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Multivalued relation-theoretic graph contraction principle with applications

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Abstract

In this paper, we present a new generalization of Nadler's fixed point theorem for multivalued relation-theoretic graph contractions on relational metric spaces. Our results extend and generalize the result of Shukla and Rodríguez-López (Questiones Mathematiae, (2019) 1-16), Nadler (Pacific J. Math. 30 (1969), 475-488), Alam and Imdad (J. Fixed Point Theory and Appl., 17(4) (2015), 693-702) and many others in the existing literature of fixed point theory. Some good examples are also included to illustrate the usefulness of our main results. Moreover, we have an application to generalized coupled fixed point problems.

Keywords: Fixed point, coupled fixed point;, contraction mapping, binary relation
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1 Introduction

The first generalization of the classical Banach contraction principle (Bcp) [4] for multivalued mappings was given by S. B. Nadler Jr. [7] in 1969. After that, several authors have obtained the extension and generalization of this principle to different kinds of multivalued contraction conditions and various spaces. Petruşel et al. [9] generalized Nadler's fixed point principle by introducing the concept of multivalued graph contraction in metric spaces. They showed the connection of their results in some variational analysis problems and gave an application to generalized coupled fixed point theorems.

In 2015, Alam and Imdad introduced a relation-theoretic contraction principle, and generalized the Bcp [4] to an amorphous binary relation. This principle weakened and generalized various fixed point results in the literature, which attracts various authors to work along this line. As a result, several extension of this principle has been obtained by many mathematicians (see [2], [3], [6], [10], [13] and [14]). Recently, Shukla and Rodríguez-López [13] generalized the

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relation-theoretic contraction principle for multivalued mappings on complete metric spaces and obtained a relation-theoretic version of the fixed point result of Nadler [7]. They also have shown that under universal relation, their result reduces to Nadler's fixed point theorem.

In this paper, we present a new variant of Nadler's fixed point theorem for multivalued mappings in the setting of metric spaces equipped with an amorphous binary relation. We do this, by introducing the notion of multivalued relation-theoretic graph contraction and establish some fixed point results for such mappings in relational metric spaces. Our results generalize the result of Nadler [7], Shukla, and Rodríguez-López [13], Alam and Imdad [1] and many others in the existing literature. We also present some examples in which all the classical fixed point theorems in metric spaces do not work, but our results remain valid. Moreover, we also gave an application to generalized coupled fixed point problems.

2 Preliminaries

Firstly, we adopt some notations which are very useful to prove our main results. Let $P(\mathcal{M})$ be the family of all nonempty subset of $\mathcal{M}$ and $(\mathcal{M}, d)$ be a metric space. Here, by $\mathbb{N}$, $\mathbb{N}_0$, $CL(\mathcal{M})$, $C(\mathcal{M})$, and $CB(\mathcal{M})$ means the set of positive integers, the set of nonnegative integers, the set of all nonempty closed subsets of $\mathcal{M}$, the set of all nonempty compact subset of $\mathcal{M}$, and the set of all nonempty, closed and bounded subsets of $\mathcal{M}$, respectively. For any nonempty subsets $\mathcal{U}, \mathcal{V}$ of $\mathcal{M}$ and $\xi \in \mathcal{M}$, we also follow the following notations:

$$D_d(\xi,\mathcal{U}) = \inf\{d(\xi, u) : u \in \mathcal{U}\},$$
$$D_d(\mathcal{U},\mathcal{V}) = \inf\{d(u, v) : u \in \mathcal{U}, v \in \mathcal{V}\},$$
$$\delta_d(\mathcal{U},\mathcal{V}) = \sup\{D_d(u,\mathcal{V}) : u \in \mathcal{U}\},$$
$$H(\mathcal{U},\mathcal{V}) = \max[\delta_d(\mathcal{U},\mathcal{V}), \delta_d(\mathcal{V},\mathcal{U})],$$

where H is Hausdorff-Pompeiu functional on $P(\mathcal{M})$ generated by metric d.

Definition 2.1. [1]. Let $\mathcal{M}$ be a nonempty set. A binary relation $\mathscr{R}$ on a nonempty set $\mathcal{M}$ is a subset of $\mathcal{M} \times \mathcal{M}$. We say that ξ relates to η under $\mathscr{R}$ if and only if $(\xi, \eta) \in \mathscr{R}$ and ξ does not relate to η under $\mathscr{R}$ if and only if $(\xi, \eta) \notin \mathscr{R}$

Definition 2.2. [1]. Let $\mathscr{R}$ be a binary relation defined on a nonempty set $\mathcal{M}$. We say ξ and η are $\mathscr{R}$-comparable if either $(\xi, \eta) \in \mathscr{R}$ or $(\eta, \xi) \in \mathscr{R}$. We denote it by $[\xi, \eta] \in \mathscr{R}$.

Definition 2.3. [1]. Let $\mathcal{M}$ be a nonempty set and $\mathscr{R}$ a binary relation on $\mathcal{M}$. A sequence $\{\xi_n\} \in \mathcal{M}$ is called $\mathscr{R}$-preserving if

$$(\xi_n, \xi_{n+1}) \in \mathscr{R}, \quad \text{for all } n \in \mathbb{N}_0.$$

Definition 2.4. [1]. Let $(\mathcal{M}, d)$ be a metric space. A binary relation $\mathscr{R}$ defined on $\mathcal{M}$ is called *d-self-closed* if whenever $\{\xi_n\}$ is $\mathscr{R}$-preserving sequence on $\mathcal{M}$ and

$$\xi_n \xrightarrow{d} \xi$$

then the sequence $\{\xi_n\}$ has a subsequence $\{\xi_{n_k}\}$ with $[\xi_{n_k}, \xi] \in \mathscr{R}$ for all $k \in \mathbb{N}_0$.

Definition 2.5. [1, 12]. Let $\mathscr{R}$ be a binary relation on a nonempty set $\mathcal{M}$ and f be a selfmapping on $\mathcal{M}$. If for all $\xi, \eta \in \mathcal{M}$ with

$$(\xi, \eta) \in \mathscr{R} \implies (f\xi, f\eta) \in \mathscr{R}$$

then the relation $\mathscr{R}$ is called f-closed and the mapping f is called comparative mapping on $\mathcal{M}$ under the binary relation $\mathscr{R}$.

Definition 2.6. [2]. Let $(\mathcal{M}, d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$. If every $\mathscr{R}$-preserving Cauchy sequence in $\mathcal{M}$ is convergent and converges to some point in $\mathcal{M}$, then say that $(\mathcal{M}, d)$ is $\mathscr{R}$-complete.

Remark 2.7. Every complete metric space is $\mathscr{R}-$complete, for any binary relation $\mathscr{R}$ on $\mathcal{M}$ and both definitions coincide under the universal relation.

Definition 2.8. [13]. Let $\mathscr{R}$ be a binary relation on a nonempty set $\mathcal{M}$ and $k \in \mathbb{N}$. Then, an element $\eta \in \mathcal{M}$ is called k-connected to $\xi \in \mathcal{M}$, if there exists a path of length k from ξ to η, that is there exist $z^i \in \mathcal{M}$, $i = 0, 1, \ldots, k$ such that $z^0 = \xi$, $z^k = \eta$ and $(z^i, z^{i+1}) \in \mathscr{R}$, $i = 0, 1, \ldots, k-1$.

Let $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a multivalued mapping and $f : \mathcal{M} \to \mathcal{M}$ be a singlevalued mapping. A sequence $\{\xi_n\} \subset \mathcal{M}$ is said to be a trajectory of $\mathcal{F}$, starting at ξ_1, if $\xi_{n+1} \in \mathcal{F}\xi_n$, for all $n \in \mathbb{N}$. By $P(\xi, k) = \{\eta \in \mathcal{M}$: there exists a path of length k from ξ to $\eta\}$, we denote the set of all $\eta \in \mathcal{M}$ such that there exists a path of length k from ξ to η, by $G(\mathcal{F}) = \{(\xi, \eta) : \xi \in \mathcal{M}, \eta \in \mathcal{F}\xi\}$ the graph of $\mathcal{F}$ and by $G(f) = \{(\xi, f\xi) : \xi \in \mathcal{M}\}$ we denote the graph of f. If $G(\mathcal{F})$ is a closed subset of $\mathcal{M} \times \mathcal{M}$, then we say $\mathcal{F}$ is closed or $\mathcal{F}$ has closed graph.

Lemma 2.9. [9]. If $(\mathcal{M}, d)$ be a metric space then we have:

(a) H is generalized metric on $CL(\mathcal{M})$.

(b) Let $\mathcal{U} \in CL(\mathcal{M})$ and $\xi \in \mathcal{M}$. Then $D_d(\xi,\ \mathcal{U}) = 0$ if and only if $\xi \in \mathcal{U}$.

(c) If $\mathcal{U}, \mathcal{V} \in P(\mathcal{M})$ and $\lambda > 1$, then for every $u \in \mathcal{U}$ there exists $v \in \mathcal{V}$ such that $d(u, v) \leq \lambda\ \delta_d(\mathcal{U}, \mathcal{V})$.

(d) If there $\epsilon > 0$ such that, for each $u \in \mathcal{U}$ there exists $v \in \mathcal{V}$ with $d(u, v) \leq \epsilon$, then $\delta_d(\mathcal{U}, \mathcal{V}) \leq \epsilon$.

Definition 2.10. [13]. Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{F} : \mathcal{M} \to CB(\mathcal{M})$ be a multivalued mapping. Then, a binary relation $\mathscr{R}$ on $\mathcal{M}$ is called $\mathcal{F}$-d-closed if,

$$(\xi, \eta) \in \mathscr{R}, \quad \mu \in \mathcal{F}\xi, \quad \nu \in \mathcal{F}\eta, \quad d(\mu, \nu) \leq d(\xi, \eta) \implies (\mu, \nu) \in \mathscr{R}. \tag{2.1}$$

If we consider $\mathcal{F} = f$ as a singlevalued selfmapping on $\mathcal{M}$ in the Definition 2.10 then $\mathscr{R}$ is called f-d-closed and condition (2.1) reduces to the following condition

$$(\xi, \eta) \in \mathscr{R}, \qquad d(f\xi, f\eta) \leq d(\xi, \eta) \implies (f\xi, f\eta) \in \mathscr{R}. \tag{2.2}$$

It is shown in [13] that the f-d-closedness assumption is weaker than the f-closedness assumption.

Definition 2.11. [13]. Let $(\mathcal{M}, d)$ be a metric space endowed with a binary relation $\mathscr{R}$. A mapping $\mathcal{F} : \mathcal{M} \to CB(\mathcal{M})$ is called multivalued relation-theoretic contraction if,

$$H(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi, \eta), \text{ for all } (\xi, \eta) \in \mathscr{R} \text{ and some } \alpha \in [0, 1). \tag{2.3}$$

3 Fixed point results for multivalued relation-theoretic graph contraction mappings

In this section, we begin with some new notions and definitions, which are essential to prove our main results.

Definition 3.1. Let $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a mapping. A relational graph of $\mathcal{F}$ under the relation $\mathscr{R}$ on $\mathcal{M}$ is denoted by $G(\mathcal{F}; \mathscr{R})$ and defined as:

$$G(\mathcal{F}; \mathscr{R}) := \{(\xi, \eta) \in \mathscr{R} : \xi \in \mathcal{M}, \quad \eta \in \mathcal{F}\xi\}.$$

Further, we say that $\mathcal{F}$ has a closed relational graph or $\mathcal{F}$ is $\mathscr{R}$-closed when $G(\mathcal{F}; \mathscr{R})$ is a closed subset of $\mathcal{M} \times \mathcal{M}$. Moreover, when $\mathcal{R}$ is universal relation then the definition of the relational graph of $\mathcal{F}$ coincides with the definition of the graph of $\mathcal{F}$.

Definition 3.2. Let $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a mapping. By $\mathcal{M}(\mathcal{F}; \mathscr{R})$ means the set of all points $\xi \in \mathcal{M}$ for which $(\xi, \eta) \in G(\mathcal{F}; \mathscr{R})$, that is

$$\mathcal{M}(\mathcal{F}; \mathscr{R}) = \{\xi \in \mathcal{M} : (\xi, \eta) \in G(\mathcal{F}; \mathscr{R})\}.$$

Definition 3.3. Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a multivalued mapping. A binary relation $\mathscr{R}$ on $\mathcal{M}$ is said to be $\mathcal{F}_G$-d-closed if the following condition satisfies:

$$(\xi, \eta) \in G(\mathcal{F}; \mathscr{R}), \qquad z \in \mathcal{F}\eta, \qquad d(\eta, z) \leq d(\xi, \eta) \implies (\eta, z) \in G(\mathcal{F}; \mathscr{R}).$$

It is obvious by the Definition 3.3 that if a relation $\mathscr{R}$ is $\mathcal{F}$-d-closed on a space $\mathcal{M}$ then it is $\mathcal{F}_G$-d-closed in the same space. However, the following example illustrates that the converse is not true.

Example 3.4. Assume that $\mathcal{M} = [0,1]$ and $d(\eta,\xi) = |\eta - \xi|$, for all $\eta, \xi \in \mathcal{M}$. Then $(\mathcal{M}, d)$ form a complete metric space. We define $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ by $\mathcal{F}(\xi) = [0, \xi/2]$, for $\xi \in \mathcal{M}$ and a binary relation $\mathscr{R} = \{(0,0),(0,1),(1,1),(1/2,1)\}$ on $\mathcal{M}$. Then, we observe $G(\mathcal{F};\mathscr{R}) = \{(0,0)\}$ and the relation $\mathscr{R}$ is $\mathcal{F}_G$-d-closed on $\mathcal{M}$. Indeed, for $(\xi,\eta) = (0,0) \in G(\mathcal{F};\mathscr{R})$, $z \in F(0)$, we have $d(\eta, z) = d(\xi,\eta)$ and $(\eta, z) \in G(\mathcal{F};\mathscr{R})$. However, $\mathscr{R}$ is not $\mathcal{F}$-d-closed on $\mathcal{M}$, as for $(\xi,\eta) = (1/2, 1) \in \mathscr{R}$, there exist $\mu = 1/4 \in \mathcal{F}\xi$ and $\nu = 1/4 \in \mathcal{F}\eta$ such that $d(\mu,\nu) < d(\mathcal{F}\xi, \mathcal{F}\eta)$ but $(\mu,\nu) \notin \mathscr{R}$.

Definition 3.5. Let $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a mapping on a metric space $(\mathcal{M}, d)$ and $\mathscr{R}$ be a binary relation on $\mathcal{M}$. Then, the mapping $\mathcal{F}$ is called a multivalued relation-theoretic graph contraction if,

$$H(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi,\eta), \text{ for some } \alpha \in [0,1) \text{ and all } (\xi,\eta) \in G(\mathcal{F};\mathscr{R}). \tag{3.1}$$

From the Definition 3.5, it is easy to conclude that the multivalued relation-theoretic graph contraction generalizes the multivalued relation-theoretic contraction due to Shukla and Rodríguez-López [13] and under the universal relation, we get the definition of the multivalued α-graph contraction due to Petrusel et al. [9].

Definition 3.6. [9]. Let $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a mapping on a metric space $(\mathcal{M}, d)$. Then, the mapping $\mathcal{F}$ is called a multivalued α-graph contraction if,

$$H(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi,\eta), \text{ for some } \alpha \in [0,1) \text{ and all } (\xi,\eta) \in G(\mathcal{F}). \tag{3.2}$$

Now, we state and prove our main results for the existence of fixed points in relational metric spaces.

Theorem 3.7. Let $(\mathcal{M}, d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$, and $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a multivalued relation-theoretic graph contraction. Assume that the relation $\mathscr{R}$ is $\mathcal{F}_G$-d-closed and $\mathcal{F}$ has a nonempty closed relational graph under the relation $\mathscr{R}$. Then, for each $\xi \in \mathcal{M}(\mathcal{F};\mathscr{R})$, there exists a trajectory $\{\xi_n\}$ of $\mathcal{F}$, starting from $\xi_1 = \xi$, which converges to fixed point of $\mathcal{F}$.

Proof . Let (ξ_0, ξ_1) is an arbitrary element in $G(\mathcal{F};\mathscr{R})$ and $1 < q < 1/\alpha$. Then, in view of Lemma 2.9, there exits $\xi_2 \in \mathcal{F}\xi_1$ such that

$$d(\xi_1, \xi_2) \leq qH(\mathcal{F}\xi_0, \mathcal{F}\xi_1). \tag{3.3}$$

As $(\xi_0, \xi_1) \in G(\mathcal{F};\mathscr{R})$ so from (3.1), we have

$$H(\mathcal{F}\xi_0, \mathcal{F}\xi_1) \leq \alpha d(\xi_0, \xi_1). \tag{3.4}$$

From inequality (3.3) and (3.4), it follows that

$$d(\xi_1, \xi_2) \leq \alpha q d(\xi_0, \xi_1) \leq d(\xi_0, \xi_1).$$

By assumption of $\mathcal{F}_G$-d-closedness of $\mathscr{R}$, we obtain that $(\xi_1, \xi_2) \in G(\mathcal{F};\mathscr{R})$. Continuing this process, we get $\{\xi_n\}$ is a $\mathscr{R}$-preserving sequence in $\mathcal{M}(\mathcal{F};\mathscr{R})$ and

$$d(\xi_n, \xi_{n+1}) \leq (\alpha q)^n \; d(\xi_0, \xi_1), \text{ for } n \in \mathbb{N}.$$

Using classical approach, we get the sequence $\{\xi_n\}$ is a $\mathscr{R}$-Cauchy and also

$$d(\xi_n, \xi_{n+p}) \leq \left(\frac{1 - (\alpha q)^p}{1 - \alpha q} \right) (\alpha q)^n d(\xi_0, \xi_1),$$

for every $n, p \in \mathbb{N}$. Since the sequence $\{\xi_n\}$ is a Cauchy in $\mathcal{M}(\mathcal{F};\mathscr{R})$ so (ξ_n, ξ_{n+1}) is a Cauchy sequence in $G(\mathcal{F};\mathscr{R})$ and by the condition of the closed relational graph of $\mathcal{F}$, there exists a pair $(\xi^*, \xi^*) \in G(\mathcal{F};\mathscr{R})$ such that the sequence $(\xi_n, \xi_{n+1}) \to (\xi^*, \xi^*)$. Thus $\xi^* \in \mathcal{F}\xi^*$ means ξ^* is a fixed point of $\mathcal{F}$. □

Theorem 3.8. Let $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $(\mathcal{M}, d)$ be a metric space. Assume that $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a multivalued relation-theoretic graph contraction and the following assertions are hold:

(a) $\mathcal{M}(\mathcal{F};\mathscr{R})$ be a nonempty set,

(b) $\mathscr{R}$ is $\mathcal{F}_G$-d-closed,

(c) one of the following condition is true:

   (i) $\mathcal{F}$ has a closed graph on $\mathcal{M}$ or

   (ii) for any sequence $\{\xi_n\}$ in $\mathcal{M}(\mathcal{F};\mathscr{R})$, if $\xi_n \to \xi$ and $\xi_{n+1} \in \mathcal{F}\xi_n$, for all $n \in \mathbb{N}$, then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $[\xi_{n_k}, \xi] \in G(\mathcal{F};\mathscr{R})$, for all $k \in \mathbb{N}$,

(d) $\mathcal{M}$ is $\mathscr{R}$-complete.

Then, for each $\xi \in \mathcal{M}(\mathcal{F};\mathscr{R})$, there exists a trajectory $\{\xi_n\}$ of $\mathcal{F}$, starting with $\xi_1 = \xi$ and converges to a fixed point of $\mathcal{F}$.

Proof . Since $\mathcal{M}(\mathcal{F};\mathscr{R})$ is nonempty set, we suppose that ξ_0 be an arbitrary point in $\mathcal{M}(\mathcal{F};\mathscr{R})$. Then, there is a point ξ_1 in $\mathcal{F}\xi_0$ such that $(\xi_0, \xi_1) \in G(\mathcal{F};\mathscr{R})$. Now, proceeding the proof of Theorem 3.7, we get the sequence $\{\xi_n\}$ is Cauchy in $\mathcal{M}(\mathcal{F};\mathscr{R})$, so in $\mathcal{M}$. And by condition (d), the sequence $\{\xi_n\}$ converges to some point (say) ξ^* in $\mathcal{M}$.

Next, we assume that assumption (c) holds and $\mathcal{F}$ has a closed graph on $\mathcal{M}$. Since $\{\xi_n\}$ is a trajectory of $\mathcal{F}$ and $\xi_n \to \xi^*$ as $n \to \infty$. Therefore, by closedness of $G(\mathcal{F})$, we have $(\xi_n, \xi_{n+1}) \to (\xi^*, \xi^*) \in G(\mathcal{F})$ and $\xi^* \in \mathcal{F}\xi^*$ means ξ^* is a fixed point of $\mathcal{F}$.

Alternatively, let assertion (ii) of (c) is true. Then, the sequence $\{\xi_n\}$ has a subsequence $\{\xi_{n_k}\}$ such that $[\xi_{n_k}, \xi^*] \in \mathscr{R}$ for all $k \in \mathbb{N}_0$. As $\xi_{n_k+1} \in \mathcal{F}\xi_{n_k}$, $k \in \mathbb{N}$. Then, for each sequence $\{\epsilon_k\}$, where $\epsilon_k > 0$, there always exists a sequence $\{\eta_k\}$ such that $\eta_k \in \mathcal{F}\xi^*$ and

$$d(\xi_{n_k+1}, \eta_k) \leq H(\mathcal{F}\xi_{n_k}, \mathcal{F}\xi^*) + \epsilon_k, \;\; k \in \mathbb{N}.$$

Since $[\xi_{n_k}, \xi^*] \in \mathbb{R}$, for all $k \in \mathbb{N}$ then by (3.1) and the above inequality, it follows that

$$d(\xi_{n_k+1}, \eta_k) \leq \alpha d(\xi_{n_k}, \xi^*) + \epsilon_k, \qquad k \in \mathbb{N}. \tag{3.5}$$

Using triangle inequality and from (3.5), we get

$$\begin{aligned} d(\eta_k, \xi^*) &\leq d(\eta_k, \xi_{n_k+1}) + d(\xi_{n_k+1}, \xi^*) \\ &\leq d(\xi_{n_k+1}, \xi^*) + \alpha d(\xi_{n_k}, \xi^*) + \epsilon_k, \qquad k \in \mathbb{N}. \end{aligned}$$

For particular choice of the sequence $\{\epsilon_k\}$ as $\epsilon_k \to 0$ when $k \to \infty$, the above inequality gives,

$$\lim_{k\to\infty} d(\eta_k, \xi^*) = 0.$$

Since $\eta_k \in \mathcal{F}\xi^*$, for all $k \in \mathbb{N}$ and $\mathcal{F}\xi^*$ is a closed subset of $\mathcal{M}$, we must have $\xi^* \in \mathcal{F}\xi^*$. Thus ξ^* is a fixed point of $\mathcal{F}$. □

The following theorem is a special case of Theorem 3.7:

Theorem 3.9. Let $(\mathcal{M}, d)$ be a metric space, $\mathscr{R}$ a binary relation on $\mathcal{M}$ and the mapping $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ has a nonempty relational graph under the binary relation $\mathscr{R}$. Supposed that $\mathscr{R}$ is $\mathcal{F}_G$-d-closed and

$$\delta_d(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi, \eta), \;\text{ for some } \alpha \in [0,1) \text{ and all } (\xi, \eta) \in G(\mathcal{F};\mathscr{R}). \tag{3.6}$$

Then, for each $\xi \in \mathcal{M}(\mathcal{F};\mathscr{R})$, there exists a trajectory $\{\xi_n\}$ of $\mathcal{F}$, starting from $\xi_1 = \xi$ and converges to fixed point of $\mathcal{F}$.

Proof . Since the multivalued relation-theoretic graph contraction condition (3.1) implies condition (3.6), the conclusion follows by Theorem 3.7. □

Similarly, we can obtain the following theorem as a direct consequence of Theorem 3.8.

Theorem 3.10. Let $(\mathcal{M}, d)$ be a metric space and $\mathscr{R}$ be binary relation on $\mathcal{M}$. Suppose $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a multivalued mapping and the following conditions are true:

(a) $\mathcal{M}(\mathcal{F};\mathscr{R}) \neq \emptyset$,

(b) $\mathscr{R}$ is $\mathcal{F}_G$-d-closed,

(c) at least one of the following assertion is true:

(i) $\mathcal{F}$ has a closed graph on $\mathcal{M}$,

(ii) for any sequence $\{\xi_n\}$ in $\mathcal{M}(\mathcal{F};\mathscr{R})$, if $\xi_n \to \xi$ and $\xi_{n+1} \in \mathcal{F}\xi_n, n \in \mathbb{N}$, then the sequence $\{\xi_n\}$ has a subsequence $\{\xi_{n_k}\}$ such that $[\xi_{n_k}, \xi] \in \mathscr{R}$, for all $k \in \mathbb{N}$,

(d) $\mathcal{F}$ satisfies the following condition:

$$\delta_d(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi, \eta), \text{ for some } \alpha \in [0,1) \text{ and all } (\xi, \eta) \in G(\mathcal{F};\mathscr{R}),$$

(e) $\mathcal{M}$ is $\mathscr{R}$-complete.

Then, for each $\xi \in \mathcal{M}(\mathcal{F};\mathscr{R})$, there exists a trajectory $\{\xi_n\}$ of $\mathcal{F}$, starting from $\xi_1 = \xi$ and converges to fixed point of $\mathcal{F}$.

If we take $\mathscr{R}$ as a complete relation (that is $\mathscr{R} = \mathcal{M} \times \mathcal{M}$ or $\mathcal{M} = \mathcal{M}(\mathcal{F};\mathscr{R})$) in Theorem 3.7 and Theorem 3.8, we obtain the following result of Nadler.

Corollary 3.11. (Nadler's fixed point theorem): Let $(\mathcal{M}, d)$ be a complete metric space. Suppose a mapping $\mathcal{F} : \mathcal{M} \to CB(\mathcal{M})$ satisfies the following condition:

$$H(\mathcal{F}\xi, \mathcal{F}\eta) \leq \alpha d(\xi, \eta), \tag{3.7}$$

for some $\alpha \in [0,1)$ and all $\xi, \eta \in \mathcal{M}$. Then $\mathcal{F}$ has a fixed point in $\mathcal{M}$.

4 Examples and Discussion

The following examples show that the multivalued relation-theoretic graph contraction is indeed more useful than the multivalued graph contraction and multivalued relation-theoretic contraction.

Example 4.1. Let $\mathcal{M} = [0,1]$ be a metric space equipped with usual metric d, that is $d(\xi, \eta) = |\xi - \eta|$, for $\xi, \eta \in \mathcal{M}$. Consider a binary relation $\mathscr{R}$ on $\mathcal{M}$ as

$$\mathscr{R} = \{(1/2, 1/2), (1/2, 1), (1, 1/2), (1, 1/3)\}$$

and the mapping $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ such that

$$\mathcal{F}(\xi) = \begin{cases} [1/2, 1-\xi], & \text{if } \xi \in [0, 1/2], \\ \{1/2\}, & \text{otherwise.} \end{cases}$$

Then, clearly $G(\mathcal{F};\mathscr{R}) = \{(1/2, 1/2), (1, 1/2)\} \neq \emptyset$ is a closed subset of $\mathcal{M} \times \mathcal{M}$ and $\mathcal{F}$ satisfies the contraction condition (3.1) for each $(\xi, \eta) \in G(\mathcal{F};\mathscr{R})$. Also, for $(\xi, \eta) \in G(\mathcal{F};\mathscr{R})$ and $z \in \mathcal{F}\eta$, we see that $d(\eta, z) \leq d(\xi, \eta)$ and $(\eta, z) \in G(\mathcal{F};\mathscr{R})$. Hence, $\mathscr{R}$ is $\mathcal{F}_G$-d-closed on $\mathcal{M}$. On the other hand, $\mathscr{R}$ is not $\mathcal{F}$-d-closed on $\mathcal{M}$, as for $(\xi, \eta) = (1, 1/3) \in \mathscr{R}$, there exists $z = 7/12 \in Fy$ such that $d(\eta, z) \leq d(\xi, \eta)$ but $(\eta, z) \notin \mathscr{R}$. Thus, all the assumptions of Theorem 3.7 are verified and point $\xi = 1/2$ is a fixed point of $\mathcal{F}$ in $\mathcal{M}$.

Example 4.2. Let $\mathcal{M} = [0,3]$ and $d(\xi, \eta) = |\xi - \eta|$, for all $\xi, \eta \in \mathcal{M}$. Define a set $S = \left\{\left(\frac{1}{5^n}, 0\right) : n \in \mathbb{N}\right\}$ and a binary relation

$$\mathscr{R} = \{(0,0), (0,1), (1,2), (2,1)\} \cup S \cup \left\{\left(\frac{n+1}{n}, \frac{n+2}{n+1}\right) : n \in \mathbb{N}\right\}$$

on $\mathcal{M}$. Suppose that $\mathcal{F} : \mathcal{M} \to P(\mathcal{M})$ be a mapping defined by

$$\mathcal{F}(\xi) = \begin{cases} \{0, \xi/5\}, & \text{if } \xi \in [0,1], \\ \{2\}, & \text{if } \xi \in (1,3]. \end{cases}$$

Then, there are the following observations:
(a) $\mathcal{M}(\mathcal{F};\mathscr{R})$ **is a nonempty set.**

As $0 \in F0$ and $(0,0) \in \mathscr{R}$ implies $0 \in \mathcal{M}(\mathcal{F};\mathscr{R})$. Hence $\mathcal{M}(\mathcal{F};\mathscr{R})$ is a nonempty set. Similarly, one can easily verify that $\xi = \dfrac{1}{5^n} \in \mathcal{M}(\mathcal{F};\mathscr{R})$, for all $n \in \mathbb{N}$ and $\mathcal{M}(\mathcal{F};\mathscr{R}) = \left\{\frac{1}{5^n} : n \in \mathbb{N}\right\} \cup \{0\}$.

(b) **$\mathscr{R}$ is $\mathcal{F}_G$-d-closed.**

Clearly, here $G(\mathcal{F};\mathscr{R}) = \{(0,0) \cup S\}$. For each $(\xi,\eta) \in G(\mathcal{F};\mathscr{R})$ and $z \in \mathcal{F}\eta$, we see that $d(\eta,z) \leq d(\xi,\eta)$ and $(\eta,z) \in G(\mathcal{F};\mathscr{R})$. Hence $\mathscr{R}$ is $\mathcal{F}$-d-closed on $\mathcal{M}$. However, $\mathscr{R}$ is not $\mathcal{F}$-d-closed on $\mathcal{M}$ as for $(\xi,\eta) = (2,3/2) \in \mathscr{R}$, there exist $\mu = 2 \in \mathcal{F}\xi$ and $\nu = 2 \in \mathcal{F}\eta$ such that $d(\mu,\nu) \leq d(\xi,\eta)$ but $(\mu,\nu) \notin \mathscr{R}$.

(c) **The condition (c)-(ii) of Theorem 3.8 is satisfied.**

Since $\mathcal{M}(\mathcal{F};\mathscr{R}) = \{\frac{1}{5^n} : n \in \mathbb{N}\} \cup \{0\}$ then for $\{\xi_n\} = \{\frac{1}{5^n}\} \in \mathcal{M}(\mathcal{F};\mathscr{R})$, we have $\{\xi_n\} \to 0$ and each subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ is such that $(\xi_{n_k},0) \in \mathscr{R}$, that is $[\xi_{n_k},0] \in \mathscr{R}$, for each $n \in \mathbb{N}_0$.

(d) **$\mathcal{F}$ is a multivalued relation-theoretic graph contraction for $\alpha = 1/5$.**

Let $\eta,\xi \in \mathcal{M}(\mathcal{F};\mathscr{R}) = \left\{\frac{1}{5^n} : n \in \mathbb{N}\right\} \cup \{0\}$ with $(\eta,\xi) \in \mathscr{R}$, then

$$\begin{aligned} H(\mathcal{F}\eta,\mathcal{F}\xi) &= H\left(\{0,\frac{\eta}{5}\},\{0,\frac{\xi}{5}\}\right) \\ &= \max\left[\min\left\{\frac{\eta}{5},\frac{1}{5}|\eta-\xi|\right\},\min\left\{\frac{\xi}{5},\frac{1}{5}|\eta-\xi|\right\}\right] \\ &\leq \frac{1}{5}|\eta-\xi| = \frac{1}{5}d(\eta,\xi). \end{aligned}$$

Hence, all the hypothesis of Theorem 3.8 are verified and $\mathcal{F}$ has fixed points at $\xi = 0$ and $\xi = 2$.

We notice that $\mathcal{F}$ does not satisfy the contraction conditions (2.3), (3.2) and (3.7) on $\mathcal{M}$ as for $(\xi,\eta) = (1,2) \in \mathscr{R}$, $H(F0,F1) > d(0,1)$. Thus, the fixed point results contain in [7, 9, 13, 14] do not work in this example, however our Theorem 3.8 remains valid in this case.

5 Fixed point results for relation-theoretic graph contraction mappings

Now, we drive some fixed point results for relation-theoretic graph contraction mappings on relational metric space. In this sequel, we define some essential notations and definitions which are very useful to present our results.

Definition 5.1. Let $\mathscr{R}$ be a binary relation and f be a self mapping on $\mathcal{M}$. We denote by $G(f;\mathscr{R})$ the relational graph of f and defined as:

$$G(f;\mathscr{R}) = \{(\xi,f\xi) \in \mathscr{R} : \xi \in \mathcal{M}\}.$$

We say f has a closed relational graph or f is $\mathscr{R}$-closed if $G(f;\mathscr{R})$ is a closed subset of $\mathcal{M} \times \mathcal{M}$.

Definition 5.2. Let $(\mathcal{M},d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ be a mapping. By $\mathcal{M}(f;\mathscr{R})$, we denotes the set of all points $\xi \in \mathcal{M}$, for which $(\xi,f\xi) \in G(f;\mathscr{R})$, that is

$$\mathcal{M}(f;\mathscr{R}) = \{\xi \in \mathcal{M} : (\xi,f\xi) \in G(f;\mathscr{R})\}.$$

The Definition 5.2 is equivalent to the Definition 2.12 of Shukla and Rodríguez-López [13], which states that $\mathcal{M}(f;\mathscr{R})$ is the set of all points $\xi \in \mathcal{M}$ for which $(\xi,f\xi) \in \mathscr{R}$, that is

$$\mathcal{M}(f;\mathscr{R}) = \{\xi \in \mathcal{M} : (\xi,f\xi) \in \mathscr{R}\}.$$

Definition 5.3. Let $(\mathcal{M},d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ be a mapping. A binary relation $\mathscr{R}$ is called f_G-d-closed, if the following condition holds:

$$(\xi,\eta) \in G(f;\mathscr{R}), \qquad d(f\xi,f\eta) \leq d(\xi,\eta) \implies (f\xi,f\eta) \in G(f;\mathscr{R}).$$

Remark 5.4. We can see that f_G-d-closedness assumption is a weaker than f-d-closedness assumption and f-closedness. The following example, however, demonstrates that the conditions of f-d-closedness and f-closedness do not imply the condition of f_G-d-closedness.

Example 5.5. Let $\mathcal{M} = [0,1]$ and $d(\xi,\eta) = |\xi - \eta|$, for all $\xi,\eta \in \mathcal{M}$. Then, $(\mathcal{M}, d)$ forms a complete metric space. We define by $\mathscr{R} = \{(0,0),(1,0),(1,1),(1/2,1)\}$ a binary relation on $\mathcal{M}$ and a mapping $f : \mathcal{M} \to \mathcal{M}$ by

$$f(\xi) = \begin{cases} \xi/3, & \text{if } \xi \in [0,1/2], \\ 1, & \text{if } \xi \in (1/2,1]. \end{cases}$$

Then $G(f;\mathscr{R}) = \{(0,0),(1,1)\}$ and for each $(\xi,\eta) \in G(f;\mathscr{R})$, we have $d(f\xi, f\eta) = d(\xi,\eta)$ and $(f\xi, f\eta) \in G(f;\mathscr{R})$. Thus, the binary relation $\mathscr{R}$ is f_G-d-closed on $\mathcal{M}$. However, $\mathscr{R}$ is neither f-closed nor f-d-closed on $\mathcal{M}$, as $(1/2,1) \in \mathscr{R}$ and $d(f1/2, f1) < d(1/2,1)$ but $(f1/2, f1) = (1/6,1) \notin \mathscr{R}$.

Definition 5.6. Let $(\mathcal{M}, d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ be a mapping. We say the mapping f is a relation-theoretic graph contraction, if

$$d(f\xi, f\eta) \le \alpha d(\xi,\eta), \text{ for all } (\xi,\eta) \in G(f;\mathscr{R}) \text{ and some } \alpha \in [0,1). \tag{5.1}$$

We can consider a singlevalued mapping f as a particular case of a multivalued mapping $\mathcal{F}$ in sense that $\mathcal{F}\xi = \{f\xi\}$, for all $\xi \in \mathcal{M}$. Let us consider $\mathcal{F} = f$ (as a singlevalued mapping) in Theorem 3.7. Then, we find the following result for relation-theoretic graph contractions as a direct consequence of Theorem 3.7.

Theorem 5.7. Let $(\mathcal{M}, d)$ be a metric space, $\mathscr{R}$ be a binary relation on $\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ be a relation-theoretic graph contraction. Suppose $\mathscr{R}$ is f_G-d-closed and f has a nonempty closed relational graph under binary relation $\mathscr{R}$. Then, for each $\xi \in \mathcal{M}(f;\mathscr{R})$, there exists a trajectory $\{\xi_n\}$ of f, starting from $\xi_1 = \xi$, which converges to fixed point of f.

Now, we present a generalized version of relation theoretic-contraction principle due to Alam and Imdad [1].

Theorem 5.8. Let $(\mathcal{M}, d)$ be a metric space and $\mathscr{R}$ be a binary relation on $\mathcal{M}$. Assume that f be a self mapping on $\mathcal{M}$ and the following assertions are true:

(i) $\mathcal{M}(f;\mathscr{R}) \neq \emptyset$,

(ii) $\mathscr{R}$ is f_G-d-closed,

(iii) either f is continuous or $\mathscr{R}$ is d-self-closed,

(iv) f satisfies the following condition:

$$d(f\xi, f\eta) \le \alpha d(\xi,\eta), \quad \text{for all } (\xi,\eta) \in G(f;\mathscr{R}) \text{ and some } \alpha \in [0,1),$$

(v) $\mathcal{M}$ is $\mathscr{R}$-complete.

Then f has a fixed point.

Proof . Since a singlevalued mapping is particular case of multivalued mapping in the sense that $\mathcal{F}\xi = \{f\xi\}$, for all $\xi \in \mathcal{M}$. Then, condition (iv) implies that, f is a multivalued relation-theoretic graph contraction and condition (i) and (ii) implies the condition (a) and (b) of Theorem 3.8 respectively. If f is continuous then the graph of f is closed and d-self-closedness of $\mathscr{R}$ reduced to condition (ii) of (c) in Theorem 3.8. Thus, all the assertions of Theorem 3.8 are true and the conclusion follows from Theorem 3.8. □

6 Applications to generalized coupled fixed point problems

Let $(\mathcal{M}, d)$ and $(\mathcal{N}, \rho)$ be two metric spaces endowed with binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ respectively. Suppose $\mathcal{F}_1 : \mathcal{M} \times \mathcal{N} \to P(\mathcal{M})$, $\mathcal{F}_2 : \mathcal{M} \times \mathcal{N} \to P(\mathcal{N})$ be multivalued mappings. We denote the relational graph of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively by $G(\mathcal{F}_1;\mathscr{R}_1)$ and $G(\mathcal{F}_2;\mathscr{R}_2)$ and defined as:

$$G(\mathcal{F}_1;\mathscr{R}_1) = \{(\xi,\mu) \in \mathscr{R}_1 : \mu \in \mathcal{F}_1(\xi,\eta),\ \xi \in \mathcal{M},\ \eta \in \mathcal{N}\}$$

and

$$G(\mathcal{F}_2;\mathscr{R}_2) = \{(\eta,\nu) \in \mathscr{R}_2 : \nu \in \mathcal{F}_2(\xi,\eta),\ \xi \in \mathcal{M},\ \eta \in \mathcal{N}\}.$$

Then, the generalized coupled fixed point problem means to find a pair $(\xi^*,\eta^*)\in\mathcal{M}\times\mathcal{N}$ such that

$$\begin{aligned}&\xi^*\in\mathcal{F}_1(\xi^*,\eta^*),\\&\eta^*\in\mathcal{F}_2(\eta^*,\xi^*).\end{aligned}\tag{6.1}$$

We have the following general results:

Theorem 6.1. Let $(\mathcal{M},d)$ and $(\mathcal{N},\rho)$ be two metric spaces endowed with binary relations $\mathscr{R}_1$ and $\mathscr{R}_2$ respectively. Suppose $\mathcal{F}_1:\mathcal{M}\times\mathcal{N}\to P(\mathcal{M})$, $\mathcal{F}_2:\mathcal{M}\times\mathcal{N}\to P(\mathcal{N})$ be two multivalued mappings and the following conditions hold:

(a) there exist $k_1,k_2\in[0,1)$ such that

$$\begin{aligned}\delta_d\left(\mathcal{F}_1(\xi,\eta),\mathcal{F}_1(\mu,\nu)\right)\le k_1\left[d(\xi,\mu)+\rho(\eta,\nu)\right],\\ \delta_\rho\left(\mathcal{F}_2(\xi,\eta),\mathcal{F}_2(\mu,\nu)\right)\le k_2\left[d(\xi,\mu)+\rho(\eta,\nu)\right],\end{aligned}$$

for every $(\xi,\eta),(\mu,\nu)\in\mathcal{M}\times\mathcal{N}$ with $(\mu,\nu)\in\mathcal{F}_1(\xi,\eta)\times\mathcal{F}_2(\xi,\eta)$ and $((\xi,\mu),(\eta,\nu))\in\mathscr{R}_1\times\mathscr{R}_2$,

(b) $\mathscr{R}_1$ and $\mathscr{R}_2$ are ${\mathcal{F}_1}_G$-d-closed and ${\mathcal{F}_2}_G$-ρ-closed respectively.

(c) $\mathcal{F}_1$ and $\mathcal{F}_2$ have nonempty closed relational graph under binary relation $\mathscr{R}_1$ and $\mathscr{R}_2$ respectively.

Then, the generalized coupled fixed point theorem (6.1) has at least one solution.

Proof . Let $\Re:=\mathscr{R}_1\times\mathscr{R}_2,\ M:=\mathcal{M}\times\mathcal{N}$ and d^* be a metric on M defined as:

$$d^*\left((\xi,\eta),(\mu,\nu)\right):=d(\xi,\mu)+\rho(\eta,\nu).$$

Then (M,d) forms a complete metric space endowed with the binary relation $\Re$. Consider $F:M\to P(M)$ be a mapping, defined as

$$F\left(\xi,\eta\right):=\mathcal{F}_1(\xi,\eta)\times\mathcal{F}_2(\xi,\eta).$$

We denote the relational graph of F by $G(F;\Re)$ and

$$G(F;\Re):=G(\mathcal{F}_1;\mathscr{R}_1)\times G(\mathcal{F}_2;\mathscr{R}_2).$$

We suppose that the assumption (b) is true. Then

$$(\xi,\mu)\in G(\mathcal{F}_1;\mathscr{R}_1),\quad r\in\mathcal{F}_1(\mu,\ .),\quad d(\mu,r)\le d(\xi,\mu)\implies(\mu,r)\in G(\mathcal{F}_1;\mathscr{R}_1)$$

and

$$(\eta,\nu)\in G(\mathcal{F}_2;\mathscr{R}_2),\quad s\in\mathcal{F}_2(.,\ \nu),\quad \rho(\nu,s)\le\rho(\eta,\nu)\implies(\nu,s)\in G(\mathcal{F}_2;\mathscr{R}_2).$$

Now, we will show that $\Re$ is F-d^*-closed on M. For this, we assume that $((\xi,\mu),(\eta,\nu))\in G(F;\Re)$ then for any $(r,s)\in F(\mu,\nu)$ such that

$$\begin{aligned}d^*\left((\mu,\nu),(r,s)\right)\le d^*\left((\xi,\eta),(\mu,\nu)\right),\\ d(\mu,r)+\rho(\nu,s)\le d(\xi,\mu)+\rho(\eta,\nu),\end{aligned}$$

we have

$$d(\mu,r)\le d(\xi,\mu)\text{ and }\rho(\nu,s)\le\rho(\eta,\nu).$$

By ${\mathcal{F}_1}_G$-d-closedness of $\mathscr{R}_1$ and ${\mathcal{F}_2}_G$-ρ-closedness of $\mathscr{R}_2$, we have $(\mu,r)\in G(\mathcal{F}_1;\mathscr{R}_1)$ and $(\nu,s)\in G(\mathcal{F}_2;\mathscr{R}_2)$ therefore $((\mu,r),(\nu,s))\in\Re$. This proves that, the relation $\Re$ is F_G-d^*-closed on M. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ have nonempty closed relational graph under the binary relation $\mathscr{R}_1$ and $\mathscr{R}_2$ respectively. Therefore, F also have a nonempty closed relational graph under the binary relation $\Re$.

Let $(\xi,\eta),(\mu,\nu)\in M$ with $((\xi,\eta),(\mu,\nu))\in\Re$ and $(\mu,\nu)\in F(\xi,\eta)$. Take any $z:=(r,s)\in F(\xi,\eta)$, then

$$\begin{aligned}D_{d^*}(z,F(\mu,\nu))&=\inf\{d^*(z,t):t=(p,q)\in F(\mu,\nu)\}\\&=\inf\{d(r,p)+\rho(s,q):p\in\mathcal{F}_1(\mu,\nu)\text{ and }q\in\mathcal{F}_2(\mu,\nu)\}\\&=\inf\{d(r,p):p\in\mathcal{F}_1(\mu,\nu)\}+\inf\{\rho(s,q):q\in\mathcal{F}_2(\mu,\nu)\}\\&=D_d\left(r,\mathcal{F}_1(\mu,\nu)\right)+D_\rho\left(s,\mathcal{F}_2(\mu,\nu)\right).\end{aligned}\tag{6.2}$$

On the other hand,

$$D_d(r, \mathcal{F}_1(\mu,\nu)) \leq \delta_d(\mathcal{F}_1(\xi,\eta), \mathcal{F}_1(\mu,\nu)) \leq k_1[d(\xi,\mu) + \rho(\eta,\nu)] \tag{6.3}$$

and

$$D_\rho(s, \mathcal{F}_2(\mu,\nu)) \leq \delta_\rho(\mathcal{F}_2(\xi,\eta), \mathcal{F}_2(\mu,\nu)) \leq k_2[d(\xi,\mu) + \rho(\eta,\nu)]. \tag{6.4}$$

Hence from (6.2), (6.3) and (6.4), we have

$$D_{d^*}(z, F(\mu,\nu)) \leq (k_1 + k_2)[d(\xi,\mu) + \rho(\eta,\nu)].$$

Thus

$$\delta_{d^*}\left(F(\xi,\eta), F(\mu,\nu)\right) \leq kd^*\left((\xi,\eta),(\mu,\nu)\right),$$

for every $(\xi,\eta),(\mu,\nu) \in M$ with $(\mu,\nu) \in F(\xi,\eta)$ and $((\xi,\eta),(\mu,\nu)) \in \Re$, where $k = k_1 + k_2$. Hence, F is a multivalued relation graph contraction with respect to δ_{d^*}. The conclusion follows by Theorem 3.9. □

If we consider $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to P(\mathcal{M})$ be a multivalued mapping then pair (ξ,η) is called a solution of the classical coupled fixed point problem if

$$\begin{aligned} &\xi \in \mathcal{F}(\xi,\eta), \\ &\eta \in \mathcal{F}(\eta,\xi). \end{aligned} \tag{6.5}$$

We have the following existence theorem.

Theorem 6.2. Let $(\mathcal{M}, d)$ be a metric spaces endowed with binary relations $\mathscr{R}$. Suppose $\mathcal{F} : \mathcal{M} \times \mathcal{M} \to P(\mathcal{M})$ be a multivalued mappings and the following conditions are hold:

(a) there exists $k \in [0,1)$ such that

$$\delta_d\left(\mathcal{F}(\xi,\eta), \mathcal{F}(\mu,\nu)\right) \leq k\left[d(\xi,\mu) + \rho(\eta,\nu)\right],$$

for every $(\xi,\eta),(\mu,\nu) \in \mathcal{M} \times \mathcal{M}$ with $(\mu,\nu) \in \mathcal{F}(\xi,\eta) \times \mathcal{F}(\xi,\eta)$ and $((\xi,\mu),(\eta,\nu)) \in \mathscr{R} \times \mathscr{R}$,

(b) $\mathscr{R}$ is $\mathcal{F}_G$-d-closed,

(c) $\mathcal{F}$ has nonempty closed relational graph under the binary relation $\mathscr{R}$.

Then, the coupled fixed point theorem (6.5) has at least one solution.

Proof . The proof is obtained by taking $\mathcal{M} = \mathcal{N}$, $d = \rho$, $k = k_1 = k_2$, $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2$ and $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$ in the proof of Theorem 6.1 □

7 Conclusion

In this paper, we introduced a multivalued relation-theoretic graph contraction mappings and generalized the Nadler fixed point theorem in the setting of relational metric spaces. We also introduced notions of $\mathcal{F}_G$-d-closedness and f_G-d-closedness, which is weaker than the notion of $\mathcal{F}$-d-closedness and f-closedness respectively. Using these notions, we extended and generalized the results of Ran and Reurings [11], Shukla and Rodríguez-López [13], Alam and Imdad [1] and many others in the existing literature. We provided a comprehensive proof of the relation-theoretic version of Nadler's fixed point theorem and gave some illustrative examples to demonstrate the significance of our main results over all the corresponding theorems presented in [7, 9, 13, 14].

References

[1] A. Alam and M. Imdad , *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl. **17** (2015), 693–702.

[2] A. Alam and M. Imdad, *Nonlinear contractions in metric spaces under locally T-transitive binary relations*, Fixed Point Theory **19** (2018), 13–23.

[3] H. Baghani and M. Ramezani, *A fixed point theorem for a new class of set-valued mappings in R-complete (not necessarily complete) metric spaces*, Filomat **31** (2017), 3875–3884.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[5] H. Ben-El-Mechaiekh , *The Ran-Reurings fixed point theorem without partial order: a simple proof*, J. Fixed Point Theory Appl. **16** (2014), 373–383.

[6] D. Khantwal, S. Aneja, G. Prasad and U.C. Gairola , *A generalization of relation-theoretic contraction principle*, TWMS J. App. Eng. Math. (Accepted).

[7] S.B. Nadler Jr, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[8] J.J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.

[9] A. Petruşel, G. Petruşel and J. Yao, *Multi-valued graph contraction principle with applications*, Optim. **69** (2020), 1541–1556.

[10] G. Prasad and R. C. Dimri, *Fixed point theorems for weakly contractive mappings in relational metric spaces with an application*, J. Anal. **26** (2018), 151–162.

[11] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.

[12] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. **13** (2012), 82–97.

[13] S. Shukla and R. Rodríguez-López, *Fixed points of multi-valued relation-theoretic contractions in metric spaces and application*, Q. Math. **43** (2020), 409–424.

[14] A. Tomar, M. Joshi, S.K. Padaliya, B. Joshi and A. Diwedi, *Fixed point under set-valued relation-theoretic nonlinear contractions and application*, Filomat **33** (2019), 4655–4664.

[15] M. Turinici, *Nieto-Lopez theorems in ordered metric spaces*, Math. Stud. **81** (2012), 219–229.