hat{\delta}_k\right) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\bar{\eta}_{n,k}.n.(r)^{n-1}\right)\right)(t)dt \\
&+\frac{r}{g(x)}.\int_0^x \frac{1}{h(t)} A(t)\, I_{0^+}^{\alpha,\sigma}(\rho)(t)\,dt
\end{aligned}$$

where

$$A(t) = \frac{1}{\sum_{i=1}^{i=N}(p_i-1)\left(\psi(N_i(t))\right)^{(p_i-2)}},$$

and from (3.6) we conclude that $\lim_{x\to\infty}|(u-v)(x)| = 0$. □ □

Remark 3.8. Assume that for all $r > 0$, (1.2), (1.3) and (3.6) hold true and one of the conditions (3.4) or (3.5) is satisfied then IVP (1.1) is asymptotically stable in K.

Now, we assume that $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and there exists a bounded function $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\epsilon_0 > 0$ such that for all $x, t \geq 0$

$$0 \leq f(t,x) \leq g_1(t).x + \epsilon_0 \tag{3.7}$$

h, δ and q are locally bounded and do not vanish identically on any subinterval of $\mathbb{R}^+$ and

$$\frac{1}{h} \in L^1_{loc}(\mathbb{R}^+,\mathbb{R}^+),\ \int_0^{+\infty}\frac{ds}{h(s)g(s)} < \infty,\ \delta \in L_\sigma^{\alpha-\beta}(\mathbb{R}^+,\mathbb{R}^+)\text{ and }\eta_n, q \in L^\alpha_\sigma(\mathbb{R}^+,\mathbb{R}^+), \tag{3.8}$$

$$\left\{\begin{array}{l} p^- \leq 2 \leq m+1 \leq p^+ \text{ and} \\ \Delta(p^+) = \left\{\begin{array}{ll} \int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}(\eta_m + q.g_1)\right)(t)dt & \text{if } m=1 \\ \int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(\eta_m)\right)(t)dt & \text{if } m>1. \end{array}\right. < 1. \end{array}\right. \tag{3.9}$$

Let

$$B_0 u = I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(u)^n + q.(g_1.u+\epsilon_0)\right),\ u \in P.$$

Remark 3.9. If the conditions (3.7) and (3.8) are satisfied then the operator $A : P \to F$ defined as

$$Au(x) = \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}(p(.,u) + q.f(.,u))\right)(t)dt$$

is completely continuous and fixed points of A are solutions of IVP (1.1). Moreover, for all $u \in P$ and all $p \in \{p_1, p_2 ... p_N\}$, we have

$$Au(x) \leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\psi_p\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.\|u\|_0) + I_{0^+}^{\alpha,\sigma}(p(.,\|u\|_0) + q.(g_1.\|u\|_0 + \epsilon_0))\right)(t)dt,\ x \geq 0.$$

Theorem 3.10. Assume that the conditions (3.7), (3.8) and (3.9) hold true, then IVP (1.1) admits a positive solution.

Proof . We use Lemma 2.6, we show that there exists $r > 0$ such that for all $u \in \partial B(0,n) \cap P$

$$\|Au\|_0 < \|u\|_0 .$$

We assume on the contrary, that for all $n \in \mathbb{N}^*$ there exists $u_n \in \partial B(0,n) \cap P$ such that $\|Au_n\|_0 \geq \|u_n\|_0$.

$$\begin{aligned} Au_n(x) &\leq \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta. \|u_n\|_0) + I_{0^+}^{\alpha,\sigma} (p(., \|u_n\|_0) + q. (g_1. \|u_n\|_0 + \epsilon_0)) \right) (t) dt \\ &\leq \int_0^x \frac{1}{h.g(t)} n^{\frac{m}{p^+-1}} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} \left(\delta n^{1-m}\right) + \frac{1}{n^m} I_{0^+}^{\alpha,\sigma} (p(.,n) + q. (g_1.n + \epsilon_0)) \right) (t) dt, x > 0 \end{aligned}$$

then

$$\|Au_n\|_0 \leq \int_0^\infty \frac{1}{h.g(t)} n^{\frac{m}{p^+-1}} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} \left(\delta n^{1-m}\right) + \frac{1}{n^m} I_{0^+}^{\alpha,\sigma} (p(.,n) + q. (g_1.n + \epsilon_0)) \right) (t) dt$$

then

$$n \leq \int_0^\infty \frac{1}{h.g(t)} n^{\frac{m}{p^+-1}} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} \left(\delta n^{1-m}\right) + \frac{1}{n^m} I_{0^+}^{\alpha,\sigma} (p(.,n) + q. (g_1.n + \epsilon_0)) \right) (t) dt$$

and so,

$$1 \leq n^{\frac{p^+-1-m}{p^+-1}} \leq \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} \left(\delta n^{1-m}\right) + \frac{1}{n^m} I_{0^+}^{\alpha,\sigma} (p(.,n) + q. (g_1.n + \epsilon_0)) \right) (t) dt. \tag{3.10}$$

By passing to the limit in (3.10), we obtain the following contradiction

$$1 \leq \lim_{n\to\infty} \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} \left(\delta n^{1-m}\right) + \frac{1}{n^m} I_{0^+}^{\alpha,\sigma} (p(.,n) + q. (g_1.n + \epsilon_0)) \right) (t) dt = \Delta\left(p^+\right) < 1.$$

Thus, we conclude that there exists $r >> 0$ such that IVP (1.1) admits a positive solution u in $\bar{B}(0,r) \cap P$. □

Now, we consider the following hypothesis

$$\begin{cases} \text{There exists a function } \rho_1 : \mathbb{R}^+ \to \mathbb{R}^+, \text{ such that for all } t > 0, \text{ if } x, y > 0 \text{ then} \\ |f(t,x) - f(t,y)| \leq \rho_1(t). |x-y| \text{ and} \\ \lim\limits_{x\to+\infty} \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \gamma(t) \left[\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma} \left(\rho_1 + \sum_{n=1}^{n=m} \eta_n.n\right) \right)(t) \right] dt = 0, \end{cases} \tag{3.11}$$

$$\gamma(t) = \frac{1}{(p^- - 1)\left(\psi\left(\gamma^-(t)\right)\right)^{(p^- - 2)}},$$

and

$$\gamma^-(t) = I_{0^+}^{\alpha-\beta,\sigma}(\delta)(t) + I_{0^+}^{\alpha,\sigma} \left(\sum_{n=1}^{n=m} \eta_n + (|g_1| + \epsilon_0).q \right)(t).$$

Theorem 3.11. Assume that Hypothesis (3.7), (3.8), (3.9) and (3.11) hold true.
Then solutions of problem (1.1) are asymptotically stable in P.

Proof . For $u, v \in P$ solutions of (1.1), Let $r = \max\{1, \|u\|, \|v\|\}$. By using the same arguments as in proof of theorem 3.7, we have for $x > 0$

$$\begin{aligned} |u-v|(x) &\leq \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \bar{A}(t) \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta. |u-v|) + I_{0^+}^{\alpha,\sigma} |p(.,u) - p(.,v)| \right)(t) dt \\ &+ \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \bar{A}(t). I_{0^+}^{\alpha,\sigma} (q. |f(.,u) - f(.,v)|)(t)\ dt \end{aligned}$$

where

$$\bar{A}(t) = \frac{1}{\sum_{i=1}^{i=N}(p_i-1)(\psi(\chi(t)))^{(p_i-2)}} \leq \frac{1}{(p^--1)(\psi(\chi(t)))^{(p^--2)}},$$

and $\chi(t) \in [\min(Bu(t), Bv(t))]$. For $w \in \{u, v\}$, $t > 0$

$$\begin{aligned} Bw(t) &= I_{0^+}^{\alpha-\beta,\sigma}(\delta.w) + I_{0^+}^{\alpha,\sigma}(p(.,w) + q.f(.,w))(t) \\ &\leq I_{0^+}^{\alpha-\beta,\sigma}(\delta.r) + I_{0^+}^{\alpha,\sigma}(p(.,r) + q.(g_1.r + \epsilon_0))(t) \\ &\leq r^m.\gamma^-(t) \end{aligned}$$

$$\gamma^-(t) = I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}(p(.,1) + q.(g_1 + \epsilon_0))(t).$$

Then

$$\begin{aligned} |u-v|(x) \leq\ & \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{\gamma(t)}{h(t)} \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|) + I_{0^+}^{\alpha,\sigma}|p(.,u) - p(.,v)|\right)(t)dt \\ & + \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{\gamma(t)}{h(t)}.I_{0^+}^{\alpha,\sigma}(q.|f(.,u) - f(.,v)|)(t)\ dt. \end{aligned}$$

Let $x_0 > 0$ such that $|u-v|(x_0) = \|u-v\|_0$ and let $r = \max\{1, \|u\|_0, \|v\|_0\}$. Hypothesis (3.11) gives

$$\begin{aligned} |u-v|(x) \leq\ & \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t) \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|) + I_{0^+}^{\alpha,\sigma}|p(.,u) - p(.,v)|\right)(t)dt \\ & + \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t)\, I_{0^+}^{\alpha,\sigma}(q.|f(.,u) - f(.,v)|)(t)\ dt \\ \leq\ & \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t) \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n n.(r)^{n-1}|u-v|\right)\right) dt \\ & + \frac{(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t)\, I_{0^+}^{\alpha,\sigma}(\rho_1.|u-v|)(t)\ dt \\ \leq\ & \frac{r.(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t) \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n.(r)^{n-1}\right)\right)(t)dt \\ & + \frac{r.(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t).I_{0^+}^{\alpha,\sigma}(\rho_1)(t)\,dt \\ \leq\ & \frac{r^m(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t) \left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} n.\eta_n\right)\right)(t)dt \\ & + \frac{r^m(\psi(r^m))^{(2-p^-)}}{g(x)} \int_0^x \frac{1}{h(t)}\gamma(t).I_{0^+}^{\alpha,\sigma}(\rho_1)(t)\,dt \end{aligned}$$

leading to

$$\lim_{x\to\infty} |u-v|(x) = 0.$$

The proof is finished. □

3.2 Hyers-Ulam and semi-Hyers-Ulam-Rassias stability results

We consider the following conditions

$$\left\{ \begin{array}{c} p^+ > m+1, \\ \Delta_1(p^+) = \int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n + q(g_1+\epsilon_0)\right)\right)(t)dt < \infty \text{ and} \\ \Delta(p^+) = \left\{ \begin{array}{ll} \int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}(\eta_m + q.g_1)\right)(t)dt & \text{if } m=1 \\ \int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(\eta_m)\right)(t)dt & \text{if } m>1 \end{array} \right. < 1. \end{array} \right. \tag{3.12}$$

Remark 3.12. Assume that the conditions (3.7) and (3.8) hold true. If (3.12) is satisfied then (3.9) also holds, and from theorem (3.10), we deduce that IVP (1.1) admits a positive solution.

In the following lemmas, we give a priori estimates for solutions of IVP (1.1) and for inequalities (2.2 and (2.6).

Lemma 3.13. Assume that the conditions (3.7), (3.8) and (3.12) hold true.
If $u \in P$ is solution of IVP (1.1) then

$$\|u\|_0 \leq R\left(p^+\right) = \max\left\{1, \left(\Delta_1\left(p^+\right)\right)^{\dfrac{p^+-1}{p^+-1-m}}\right\}.$$

Proof . Let $u \in P$ be a solution of IVP (1.1) and assume that $\|u\|_0 > 1$. Let $x_0 > 0$ such that $u(x_0) = \|u\|_0$. We have

$$\begin{aligned} u(x_0) &= \frac{1}{g(x_0)} \int_0^{x_0} \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}\left(p\left(.,u\right) + qf\left(.,u\right)\right)\right)(t)dt \\ &\leq \int_0^{x_0} \frac{1}{hg(t)} \psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.u) + I_{0^+}^{\alpha,\sigma}\left(p\left(.,u\right) + q\left(g_1.u + \epsilon_0\right)\right)\right)(t)dt \\ &\leq \psi_{p^+}\left(\|u\|_0^m\right) \int_0^{x_0} \frac{1}{hg(t)} \psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \eta_n + q\left(g_1 + \epsilon_0\right)\right)\right)(t)dt \\ &\leq \left(\|u\|_0\right)^{\dfrac{m}{p^+-1}} \Delta_1\left(p^+\right) \end{aligned}$$

thus

$$\|u\|_0 \leq \left(\Delta_1\left(p^+\right)\right)^{\dfrac{p^+-1}{p^+-1-m}} \leq R\left(p^+\right).$$

□

Lemma 3.14. Let $\varphi \in L_\sigma^\alpha\left(\mathbb{R}^+, \mathbb{R}^+\right)$ be a positive function and let $v \in P$ such that

$$v(x) \leq \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.v) + I_{0^+}^{\alpha,\sigma}\left(p\left(.,v\right) + qf\left(.,v\right) + \varphi\right)\right)(t)dt,$$

then

$$\|v\|_0 \leq F_{p^+}(\varphi) = \max\left\{1, \left(\Delta_2\left(p^+,\varphi\right)\right)^{\dfrac{p^+-1}{p^+-1-m}}\right\} \tag{3.13}$$

where

$$\Delta_2\left(p^+,\varphi\right) = \int_0^\infty \frac{1}{hg(t)} \psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \eta_n + q\left(g_1 + \epsilon_0\right) + \varphi\right)\right)(t)dt < \infty.$$

Proof . Let $x_0 > 0$ such that $v(x_0) = \|v\|_0$ and assume that $\|v\|_0 > 1$. We have

$$\begin{aligned} v(x_0) &\leq \frac{1}{g(x_0)} \int_0^{x_0} \frac{1}{h(t)} \psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.v) + I_{0^+}^{\alpha,\sigma}\left(p\left(.,v\right) + qf\left(.,v\right) + \varphi\right)\right)(t)dt \\ &\leq \int_0^{x_0} \frac{1}{hg(t)} \psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.v) + I_{0^+}^{\alpha,\sigma}\left(p\left(.,v\right) + q\left(g_1.v + \epsilon_0\right) + \varphi\right)\right)(t)dt \\ &\leq \psi_{p^+}\left(\|u\|_0^m\right) \int_0^{x_0} \frac{1}{hg(t)} \psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m} \eta_n + q\left(g_1 + \epsilon_0\right) + \varphi\right)\right)(t)dt \\ &\leq \left(\|u\|_0\right)^{\dfrac{m}{p^+-1}} \Delta_2\left(p^+,\varphi\right) \end{aligned}$$

thus

$$\|u\|_0 \leq \left(\Delta\left(p^+,\varphi\right)\right)^{\dfrac{p^+-1}{p^+-1-m}} \leq F_{p^+}(\varphi).$$

□

Remark 3.15. Assume that $v \in P$ verifies

$$\left| -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.v'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t) + p(t,v(t)) + q(t)\, f(t,v(t)) \right| < \varphi(t),$$

Let $z(t) = -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.v'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t) + p(t,v(t)) + q(t)\, f(t,v(t))$. Then

$$v = \int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.v) + I_{0^+}^{\alpha,\sigma}\left(p(t,v) + q.f(.,v) + z\right)\right)(t)dt.$$

Theorem 3.16. Assume that the conditions (3.7), (3.8) and (3.12) hold true. Then IVP (1.1) has the generalized semi-Hyers-Ulam-Rassias stability.

Proof . Assume that $v \in P$, $\varphi \geq 0$ such that

$$\left| -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.v\right)'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t) + p(t,v(t)) + q(t)\, f(t,v(t)) \right| < \varphi(t).$$

Let $z(t) = -\sum_{i=1}^{i=N} D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.\left(g.v\right)'\right)\right)(t) + D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t) + p(t,v(t)) + q(t)\, f(t,v(t))$. Then for $x > 0$, we have $|z(x)| < \varphi(x)$ and

$$v = \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\psi\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta.v) + I_{0^+}^{\alpha,\sigma}\left(p(t,v) + q.f(.,v) + z\right)\right)(t)dt.$$

Let $u \in P$ be a solution of (1.1). For $x > 0$,

$$u(x) - v(x) = \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\left(\psi(Bu) - \psi(Bv + Z)\right)(t)\,dt$$

where $Z = I_{0^+}^{\alpha,\sigma}(z)$. It follows from lemmas 3.13 and (3.14) that $u, v \in \left[0, \Pi_{p^+}(\varphi)\right]$, where

$$\Pi_{p^+}(\varphi) = \max\left\{R\left(p^+\right), F_{p^+}(\varphi)\right\} \geq 1.$$

Let $w \in \{u, v\}$. We have

$$\begin{aligned} Bw(t) &= I_{0^+}^{\alpha-\beta,\sigma}(\delta.w) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(w)^n + q.f(t,w)\right) \\ &\leq I_{0^+}^{\alpha-\beta,\sigma}(\delta.w) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.(w)^n + q.\left(g_1 w + \epsilon_0\right)\right) \\ &\leq \left(\Pi_{p^+}(\varphi)\right)^m\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta) + I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n + q.\left(g_1 + \epsilon_0\right)\right)\right]. \end{aligned}$$

Then

$$\begin{aligned} |u(x) - v(x)| &\leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\left(\psi_{p^+}(Bu) + \psi_{p^+}(Bv) + \psi_{p^+}(|Z|)\right)(t)\,dt \\ &\leq \frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\left(\psi_{p^+}(Bu) + \psi_{p^+}(Bv) + \psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(\varphi)\right)\right)(t)\,dt \\ &\leq \Phi(x) \end{aligned}$$

where

$$\Phi(x)=\frac{1}{g(x)}\int_0^x \frac{1}{h(t)}\left(2.\psi_{p^+}\left(G(\varphi)\right)+\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(\varphi)\right)\right)(t)\,dt,$$

and

$$G(\varphi)=\left(\Pi_{p^+}(\varphi)\right)^m\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n+q.(g_1+\epsilon_0)\right)\right].$$

Thus, IVP (1.1) has the generalized semi-Hyers-Ulam-Rassias stability. □ □

Corollary 3.17. Assume that the function σ is bounded and the conditions (3.7), (3.8) and (3.12) hold true. Then IVP (1.1) has the semi-Hyers-Ulam-Rassias stability.

Proof . It is the direct consequence of the theorem 3.16 and remark 2.11. □

Now, we consider the conditions

$$\begin{cases} 1<p^-\le 2\le m+1<p^+, \\ \Delta_1(p^+)=\int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n+q.(g_1+\epsilon_0)\right)\right)(t)dt<1, \end{cases} \tag{3.14}$$

and

$$\left\{\begin{array}{c} \text{There exists a function } g_3:\mathbb{R}^+\to\mathbb{R}^+ \text{ such that for } x>0 \\ |f(t,x)-f(t,y)|\le g_3(t).|x-y| \text{ and} \\ \Pi=\frac{1}{(p^--1)}\int_0^\infty \frac{\left(\psi_{p^+}\left(B_0(1)(t)\right)\right)^{2-p^-}}{h.g(t)}\left(I_{0^+}^{\alpha-\beta,\sigma}(\delta)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n+q.g_3\right)\right)(t)\,dt<1 \end{array}\right\} \tag{3.15}$$

where

$$B_0(1)=I_{0^+}^{\alpha-\beta,\sigma}(\delta)+I_{0^+}^{\alpha,\sigma}\left(p(.,1)+q.(g_1+\epsilon_0)\right).$$

Theorem 3.18. Assume that the function σ is bounded and the conditions (3.7), (3.8), (3.14) and (3.15) hold true. Then IVP (1.1) is Hyers-Ulam stable, with the Hyers-Ulam stability constant

$$M=\max\left\{2\left(\frac{1-\Delta_1(p^+)}{\int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt}\right)^{1-p^+},\frac{\int_0^\infty \frac{1}{h.g(t)}\left(\psi_{p^+}(\lambda_4)(t)\right)^{2-p^-}.I_{0^+}^{\alpha,\sigma}(1)(t)\,dt}{(1-\Pi)(p^--1)}\right\}$$

where

$$\lambda_4(t)=B_0(1)(t)+\frac{I_{0^+}^{\alpha,\sigma}(1)(t)}{\phi_{p^+}\left(\int_0^\infty \frac{1}{h.g(s)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(s)ds\right)}.$$

.

Proof . The condition (3.14) means that (3.9) holds since

$$\Delta\left(p^+\right)\le\Delta_1\left(p^+\right)<1,$$

and so, we deduce from theorem 3.10 the existence of positive solution. Now, let $v\in P$, $\epsilon>0$ such that

$$\left|-\sum_{i=1}^{i=N}D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.(g.v)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t)+p(t,v(t))+q(t)f(t,v(t))\right|<\epsilon.$$

First, we show that

$$\|v\|_0\le\Sigma(\epsilon)=\max\left\{1,\frac{\psi_{p^+}(\epsilon)}{1-\Delta_1(p^+)}\int_0^\infty \frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt\right\}.$$

Let

$$z(t)=-\sum_{i=1}^{i=N}D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.(g.v)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}(\delta.v)(t)+p(t,v(t))+q(t)f(t,v(t)),$$

then $|z(x)| < \epsilon$, for $x > 0$. If $\|v\|_0 \geq 1$:

$$\begin{aligned} v(x) &= \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} \psi \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta.v) + I_{0^+}^{\alpha,\sigma} (p(t,v) + q.f(.,v) + z) \right) (t) dt \\ &\leq \int_0^x \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta.v) + I_{0^+}^{\alpha,\sigma} (p(t,v) + q.(g_1 v + \epsilon_0) + \epsilon) \right) (t) dt \\ &\leq \int_0^x \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta. \|v\|_0) + I_{0^+}^{\alpha,\sigma} (p(t, \|v\|_0) + q.(g_1 \|v\|_0 + \epsilon_0)) \right) (t) dt \\ &+ \int_0^x \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha,\sigma} (\epsilon) \right) (t) dt. \end{aligned}$$

$$\begin{aligned} \|v\|_0 &\leq \psi_{p^+} (\|v\|_0^m) \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha-\beta,\sigma} (\delta) + I_{0^+}^{\alpha,\sigma} (p(t,1) + q.(g_1 + \epsilon_0)) \right) (t) dt \\ &+ \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha,\sigma} (\epsilon) \right) (t) dt. \end{aligned}$$

The conditions $p^+ > m+1$ and (3.14) make that

$$\|v\|_0 \leq \frac{1}{1 - \Delta_1 (p^+)} \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha,\sigma} (\epsilon) \right) (t) dt.$$

Thus,

$$\|v\|_0 \leq \Sigma(\epsilon).$$

Now, let $u \in P$ be a solution of (1.1). It follows from Lemma 3.13 and condition (3.14) that

$$\|u\|_0 \leq R(p^+) = 1.$$

We distinguish two cases.

Case 1. If $\frac{\psi_{p^+}(\epsilon)}{1-\Delta_1(p^+)} \int_0^\infty \frac{1}{h.g(t)} \psi_{p^+} \left(I_{0^+}^{\alpha,\sigma} (1) \right) (t) dt \leq 1$, then $\max\{\|u\|_0, \|v\|_0\} \leq 1$. In other hand, for $x > 0$,

$$\begin{aligned} |u(x) - v(x)| &\leq \frac{1}{g(x)} \int_0^x \frac{1}{h(t)} |\psi(Bu) - \psi(Bv + Z)| (t) \, dt \\ &\leq \int_0^x \frac{1}{h.g(t)} |\psi(Bu) - \psi(Bv)| (t) \, dt \\ &+ \int_0^x \frac{1}{h.g(t)} |\psi(Bv) - \psi(Bv + Z)| (t) \, dt. \end{aligned}$$

So, there exist $(\chi_1, \chi_2) \in \left[\min\left(Bv, Bv + I_{0^+}^{\alpha,\sigma}(\epsilon)\right), \max\left(Bv, Bv + I_{0^+}^{\alpha,\sigma}(\epsilon)\right)\right] \times [\min(Bu, Bv), \max(Bu, Bv)]$ such that for $t > 0$

$$\begin{aligned} |\psi(Bv) - \psi(Bv + Z)| (t) &\leq \frac{1}{\sum_{i=1}^{i=N} (p_i - 1) (\psi(\chi_1(t)))^{(p_i - 2)}} . I_{0^+}^{\alpha,\sigma} (\epsilon) (t) \\ &\leq \frac{1}{(p^- - 1)} \left(\psi \left(Bv + I_{0^+}^{\alpha,\sigma} (\epsilon) \right) \right)^{2-p^-} . I_{0^+}^{\alpha,\sigma} (\epsilon) (t) \\ &\leq \frac{1}{(p^- - 1)} \left(\psi_{p^+} \left(B_0(1) + I_{0^+}^{\alpha,\sigma} (\epsilon) \right) \right)^{2-p^-} . I_{0^+}^{\alpha,\sigma} (\epsilon) (t) \end{aligned}$$

and

$$\begin{aligned} |\psi(Bu) - \psi(Bv)| &\leq \frac{1}{\sum_{i=1}^{i=N} (p_i - 1) (\psi(\chi_2(t)))^{(p_i - 2)}} . |Bu - Bv| \\ &\leq \frac{1}{(p^- - 1)} [\psi(B(1))]^{2-p^-} |Bu - Bv| \\ &\leq \frac{1}{(p^- - 1)} \left[\psi_{p^+} (B_0(1)) \right]^{2-p^-} |Bu - Bv| \end{aligned}$$

with

$$\begin{aligned}
|Bu-Bv|(t) &\le I_{0^+}^{\alpha-\beta,\sigma}(\delta.|u-v|)(t)+I_{0^+}^{\alpha,\sigma}(|p(.,u)-p(.,v)|+q.|f(.,u)-f(.,v)|)(t)\\
&\le \|u-v\|_0\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta)(t)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n(\max(u,v))^{n-1}+q.g_3\right)(t)\right]\\
&\le \|u-v\|_0\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta)(t)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n+q.g_3\right)(t)\right].
\end{aligned}$$

This means that

$$|\psi(Bu)-\psi(Bv)|\le\frac{\left(\psi_{p^+}(B_0(1)(t))\right)^{2-p^-}}{(p^--1)}.\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta)(t)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n+q.g_3\right)(t)\right]\|u-v\|_0 .$$

Then

$$\begin{aligned}
|u(x)-v(x)| &\le \frac{\|u-v\|_0}{(p^--1)}\int_0^x\frac{\left(\psi_{p^+}(B_0(1)(t))\right)^{2-p^-}}{h.g(t)}\left[I_{0^+}^{\alpha-\beta,\sigma}(\delta)(t)+I_{0^+}^{\alpha,\sigma}\left(\sum_{n=1}^{n=m}\eta_n.n+q.g_3\right)(t)\right]dt\\
&\quad+\int_0^x\frac{1}{h.g(t)}\frac{1}{(p^--1)}\left(\psi_{p^+}\left(B_0(1)+I_{0^+}^{\alpha,\sigma}(\epsilon)\right)(t)\right)^{2-p^-}.I_{0^+}^{\alpha,\sigma}(\epsilon)(t)\,dt.
\end{aligned}$$

Hypothesis (3.15) leads.

$$\begin{aligned}
\|u-v\| &\le \frac{\epsilon}{(1-\Pi)(p^--1)}\int_0^\infty\frac{1}{h.g(t)}\left(\psi_{p^+}\left(B_0(1)+I_{0^+}^{\alpha,\sigma}(\epsilon)\right)(t)\right)^{2-p^-}.I_{0^+}^{\alpha,\sigma}(1)(t)\,dt\\
&\le \frac{\epsilon}{(1-\Pi)(p^--1)}\int_0^\infty\frac{1}{h.g(t)}\left(\psi_{p^+}(\lambda_4(t))\right)^{2-p^-}.I_{0^+}^{\alpha,\sigma}(1)(t)\,dt\\
&\le M.\epsilon.
\end{aligned}$$

Case 2. If $\frac{\psi_{p^+}(\epsilon)}{1-\Delta_1(p^+)}\int_0^\infty\frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt>1$, then

$$\epsilon>\phi_{p^+}(\theta),\ \theta=\frac{1-\Delta_1(p^+)}{\int_0^\infty\frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt}$$

and so, for $x>0$

$$\begin{aligned}
|u(x)-v(x)| &\le 1+\frac{\psi_{p^+}(\epsilon)}{1-\Delta_1(p^+)}\int_0^\infty\frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt\\
&< \frac{2.\psi_{p^+}(\epsilon)}{1-\Delta_1(p^+)}\int_0^\infty\frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt\\
&\le 2\epsilon\left(\frac{1-\Delta_1(p^+)}{\int_0^\infty\frac{1}{h.g(t)}\psi_{p^+}\left(I_{0^+}^{\alpha,\sigma}(1)\right)(t)dt}\right)^{1-p^+}\\
&\le \epsilon.M
\end{aligned}$$

The proof is finished. □

Example 3.19. We consider the problem

$$\begin{cases}-\sum_{i=1}^{i=N}D_{0^+}^{\alpha,\omega,\sigma}\left(\phi_{p_i}\left(h.(g.u)'\right)\right)(t)+D_{0^+}^{\beta,\omega,\sigma}(\delta.u)(t)+2\delta(t).u(t)=0,\ t>0,\\ u(0)=0,\end{cases}\tag{3.16}$$

with $\phi(x) = x + |x|.x$, $\sigma(t) = 1 - e^{-t}$ and $g(t) = \dfrac{a.e^t}{h(t)}$, where

$$\begin{aligned} a &> 2\sup\left\{ I_{0^+}^{\alpha-\beta,\sigma}(1)(t) + I_{0^+}^{\alpha,\sigma}(2)(t), \sqrt{I_{0^+}^{\alpha-\beta,\sigma}(1)(t) + I_{0^+}^{\alpha,\sigma}(2)}, t>0 \right\} \\ &= 2\sqrt{\frac{1}{\Gamma(\alpha-\beta)} + \frac{2}{\Gamma(\alpha)}} \end{aligned}$$

and

$$\begin{cases} \dfrac{1}{h} \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^+), \lim\limits_{x\to+\infty} \frac{1}{g(x)} \int_0^x \dfrac{ds}{h(s)} = l < \infty, \text{ and} \\ \text{the functions } \delta,\ \eta_n, \text{ and } q \text{ are continuous, positive and bounded.} \end{cases}$$

We get $f(t,x) = x$, $q = \delta$, $p(t,x) = \delta.x$. Hypothesis (3.8) and (3.7) hold, since σ is bounded, with $\epsilon_0 = 0$ and $g_1(t) = 1$. Moreover,

$$I_{0^+}^{\alpha-\beta,\sigma}(1)(t) = \frac{(\sigma(t))^{\alpha-\beta}}{\Gamma(\alpha-\beta)}, I_{0^+}^{\alpha,\sigma}(1)(t) = \frac{(\sigma(t))^{\alpha}}{\Gamma(\alpha)}.$$

and

$$\begin{aligned} p^- &= 2 = m+1 < p^+ = 3,\ g_1 = g_3 = 1, \\ \Delta_1(3) &= \int_0^\infty \frac{1}{h.g}\psi_3\left(I_{0^+}^{\alpha-\beta}(\delta) + I_{0^+}^{\alpha}(2\delta)\right)(t)\,dt = \sqrt{\delta^+}\int_0^\infty \frac{1}{h.g}\sqrt{\left(I_{0^+}^{\alpha-\beta}(1) + I_{0^+}^{\alpha}(2)\right)(t)}\,dt \\ &< \frac{\sqrt{\delta^+}}{2}, \text{ and} \\ \Pi &= \int_0^\infty \frac{1}{h.g}\left(I_{0^+}^{\alpha-\beta}(\delta) + I_{0^+}^{\alpha}(2\delta)\right)(t)\,dt \le \delta^+\int_0^\infty \frac{1}{h.g}\left(I_{0^+}^{\alpha-\beta}(1) + I_{0^+}^{\alpha}(2)\right)(t)\,dt < \frac{\delta^+}{2} \end{aligned}$$

where $\delta^+ = \sup\{\delta(t),\ t \ge 0\}$. Then the conditions (3.7), (3.8), (3.14) and (3.15) hold when $\delta^+ \le 2$. Thus, we deduce fom theorem 3.18 that for all $\delta \in C(\mathbb{R}^+, [0,2])$, IVP (3.16) is Hyers-Ulam stable and so, it is semi-Hyers-Ulam-Rassias stable.

4 Conclusion

In this work, we have discussed about the existence of the positive solutions of Hilfer fractional IVP (1.1) involving a (p_1,p_2...p_n)-Laplacian operator. Also, we presented sufficient conditions for the asymptotic, Hyers-Ulam and semi-Hyers-Ulam-Rassias stability of mentioned IVP (1.1).

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