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Unified duality for mathematical programming problems with vanishing constraints

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Abstract

In this article, we formulate a new mixed-type dual problem for a mathematical program with vanishing constraints. The presented dual problem does not involve the index set, however, the dual models contain the calculations of index sets, which makes it difficult to solve these models from an algorithm point of view. The weak, strong and strict converse duality theorems are discussed in order to establish the relationships between the mathematical program with vanishing constraints and its mixed type dual under generalized convexity. To validate the results, a non-trivial example is discussed. Our dual model unifies the dual models discussed in [Q. Hu, J. Wang, Y. Chen, New dualities for mathematical programs with vanishing constraints, Annals of Operations Research, 287 (2020) 233-255].

Keywords: Mathematical programming problem, vanishing constraints, constraint qualifications, duality
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1 Introduction

Mathematical programming problems with vanishing constraints is indeed an interesting topic due to their applications in several area of modern research such as in topology design problems [6], economic dispatch problems [11], robot motion planning problems [15], optimal control and structural optimization [16]. Many excellent articles have recently attracted people's curiosity in this challenging class of optimization problems. In particular, Achtziger and Kanzow [1] proposed a suitable version of the Abadie constraint qualification, as well as a corresponding optimality condition, and demonstrated that this revised constraint qualification holds under reasonable assumptions. In [13], Kazemi and Kanzi extended Achtziger and Kanzow's [1] work by introducing some constraint qualifications for a system with non-smooth vanishing constraints. They then discussed the application of these constraint qualifications to various types of stationary conditions for mathematical programming problems with vanishing constraints.

Khare and Nath [14] gave the Fritz-John type stationary conditions for the existence of an optimal solution for the single objective mathematical programming problem with vanishing constraints. Further, Dussault et al. in [8]

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proposed a new method and demonstrated that it achieves S-stationary points for a class of problems with vanishing constraints under linear independence constraint qualifications. They furthermore defined new weak constraint qualifications for the class of problems with vanishing constraints, all of which have direct algorithmic applications.

In recent years, there has been a lot of focus on developing new approaches for determining the solvability of the original multiobjective programming problem using some associated vector optimization problem/modified objective function method. Jayswal and Singh [12] concentrated on vanishing constraint multiobjective optimization problems with invex functions using the modified objective function approach. Also, they discussed saddle point criteria for the modified objective function problem. Under generalised convexity assumptions, Ahmad et al. [2] derived sufficient optimality conditions for interval-valued optimization problems with vanishing constraints. They also established duality results for a Mond-Weir type dual model. Later on, in [3], Antczak studied differentiable semi-infinite multiobjective programming problems with vanishing constraints. Optimal conditions were determined under the appropriate invexity hypotheses for these nonconvex smooth vector optimization problems. As well, vector duals in the sense of Mond–Weir are defined for the considered primal problem, and duality results are established under invexity hypotheses. Tung [18] investigated optimality conditions for Pareto efficient and weakly efficient solutions, as well as duality theorems of the Wolfe and Mond-Weir types for nonsmooth multiobjective semi-infinite programming.

Hoheisel and Kanzow [9] investigated the Abadie and Guignard constraint qualifications and demonstrated that the Abadie constraint qualifications are too strong for mathematical programming problem with vanishing constraints, whereas the Guignard constraint qualifications are held in many situations. Ardali et al. [4] recently been extended the Guignard constraint qualification to nonsmooth multiobjective optimization problems containing equilibrium constraints. Motivated by various concepts of generalize convexity, Dubey and Mishra [7] investigated a mixed type second-order fractional dual model and demonstrated duality theorems under generalized $(V,\ \rho,\ \theta)$-bonvex type-I assumptions.

In [17], Mishra et al. developed and explored Wolfe and Mond-Weir type dual models for the mathematical programming problem with vanishing constraints. These models are not suitable for numerical solutions to dual problems since they need to calculate index sets. As a result, Hu et al. [10] recently proposed new Wolfe and Mond-Weir type dual models for a mathematical programming problem with vanishing constraints and established duality results under generalized convexity assumptions that do not require index set calculations.

The motivation for writing the dual precisely in the more general dual constitution model can be challenging to understand at times, but this is only for mathematical analysis. As a result of these numerous type dualities, the question of whether we can develop a mixed type dual to incorporate these dualities arises. As a direct result of the work done by Hu et al. [10], we apply the technique of incomplete Lagrangian multipliers (that is, we use part of the primal problem's constraints in the objective of the dual problem and leave the rest of the primal problem's constraints in the dual problem's constraints) in order to construct a mixed dual model for vanishing constraints. This model unifies the Wolfe and Mond-Weir type dual models, which is something that has never been accomplished in the scientific literature before. The proposed mixed dual problem does not involve an index set; however, the dual models contain index set calculations, making it difficult to solve these models from an algorithmic standpoint (see, Remark 3.2 and Remark 3.4 of [10]). Further, in order to relate the primal and the dual problems, several duality theorems are established.

The structure of this document is as follows: The second section covers some fundamental principles and preliminary information. The third section discusses the duality results for a novel mixed type dual problem for mathematical programming with vanishing conditions. Section 4 discusses the exceptional cases of the proposed mixed type dual problem. Finally, Section 5 contains the paper's conclusion.

2 Preliminaries

In the present analysis we consider the following mathematical programming problem with vanishing constraints, which was introduced by Achtziger and Kanzow [1]:

$$\text{(MPVC)} \qquad \min_{x\in\mathbb{F}} \quad \Psi(x)$$

subject to

$$\varphi_i(x) \leq 0,\ \forall i \in \mathbb{I} = \{1, 2, ..., p\},$$

$$\zeta_i(x) = 0,\ \forall i \in \mathbb{J} = \{1, 2, ..., q\},$$

$$\ell_i(x) \geq 0, \ \forall i \in \mathbb{K} = \{1, 2, ..., r\},$$
$$\Phi_i(x)\ell_i(x) \leq 0, \ \forall i \in \mathbb{K} = \{1, 2, ..., r\},$$

where $\Psi : R^n \to R$ is Lipschitz continuous function and $\varphi_i, \zeta_i, \ell_i, \Phi_i : R^n \to R$ are continuously differentiable functions. The feasible region is given by

$$\begin{aligned}\mathbb{F} = \{x \in R^n | \varphi_i(x) \leq 0, \ \forall i \in \mathbb{I},\\ \zeta_i(x) = 0, \ \forall i \in \mathbb{J},\\ \ell_i(x) \geq 0, \ \forall i \in \mathbb{K},\\ \Phi_i(x)\ell_i(x) \leq 0, \ \forall i \in \mathbb{K}\}.\end{aligned}$$

Let $x^* \in \mathbb{F}$ be any feasible solution of the (MPVC). The following index sets will be used in the sequel.

$$\Lambda_\varphi = \{i \in \mathbb{I} | \varphi_i(x^*) = 0\},$$
$$\Lambda_+ = \{i \in \mathbb{K} | \ell_i(x^*) > 0\},$$
$$\Lambda_0 = \{i \in \mathbb{K} | \ell_i(x^*) = 0\},$$
$$\Lambda_{+0} = \{i \in \mathbb{K} | \ell_i(x^*) > 0, \Phi_i(x^*) = 0\},$$
$$\Lambda_{+-} = \{i \in \mathbb{K} | \ell_i(x^*) > 0, \Phi_i(x^*) < 0\},$$
$$\Lambda_{0+} = \{i \in \mathbb{K} | \ell_i(x^*) = 0, \Phi_i(x^*) > 0\},$$
$$\Lambda_{00} = \{i \in \mathbb{K} | \ell_i(x^*) = 0, \Phi_i(x^*) = 0\},$$
$$\Lambda_{0-} = \{i \in \mathbb{K} | \ell_i(x^*) = 0, \Phi_i(x^*) < 0\}.$$

The following definitions are taken from Achtziger and Kanzow [1].

Definition 2.1. The linearizing cone of the (MPVC) at $x^* \in \mathbb{F}$ is given by

$$\begin{aligned}\mathbb{L}(x^*) = \Big\{\delta \in R^n | \nabla\varphi_i(x^*)^T\delta \leq 0, \ \forall i \in \Lambda_\varphi,\\ \nabla\zeta_i(x^*)^T\delta = 0, \ \forall i \in \mathbb{J},\\ \nabla\ell_i(x^*)^T\delta = 0, \ \forall i \in \Lambda_{0+},\\ \nabla\ell_i(x^*)^T\delta \geq 0, \ \forall i \in \Lambda_{00} \cup \Lambda_{0-},\\ \nabla\Phi_i(x^*)^T\delta \leq 0, \ \forall i \in \Lambda_{+0}\Big\}.\end{aligned}$$

and the symbol T denotes the transpose of a matrix.

Definition 2.2. The Abadie constraint qualification (ACQ) is said to hold at $x^* \in \mathbb{F}$ if $\mathbb{T}(x^*) = \mathbb{L}(x^*)$, where

$$\mathbb{T}(x^*) = \Big\{\delta \in R^n | \exists \{x^n\} \subseteq \mathbb{F}, \{t_n\} \downarrow 0 : x^n \to x^* \text{ and } \frac{x^n - x^*}{t_n} \to \delta\Big\}$$

is the tangent cone of (MPVC) at $x^* \in \mathbb{F}$.

Definition 2.3. The modified Abadie constraint qualification (VC-ACQ) is said to hold at $x^* \in \mathbb{F}$ if $\mathbb{L}^{VC}(x^*) \subseteq \mathbb{T}(x^*)$, where

$$\begin{aligned}\mathbb{L}^{VC}(x^*) = \Big\{\delta \in R^n | \nabla\varphi_i(x^*)^T\delta \leq 0, \ \forall i \in \Lambda_\varphi,\\ \nabla\zeta_i(x^*)^T\delta = 0, \ \forall i \in \mathbb{J},\end{aligned}$$

$$\left.\begin{array}{l}\nabla \ell_i(x^*)^T\delta = 0,\ \forall i \in \Lambda_{0+},\\ \nabla \ell_i(x^*)^T\delta \geq 0,\ \forall i \in \Lambda_{00}\cup\Lambda_{0-},\\ \nabla \Phi_i(x^*)^T\delta \leq 0,\ \forall i \in \Lambda_{+0}\cup\Lambda_{00}\end{array}\right\}.$$

is the modified linearized cone of (MPVC) at $x^* \in \mathbb{F}$.

The following Karush-Kuhn-Tucker type necessary optimality conditions for the considered mathematical programming problem with vanishing constraints were derived by Achtziger and Kanzow [1] under modified Abadie constraint qualification (VC-ACQ).

Theorem 2.4. *Let $x^* \in \mathbb{F}$ be a local minimum of (MPVC) such that (VC-ACQ) holds at x^*. Then there exist $\mu_i \in R_+, i \in \mathbb{I}$, $\gamma_i \in R, i \in \mathbb{J}$ and $\eta_i^\ell, \eta_i^\Phi \in R, i \in \mathbb{K}$ such that*

$$\nabla\Psi(x^*) + \sum_{i\in\mathbb{I}}\mu_i\nabla\varphi_i(x^*) + \sum_{i\in\mathbb{J}}\gamma_i\nabla\zeta_i(x^*) - \sum_{i\in\mathbb{K}}\eta_i^\ell\nabla\ell_i(x^*) + \sum_{i\in\mathbb{K}}\eta_i^\Phi\nabla\Phi_i(x^*) = 0, \tag{2.1}$$

$$\varphi_i(x^*) \leq 0,\ \mu_i\varphi_i(x^*) = 0,\ \forall i \in \mathbb{I}, \tag{2.2}$$

$$\zeta_i(x^*) = 0,\ \forall i \in \mathbb{J}, \tag{2.3}$$

$$\eta_i^\ell = 0,\ i \in \Lambda_+,\ \eta_i^\ell \geq 0,\ i \in \Lambda_{00}\cup\Lambda_{0-},\ \eta_i^\ell \textit{ free},\ \ i \in \Lambda_{0+}, \tag{2.4}$$

$$\eta_i^\Phi = 0,\ i \in \Lambda_{+-}\cup\Lambda_{0-}\cup\Lambda_{0+},\ \eta_i^\Phi \geq 0,\ i \in \Lambda_{+0}\cup\Lambda_{00}. \tag{2.5}$$

We now turn our attention to use some well-known concepts of generalized convexity for a real valued differentiable function (see, for example, [5]).

Definition 2.5. Let $\Omega : X \subseteq R^n \to R$ be a differentiable function. Then, Ω is said to be a quasiconvex at $x^* \in X$ if for any $x \in X$, we have

$$\Omega(x) \leq \Omega(x^*) \Rightarrow (x - x^*)^T\nabla\Omega(x^*) \leq 0,$$

equivalently

$$(x - x^*)^T\nabla\Omega(x^*) > 0 \Rightarrow \Omega(x) > \Omega(x^*).$$

Definition 2.6. Let $\Omega : X \subseteq R^n \to R$ be a differentiable function. Then, Ω is said to be a (strictly) pseudoconvex at $x^* \in X$ if for any $x \in X$, we have

$$(x - x^*)^T\nabla\Omega(x^*) \geq 0 \Rightarrow \Omega(x)(>) \geq \Omega(x^*),$$

equivalently

$$\Omega(x)(\leq) < \Omega(x^*) \Rightarrow (x - x^*)^T\nabla\Omega(x^*) < 0.$$

3 Unified duality

Let $\mathbb{I}_{\hat{s}} \subset \mathbb{I}, \hat{s} = 0, 1, ..., k$ with $\bigcup_{\hat{s}=0}^{k}\mathbb{I}_{\hat{s}} = \mathbb{I}$ and $\mathbb{I}_{\hat{s}} \cap \mathbb{I}_{\bar{s}} = \emptyset$ if $\hat{s} \neq \bar{s}$ and let $\mathbb{J}_{\hat{t}} \subset \mathbb{J}, \hat{t} = 0, 1, ..., l$ with $\bigcup_{\hat{t}=0}^{l}\mathbb{J}_{\hat{t}} = \mathbb{J}$ and $\mathbb{J}_{\hat{t}} \cap \mathbb{J}_{\bar{t}} = \emptyset$ if $\hat{t} \neq \bar{t}$. We assume further that $\mathbb{K}_{\hat{\alpha}} \subset \mathbb{K}, \hat{\alpha} = 0, 1, ..., m$ with $\bigcup_{\hat{\alpha}=0}^{m}\mathbb{J}_{\hat{\alpha}} = \mathbb{K}$ and $\mathbb{K}_{\hat{\alpha}} \cap \mathbb{K}_{\bar{\alpha}} = \emptyset$ if $\hat{\alpha} \neq \bar{\alpha}$. Now, for $x \in \mathbb{F}$, we propose the following new unified dual MDVC(x) for (MPVC):

$$\max\left[\Psi(y) + \sum_{i\in\mathbb{I}_0}\mu_i\varphi_i(y) + \sum_{i\in\mathbb{J}_0}\gamma_i\zeta_i(y) - \sum_{i\in\mathbb{K}_0}\eta_i^\ell\ell_i(y) + \sum_{i\in\mathbb{K}_0}\eta_i^\Phi\Phi_i(y)\right]$$

subject to

$$\nabla\Psi(y) + \sum_{i\in\mathbb{I}} \mu_i \nabla\varphi_i(y) + \sum_{i\in\mathbb{J}} \gamma_i \nabla\zeta_i(y) - \sum_{i\in\mathbb{K}} \eta_i^{\ell} \nabla\ell_i(y) + \sum_{i\in\mathbb{K}} \eta_i^{\Phi} \nabla\Phi_i(y) = 0, \tag{3.1}$$

$$\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i \varphi_i(y) \geq 0, \;\; \hat{s} = 1, 2, ..., k, \tag{3.2}$$

$$\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i \zeta_i(y) = 0, \hat{t} = 1, 2, ..., l, \tag{3.3}$$

$$\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi} \Phi_i(y) \geq 0, \hat{\alpha} = 1, 2, ..., m, \tag{3.4}$$

$$-\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell} \ell_i(y) \geq 0, \hat{\alpha} = 1, 2, ..., m, \tag{3.5}$$

$$y \in R^n, \quad \mu \in R^p_+, \quad \gamma \in R^q, \quad \eta^{\ell} \in R^r, \quad \eta^{\Phi} \in R^r, \tag{3.6}$$

where $\eta_i^{\Phi} = v_i \ell_i(x), v_i \in R_+, i \in \mathbb{K}$ and $\eta_i^{\ell} = \rho_i - v_i \Phi_i(x), \rho_i \in R_+, i \in \mathbb{K}$. We denote by $\mathbb{W}_1(x)$ the set of all feasible solutions of the problem MDVC(x) and let $pr\mathbb{W}_1(x) = \{y \in R^n | (y, \mu, \gamma, \eta^{\ell}, \eta^{\Phi}, v, \rho) \in \mathbb{W}_1(x)\}$ be the projection of the set $\mathbb{W}_1(x)$ on R^n.

To be independent of the (MPVC), we consider another dual problem which is denoted by (MDVC) as follows:

$$\text{(MDVC)} \qquad \max \left[\Psi(y) + \sum_{i\in\mathbb{I}_0} \mu_i \varphi_i(y) + \sum_{i\in\mathbb{J}_0} \gamma_i \zeta_i(y) - \sum_{i\in\mathbb{K}_0} \eta_i^{\ell} \ell_i(y) + \sum_{i\in\mathbb{K}_0} \eta_i^{\Phi} \Phi_i(y) \right]$$

subject to

$$(y, \mu, \gamma, \eta^{\ell}, \eta^{\Phi}, v, \rho) \in \bigcap_{x\in\mathbb{F}} \mathbb{W}_1(x).$$

The set of all feasible points of the (MDVC) is denoted by $\mathbb{W}_1 = \bigcap_{x\in\mathbb{F}} \mathbb{W}_1(x)$ and the projection of the set $\mathbb{W}_1$ on R^n is denoted by $pr\mathbb{W}_1$.

For $x^* \in \mathbb{F}$, we define the following index sets:

$$\Lambda_+^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) > 0\}, \qquad \Lambda_+^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) > 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_0^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) = 0\}, \qquad \Lambda_0^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) = 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_{+0}^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) > 0, \Phi_i(x^*) = 0\},$$

$$\Lambda_{+0}^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) > 0, \Phi_i(x^*) = 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_{+-}^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) > 0, \Phi_i(x^*) < 0\},$$

$$\Lambda_{+-}^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) > 0, \Phi_i(x^*) < 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_{0+}^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) = 0, \Phi_i(x^*) > 0\},$$

$$\Lambda_{0+}^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) = 0, \Phi_i(x^*) > 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_{00}^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) = 0, \Phi_i(x^*) = 0\},$$

$$\Lambda_{00}^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) = 0, \Phi_i(x^*) = 0\}, \hat{\alpha} = 1, 2, ..., m,$$

$$\Lambda_{0-}^{\mathbb{K}_0} = \{i \in \mathbb{K}_0 | \ell_i(x^*) = 0, \Phi_i(x^*) < 0\},$$

$$\Lambda_{0-}^{\mathbb{K}_{\hat{\alpha}}} = \{i \in \mathbb{K}_{\hat{\alpha}} | \ell_i(x^*) = 0, \Phi_i(x^*) < 0\}, \hat{\alpha} = 1, 2, ..., m.$$

Now, we prove duality results between problems (MPVC) and (MDVC) under certain generalized convexity assumptions imposed on the involved functions.

Theorem 3.1.(Weak Duality). *Let* $x \in \mathbb{F}$ *and* $(y, \mu, \gamma, \eta^\ell, \eta^\Phi, v, \rho) \in \mathbb{W}_1$. *Further, assume that* $\Psi(.) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(.) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(.) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(.) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(.)$ *is pseudoconvex at* y *on* $\mathbb{F} \cup pr\mathbb{W}_1$ *and that* $\sum_{\hat{s}=1}^{k} \sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\varphi_i(.) + \sum_{\hat{t}=1}^{l} \sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\zeta_i(.) - \sum_{\hat{\alpha}=1}^{m} \sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^\ell \ell_i(.) + \sum_{\hat{\alpha}=1}^{m} \sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^\Phi \Phi_i(.)$ *is quasiconvex at* y *on* $\mathbb{F} \cup pr\mathbb{W}_1$, *then*

$$\Psi(x) \geq \Psi(y) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(y) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(y) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(y) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(y)$$

.

Proof. Suppose to the contrary that

$$\Psi(x) < \Psi(y) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(y) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(y) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(y) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(y). \tag{3.7}$$

Since $x \in \mathbb{F}$ and $(y, \mu, \gamma, \eta^\ell, \eta^\Phi, v, \rho) \in \mathbb{W}_1$, it follows that

$$\varphi_i(x) \leq 0,\ \mu_i \geq 0,\ \forall i \in \mathbb{I}_0,$$
$$\zeta_i(x) = 0,\ \gamma_i \in R,\ \forall i \in \mathbb{J}_0,$$
$$-\ell_i(x) < 0,\ \eta_i^\ell \geq 0,\ \forall i \in \Lambda_+^{\mathbb{K}_0},$$
$$-\ell_i(x) = 0,\ \eta_i^\ell \in R,\ \forall i \in \Lambda_0^{\mathbb{K}_0},$$
$$\Phi_i(x) > 0,\ \eta_i^\Phi = 0,\ \forall i \in \Lambda_{0+}^{\mathbb{K}_0},$$
$$\Phi_i(x) = 0,\ \eta_i^\Phi \geq 0,\ \forall i \in \Lambda_{00}^{\mathbb{K}_0} \cup \Lambda_{+0}^{\mathbb{K}_0},$$
$$\Phi_i(x) < 0,\ \eta_i^\Phi \geq 0\ , \forall i \in \Lambda_{0-}^{\mathbb{K}_0} \cup \Lambda_{+-}^{\mathbb{K}_0},$$

that is

$$\sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(x) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(x) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(x) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(x) \leq 0. \tag{3.8}$$

On adding (3.7) and (3.8), we get

$$\Psi(x) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(x) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(x) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(x) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(x)$$
$$< \Psi(y) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(y) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(y) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(y) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(y),$$

which by pseudoconvexity of $\Psi(.) + \sum_{i\in\mathbb{I}_0} \mu_i\varphi_i(.) + \sum_{i\in\mathbb{J}_0} \gamma_i\zeta_i(.) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \ell_i(.) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \Phi_i(.)$ at y on $\mathbb{F} \cup pr\mathbb{W}_1$, one has

$$(x-y)^T \left[\nabla\Psi(y) + \sum_{i\in\mathbb{I}_0} \mu_i\nabla\varphi_i(y) + \sum_{i\in\mathbb{J}_0} \gamma_i\nabla\zeta_i(y) - \sum_{i\in\mathbb{K}_0} \eta_i^\ell \nabla\ell_i(y) + \sum_{i\in\mathbb{K}_0} \eta_i^\Phi \nabla\Phi_i(y)\right] < 0. \tag{3.9}$$

On the other hand, from $x \in \mathbb{F}$ and $(y, \mu, \gamma, \eta^\ell, \eta^\Phi, v, \rho) \in \mathbb{W}_1$, it follows that

$$\varphi_i(x) \leq 0,\ \mu_i \geq 0,\ i \in \mathbb{I}_{\hat{s}}, \hat{s} = 1, 2, ..., k,$$
$$\zeta_i(x) = 0,\ \gamma_i \in R,\ i \in \mathbb{J}_{\hat{t}},\ \hat{t} = 1, 2, ..., l,$$
$$-\ell_i(x) < 0,\ \eta_i^\ell \geq 0,\ i \in \Lambda_+^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$
$$-\ell_i(x) = 0,\ \eta_i^\ell \in R,\ i \in \Lambda_0^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$
$$\Phi_i(x) > 0,\ \eta_i^\Phi = 0,\ i \in \Lambda_{0+}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$\Phi_i(x) = 0,\ \eta_i^{\Phi} \geq 0,\ i \in \Lambda_{00}^{\mathbb{K}_{\hat{\alpha}}} \cup \Lambda_{+0}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$\Phi_i(x) < 0,\ \eta_i^{\Phi} \geq 0,\ i \in \Lambda_{0-}^{\mathbb{K}_{\hat{\alpha}}} \cup \Lambda_{+-}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

that is,

$$\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\varphi_i(x) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\zeta_i(x) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell}\ell_i(x) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi}\Phi_i(x) \leq 0.$$

Now, by (3.2)-(3.5), above inequality implies that

$$\left\{\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\varphi_i(x) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\zeta_i(x) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell}\ell_i(x) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi}\Phi_i(x)\right\}$$

$$-\left\{\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\varphi_i(y) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\zeta_i(y) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell}\ell_i(y) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi}\Phi_i(y)\right\} \leq 0,$$

which by quasiconvexity of $\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\varphi_i(.) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\zeta_i(.) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell}\ell_i(.) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi}\ \Phi_i(.)$ at y on $\mathbb{F}\cup pr\mathbb{W}_1$, one has

$$(x-y)^T\left[\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}} \mu_i\nabla\varphi_i(y) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}} \gamma_i\nabla\zeta_i(y) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\ell}\nabla\ell_i(y) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}} \eta_i^{\Phi}\nabla\Phi_i(y)\right] \leq 0. \tag{3.10}$$

On combining (3.9) and (3.10), we get

$$(x-y)^T\Bigg[\nabla\Psi(y) + \sum_{i\in\mathbb{I}_0}\mu_i\nabla\varphi_i(y) + \sum_{i\in\mathbb{J}_0}\gamma_i\nabla\zeta_i(y) - \sum_{i\in\mathbb{K}_0}\eta_i^{\ell}\nabla\ell_i(y) + \sum_{i\in\mathbb{K}_0}\eta_i^{\Phi}\nabla\Phi_i(y) + \sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}}\mu_i\nabla\varphi_i(y) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}}\gamma_i\nabla\zeta_i(y) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}}\eta_i^{\ell}\nabla\ell_i(y) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}}\eta_i^{\Phi}\nabla\Phi_i(y)\Bigg] < 0,$$

which means that

$$(x-y)^T\left[\nabla\Psi(y) + \sum_{i\in\mathbb{I}}\mu_i\nabla\varphi_i(y) + \sum_{i\in\mathbb{J}}\gamma_i\nabla\zeta_i(y) - \sum_{i\in\mathbb{K}}\eta_i^{\ell}\nabla\ell_i(y) + \sum_{i\in\mathbb{K}}\eta_i^{\Phi}\nabla\Phi_i(y)\right] < 0.$$

This is a contradiction to (3.1). Hence the result. □

Theorem 3.2.(Strong Duality). *Let $\tilde{x}$ be a local minimum of (MPVC) such that (VC-ACQ) holds at $\tilde{x}$. Then there exist $(\tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho}) \in R_+^p \times R^q \times R^r \times R^r \times R^r \times R^r$ such that $(\tilde{x}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ is a feasible solution of (MDVC) and the two objective values are same. Furthermore, if the assumptions of Theorem 3.1 hold for all feasible solutions of (MDVC), then $(\tilde{x}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ is a global maximum of (MDVC).*

Proof. By assumption, $\tilde{x}$ is a local minimum of (MPVC) and (VC-ACQ) is satisfied at $\tilde{x}$. Then, there exist $\tilde{\mu} \in R_+^p, \tilde{\gamma} \in R^q, \tilde{\eta}^{\ell} \in R^r, \tilde{\eta}^{\Phi} \in R^r, \tilde{v} \in R^r, \tilde{\rho} \in R^r$ such that the the conditions (2.1)-(2.5) hold. Therefore, $(\tilde{x}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ is a feasible solution of (MDVC), moreover, the corresponding objective values of (MPVC) and (MDVC) are equal. The global maximum of $(\tilde{x}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ for (MDVC) follows from weak duality (Theorem 3.1). □

Theorem 3.3. (Strict Converse Duality). *Let $\tilde{x}$ be a local minimum of (MPVC) and $(\tilde{y}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ be a global maximum of (MDVC). Assume that the assumptions of Theorem 3.2 are fulfilled. Further, assume that $\Psi(.) + \sum_{i\in\mathbb{I}_0}\tilde{\mu}_i\varphi_i(.) + \sum_{i\in\mathbb{J}_0}\tilde{\gamma}_i\zeta_i(.) - \sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\ell}\ell_i(.) + \sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\Phi}\Phi_i(.)$ is strictly pseudoconvex at $\tilde{y}$ on $\mathbb{F}\cup pr\mathbb{W}_1$ and that $\sum_{\hat{s}=1}^{k}\sum_{i\in\mathbb{I}_{\hat{s}}}\tilde{\mu}_i\ \varphi_i(.) + \sum_{\hat{t}=1}^{l}\sum_{i\in\mathbb{J}_{\hat{t}}}\tilde{\gamma}_i\zeta_i(.) - \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}}\tilde{\eta}_i^{\ell}\ell_i(.) + \sum_{\hat{\alpha}=1}^{m}\sum_{i\in\mathbb{K}_{\hat{\alpha}}}\tilde{\eta}_i^{\Phi}\Phi_i(.)$ is quasiconvex at $\tilde{y}$ on $\mathbb{F}\cup pr\mathbb{W}_1$, then $\tilde{x} = \tilde{y}$.*

Proof. Suppose to the contrary that $\tilde{x} \neq \tilde{y}$. Then by Theorem 3.2, there exist $\tilde{\mu} \in R_+^p, \tilde{\gamma} \in R^q, \tilde{\eta}^{\ell} \in R^r, \tilde{\eta}^{\Phi} \in R^r, \tilde{v} \in R^r, \tilde{\rho} \in R^r$ such that the $(\tilde{x}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho})$ is a global maximum of (MDVC) and hence

$$\Psi(\tilde{x}) = \left[\Psi(\tilde{y}) + \sum_{i\in\mathbb{I}_0}\tilde{\mu}_i\varphi_i(\tilde{y}) + \sum_{i\in\mathbb{J}_0}\tilde{\gamma}_i\zeta_i(\tilde{y}) - \sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\ell}\ell_i(\tilde{y}) + \sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\Phi}\Phi_i(\tilde{y})\right]. \tag{3.11}$$

On the other hand, from $\tilde{x} \in \mathbb{F}$ and $(\tilde{y}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho}) \in \mathbb{W}_1$, it follows that

$$\varphi_i(\tilde{x}) \leq 0,\ \tilde{\mu}_i \geq 0,\ i \in \mathbb{I}_{\hat{s}}, \hat{s} = 1, 2, ..., k,$$

$$\zeta_i(\tilde{x}) = 0,\ \tilde{\gamma}_i \in R,\ i \in \mathbb{J}_{\hat{t}},\ \hat{t} = 1, 2, ..., l,$$

$$-\ell_i(\tilde{x}) < 0,\ \tilde{\eta}_i^{\ell} \geq 0,\ i \in \Lambda_{+}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$-\ell_i(\tilde{x}) = 0,\ \tilde{\eta}_i^{\ell} \in R,\ i \in \Lambda_{0}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$\Phi_i(\tilde{x}) > 0,\ \tilde{\eta}_i^{\Phi} = 0,\ i \in \Lambda_{0+}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$\Phi_i(\tilde{x}) = 0,\ \tilde{\eta}_i^{\Phi} \geq 0,\ i \in \Lambda_{00}^{\mathbb{K}_{\hat{\alpha}}} \cup \Lambda_{+0}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

$$\Phi_i(\tilde{x}) < 0,\ \tilde{\eta}_i^{\Phi} \geq 0,\ i \in \Lambda_{0-}^{\mathbb{K}_{\hat{\alpha}}} \cup \Lambda_{+-}^{\mathbb{K}_{\hat{\alpha}}},\ \hat{\alpha} = 1, 2, ..., m,$$

that is,

$$\sum_{\hat{s}=1}^{k} \sum_{i \in \mathbb{I}_{\hat{s}}} \tilde{\mu}_i \varphi_i(\tilde{x}) + \sum_{\hat{t}=1}^{l} \sum_{i \in \mathbb{J}_{\hat{t}}} \tilde{\gamma}_i \zeta_i(\tilde{x}) - \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\ell} \ell_i(\tilde{x}) + \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\Phi} \Phi_i(\tilde{x}) \leq 0.$$

Now, by (3.2)-(3.5), above inequality implies that

$$\left\{ \sum_{\hat{s}=1}^{k} \sum_{i \in \mathbb{I}_{\hat{s}}} \tilde{\mu}_i \varphi_i(\tilde{x}) + \sum_{\hat{t}=1}^{l} \sum_{i \in \mathbb{J}_{\hat{t}}} \tilde{\gamma}_i \zeta_i(\tilde{x}) - \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\ell} \ell_i(\tilde{x}) + \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\Phi} \Phi_i(\tilde{x}) \right\}$$
$$- \left\{ \sum_{\hat{s}=1}^{k} \sum_{i \in \mathbb{I}_{\hat{s}}} \tilde{\mu}_i \varphi_i(\tilde{y}) + \sum_{\hat{t}=1}^{l} \sum_{i \in \mathbb{J}_{\hat{t}}} \tilde{\gamma}_i \zeta_i(\tilde{y}) - \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\ell} \ell_i(\tilde{y}) + \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\Phi} \Phi_i(\tilde{y}) \right\} \leq 0,$$

which by quasiconvexity of $\sum_{\hat{s}=1}^{k} \sum_{i \in \mathbb{I}_{\hat{s}}} \tilde{\mu}_i \varphi_i(.) + \sum_{\hat{t}=1}^{l} \sum_{i \in \mathbb{J}_{\hat{t}}} \tilde{\gamma}_i \zeta_i(.) - \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\ell} \ell_i(.) + \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\Phi}\ \Phi_i(.)$ at $\tilde{y}$ on $\mathbb{F} \cup pr\mathbb{W}_1$, one has

$$(\tilde{x} - \tilde{y})^T \left[\sum_{\hat{s}=1}^{k} \sum_{i \in \mathbb{I}_{\hat{s}}} \tilde{\mu}_i \nabla \varphi_i(\tilde{y}) + \sum_{\hat{t}=1}^{l} \sum_{i \in \mathbb{J}_{\hat{t}}} \tilde{\gamma}_i \nabla \zeta_i(\tilde{y}) - \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\ell} \nabla \ell_i(\tilde{y}) + \sum_{\hat{\alpha}=1}^{m} \sum_{i \in \mathbb{K}_{\hat{\alpha}}} \tilde{\eta}_i^{\Phi} \nabla \Phi_i(\tilde{y}) \right] \leq 0. \tag{3.12}$$

By (3.1) and (3.12), we get

$$(\tilde{x} - \tilde{y})^T \left[\nabla \Psi(\tilde{y}) + \sum_{i \in \mathbb{I}_0} \tilde{\mu}_i \nabla \varphi_i(\tilde{y}) + \sum_{i \in \mathbb{J}_0} \tilde{\gamma}_i \nabla \zeta_i(\tilde{y}) - \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\ell} \nabla \ell_i(\tilde{y}) + \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\Phi} \nabla \Phi_i(\tilde{y}) \right] \geq 0,$$

which by strict pseudoconvexity of $\Psi(.) + \sum_{i \in \mathbb{I}_0} \tilde{\mu}_i \varphi_i(.) + \sum_{i \in \mathbb{J}_0} \tilde{\gamma}_i \zeta_i(.) - \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\ell} \ell_i(.) + \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\Phi}\ \Phi_i(.)$ at $\tilde{y}$ on $\mathbb{F} \cup pr\mathbb{W}_1$, one has

$$\Psi(\tilde{x}) + \sum_{i \in \mathbb{I}_0} \tilde{\mu}_i \varphi_i(\tilde{x}) + \sum_{i \in \mathbb{J}_0} \tilde{\gamma}_i \zeta_i(\tilde{x}) - \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\ell} \ell_i(\tilde{x}) + \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\Phi} \Phi_i(\tilde{x})$$
$$< \Psi(\tilde{y}) + \sum_{i \in \mathbb{I}_0} \tilde{\mu}_i \varphi_i(\tilde{y}) + \sum_{i \in \mathbb{J}_0} \tilde{\gamma}_i \zeta_i(\tilde{y}) - \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\ell} \ell_i(\tilde{y}) + \sum_{i \in \mathbb{K}_0} \tilde{\eta}_i^{\Phi} \Phi_i(\tilde{y}). \tag{3.13}$$

Since $\tilde{x} \in \mathbb{F}$ and $(\tilde{y}, \tilde{\mu}, \tilde{\gamma}, \tilde{\eta}^{\ell}, \tilde{\eta}^{\Phi}, \tilde{v}, \tilde{\rho}) \in \mathbb{W}_1$, it follows that

$$\varphi_i(\tilde{x}) \leq 0,\ \tilde{\mu}_i \geq 0,\ \forall i \in \mathbb{I}_0,$$

$$\zeta_i(\tilde{x}) = 0,\ \tilde{\gamma}_i \in R,\ \forall i \in \mathbb{J}_0,$$

$$-\ell_i(\tilde{x}) < 0,\ \tilde{\eta}_i^{\ell} \geq 0,\ \forall i \in \Lambda_{+}^{\mathbb{K}_0},$$

$$-\ell_i(\tilde{x}) = 0,\ \tilde{\eta}_i^{\ell} \in R,\ \forall i \in \Lambda_{0}^{\mathbb{K}_0},$$

$$\Phi_i(\tilde{x}) > 0,\ \tilde{\eta}_i^{\Phi} = 0,\ \forall i \in \Lambda_{0+}^{\mathbb{K}_0},$$

$$\Phi_i(\tilde{x}) = 0,\ \tilde{\eta}_i^{\Phi} \geq 0,\ \forall i \in \Lambda_{00}^{\mathbb{K}_0} \cup \Lambda_{+0}^{\mathbb{K}_0},$$

$$\Phi_i(\tilde{x}) < 0,\ \tilde{\eta}_i^{\Phi} \geq 0\ , \forall i \in \Lambda_{0-}^{\mathbb{K}_0} \cup \Lambda_{+-}^{\mathbb{K}_0},$$

that is

$$\sum_{i\in\mathbb{I}_0}\tilde{\mu}_i\varphi_i(\tilde{x})+\sum_{i\in\mathbb{J}_0}\tilde{\gamma}_i\zeta_i(\tilde{x})-\sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\ell}\ell_i(\tilde{x})+\sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\Phi}\Phi_i(\tilde{x})\leq 0. \tag{3.14}$$

By (3.13) and (3.14), we get

$$\Psi(\tilde{x})<\Psi(\tilde{y})+\sum_{i\in\mathbb{I}_0}\tilde{\mu}_i\varphi_i(\tilde{y})+\sum_{i\in\mathbb{J}_0}\tilde{\gamma}_i\zeta_i(\tilde{y})-\sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\ell}\ell_i(\tilde{y})+\sum_{i\in\mathbb{K}_0}\tilde{\eta}_i^{\Phi}\Phi_i(\tilde{y}).$$

This is a contradiction to (3.11). Hence the result. □

The following example is provided to demonstrate the validity of the new Mixed dual model and associated theorems.

Example 3.4. Consider the following mathematical programming problem with vanishing constraints:

$$\text{(MPVC-1)}\qquad \min_{x\in\mathbb{F}}\ \Psi(x)=x_1^2+x_2^2+x_3^2+x_4^2$$
$$\text{subject to}$$
$$\ell_i(x)\geq 0,\ i=1,2,$$
$$\Phi_i(x)\ell_i(x)\leq 0,\ i=1,2,$$

with $n=4$, $p=0$, $q=0$ and $r=2$, where $\ell_1(x)=x_2$, $\ell_2(x)=x_2+x_3$, $\Phi_1(x)=x_3$ and $\Phi_2(x)=x_4$. The feasible region is given by $\mathbb{F}_1=\{x\in R^4|\ell_i(x)\geq 0,\ \Phi_i(x)\ell_i(x)\leq 0,\ \forall i=1,2\}$. The new mixed type dual model to (MPVC-1) is given by

$$\text{(MDVC-1)}\qquad \max\left[\Psi(y)-\eta_1^{\ell}\ell_1(y)+\eta_1^{\Phi}\Phi_1(y)\right]=\left[y_1^2+y_2^2+y_3^2+y_4^2-\eta_1^{\ell}y_2+\eta_1^{\Phi}y_3\right]$$
$$\text{subject to}$$
$$\nabla\Psi(y)-\sum_{i=1}^{2}\eta_i^{\ell}\nabla\ell_i(y)+\sum_{i=1}^{2}\eta_i^{\Phi}\nabla\Phi_i(y)$$
$$=\left(2y_1-\eta_2^{\ell}+\eta_1^{\Phi}+\eta_2^{\Phi},2y_2-\eta_1^{l}-\eta_2^{\ell}\right)=(0,0),$$
$$\eta_2^{\Phi}\Phi_2(y)=\eta_2^{\Phi}y_1\geq 0,$$
$$-\eta_2^{\ell}\ell_2(y)=-\eta_2^{\ell}(y_1+y_2)\geq 0,$$
$$y\in R^4,\quad \eta^{\ell}\in R^4,\quad \eta^{\Phi}\in R^4,$$

where $\eta_i^{\Phi}=v_i\ell_i(x), v_i\in R_+, i=1,2$ and $\eta_i^{\ell}=\rho_i-v_i\Phi_i(x),\rho_i\in R_+,i=1,2$.

(*i*) The set of all feasible points of the (MDVC-1) is denoted by $\overline{\mathbb{W}}_1$ and note that, $(y,\eta^{\ell},\eta^{\Phi},v,\rho)=(0,0,0,0,0)\in\overline{\mathbb{W}}_1$ is a feasible solution for (MDVC-1). Furthermore, it is not difficult to see that $\Psi(.)-\eta_1^{\ell}\ell_1(.)+\eta_1^{\Phi}\Phi_1(.)$ is pseudoconvex at y on $\mathbb{F}\cup pr\mathbb{W}_1$ and $-\eta_2^{\ell}\ell_2(.)+\eta_2^{\Phi}\Phi_2(.)$ is quasiconvex at y on $\mathbb{F}\cup pr\mathbb{W}_1$. Also, for the feasible solutions $x=(0,0,0,0)$ for (MPVC-1) and $(y,\eta^{\ell},\eta^{\Phi},v,\rho)$ for (MDVC-1), we observe that $\Psi(x)\geq\Psi(y)-\eta_2^{\ell}\ell_2(y)+\eta_2^{\Phi}\Phi_2(y)$. Hence, the weak duality Theorem 3.1 is verified.

(*ii*) It is easy to see that the linear independence constraint qualifications are met by $\nabla\ell_1=(0,1,0,0)^T$, $\nabla\ell_2=(0,1,1,0)^T$, $\nabla\Phi_1=(0,0,1,0)^T$ and $\nabla\Phi_2=(0,0,0,1)^T$. As a result, (MPVC-1) is satisfied (VC-ACQ). According to Theorem 2.4, Lagrange multipliers $(\eta^{\ell},\eta^{\Phi},v,\rho)\in R^r\times R^r\times R^r\times R^r$ exist such that $(x=0,\eta^{\ell},\eta^{\Phi},v,\rho)$ is a feasible point of the (MDVC-1). Therefore, it is self-evident that the (MDVC-1)'s global maximum is $(x=0,\eta^{\ell},\eta^{\Phi},v,\rho)$ and that $\Psi(0)=\left[\Psi(0)-\eta_1^{\ell}\ell_1(0)+\eta_1^{\Phi}\Phi_1(0)\right]$; this demonstrates that Theorem 3.2 is true.

(*iii*) It is easy to demonstrate that the Theorem 3.3 hypothesis is correct. In consideration of (MPVC-1) and (MDVC-1), let x represent a local minimum of the (MPVC-1), and let $(y,\mu,\gamma,\eta^{\ell},\eta^{\Phi},v,\rho)$ represent a global maximum of (MDVC-1). Consequently, x equals y, proving that Theorem 3.4 is correct.

4 Special Cases

The varieties $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ in (MDVC) will create a variety of dualities. It will reduce a mixed type dual involving several well-known duality forms, such as the Wolfe type and Mond-Weir type duals, which are exceptional cases of the mixed type dual. As an illustration, take into consideration the following:

(*i*) If $\mathbb{I}_0 = \mathbb{I}$, $\mathbb{J}_0 = \mathbb{J}$ and $\mathbb{K}_0 = \mathbb{K}$ in problem (MDVC), then unified dual problem (MDVC) reduces to the Wolfe type dual [10] problem (D1):

$$\text{(D1)} \qquad \max \left[\Psi(y) + \sum_{i \in \mathbb{I}} \mu_i \varphi_i(y) + \sum_{i \in \mathbb{J}} \gamma_i \zeta_i(y) - \sum_{i \in \mathbb{K}} \eta_i^{\ell} \ell_i(y) + \sum_{i \in \mathbb{K}} \eta_i^{\Phi} \Phi_i(y) \right]$$
$$\text{subject to}$$
$$(y, \mu, \gamma, \eta^{\ell}, \eta^{\Phi}, v, \rho) \in \bigcap_{x \in \mathbb{F}} \mathbb{W}_1(x).$$

(*ii*) If $\mathbb{I}_0 = \emptyset$, $\mathbb{J}_0 = \emptyset$ and $\mathbb{K}_0 = \emptyset$ in problem (MDVC), then unified dual problem (MDVC) reduces to the Mond-Weir type dual [10] problem (D2):

$$\text{(D2)} \qquad \max \Psi(y)$$
$$\text{subject to}$$
$$(y, \mu, \gamma, \eta^{\ell}, \eta^{\Phi}, v, \rho) \in \bigcap_{x \in \mathbb{F}} \mathbb{W}_1(x).$$

5 Conclusion

In this article, by using incomplete Lagrangian multipliers technique we have proposed a mixed type dual problem for a mathematical programming problem with vanishing constraints and discussed duality results. Furthermore, the methods provided in the work have been demonstrated by an example to be relevant for an interval-valued programming problem with vanishing constraints. Our results apparently generalize the duality results of Hu et al. [10] for the considered mathematical programming problem with vanishing constraints under generalized convexity. It would be interesting to look at a broader class of nonsmooth vector optimization problems with multiple interval-valued objective functions than convex ones to see whether the optimality conditions and duality results can be proven using modified Abadie constraint qualification (VC-ACQ). In forthcoming publications, we shall examine these issues.

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References

[1] W. Achtziger and C. Kanzow, *Mathematical programs with vanishing constraints: Optimality conditions and constraint qualifications*, Math. Program. **114** (2008), 69–99.

[2] I. Ahmad, K. Kummari and S. Al-Homidan, *Sufficiency and duality for interval-valued optimization problems with vanishing constraints using weak constraint qualification*, Int. J. Anal. Appl. **18** (2020), no. 5, 784–798.

[3] T. Antczak, *Optimality conditions and Mond–Weir duality for a class of differentiable semi-infinite multiobjective programming problems with vanishing constraints*, 4OR **20** (2022), no. 3, 417–442.

[4] A.A. Ardali, M. Zeinali and G.H. Shirdel, *Some Pareto optimality results for nonsmooth multiobjective optimization problems with equilibrium constraints*, Int. J. Nonlinear Anal. Appl. 13 (2022), no. 2, 2185–2196.

[5] M.S. Bazaraa, H.D. Sherali and C. M. Shetty, *Nonlinear Programming. Theory and Algorithms*, 2nd edition, John Wiley & Sons, Hoboken, 1993.

[6] M.P. Bendsøe and O. Sigmund, *Topology Optimization-Theory, Methods and Applications*, 2nd ed., Springer, Heidelberg, Germany, 2003.

[7] R. Dubey and V.N. Mishra, *Second-order nondifferentiable multiobjective mixed type fractional programming problems*, Int. J. Nonlinear Anal. Appl. **11** (2020), no. 1, 439–451.

[8] J.-P. Dussault, M. Haddou and T. Migot, *Mathematical programs with vanishing constraints: constraint qualifications, their applications, and a new regularization method*, Optim. **68** (2019), no. 2-3, 509–538.

[9] T. Hoheisel and C. Kanzow, *On the Abadie and Guignard constraint qualifications for Mathematical Programmes with Vanishing Constraints*, Optim. **58** (2009), no. 4, 431–448.

[10] Q. Hu, J. Wang and Y. Chen, *New dualities for mathematical programs with vanishing constraints*, Ann. Oper. Res., **287** (2020), 233–255.

[11] R.A. Jabr, *Solution to economic dispatching with disjoint feasible regions via semidefinite programming*, IEEE T. Power Syst. **27** (2012), no. 1, 572–573.

[12] A. Jayswal and V. Singh, *The Characterization of efficiency and saddle point criteria for multiobjective optimization problem with vanishing constraints*, Acta Math. Sci. **39** (2019), no. 2, 382–394.

[13] S. Kazemi and N. Kanzi, *Constraint qualifications andsStationary conditions for mathematical programming with non-differentiable vanishing constraints*, J. Optim. Theory Appl. **179** (2018), no. 3, 800–819.

[14] A. Khare and T. Nath, *Enhanced Fritz-John stationarity, new constraint qualifications and local error bound for mathematical programs with vanishing constraints*, J. Math. Anal. Appl. **472** (2019), no. , 1042–1077.

[15] C. Kirches, A. Potschka, H. G. Bock and S. Sager, *A parametric active set method for quadratic programs with vanishing constraints*, Pacific J. Optim. **9** (2013), no. 2, 275–299.

[16] N. J. Michael, C. Kirches and S. Sager, *On perspective functions and vanishing constraints in mixed-integer nonlinear optimal control*, In: M. Jünger & G. Reinelt (Eds.), Facets of combinatorial optimization (pp. 387-417), Springer, Berlin, 2013.

[17] S.K. Mishra, V. Singh and V. Laha, *On duality for mathematical programs with vanishing constraints*. Ann. Oper. Res. **243** (2016), no. 1, 249–272.

[18] L.T. Tung, *Karush–Kuhn–Tucker optimality conditions and duality for multiobjective semi-infinite programming with vanishing constraints* , Annal. Oper. Res. **311** (2022), 1307–1334.