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Optimality conditions for multi-objective interval-valued E-convex functions with the use of gH-symmetrical differentiation

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Abstract

In this paper, we introduce and discuss multi-objective interval-valued E-convex programming using gH-symmetrical differentiability. We prove nonlinear optimality conditions of Fritz John type for this context and construct an example to verify our results. Furthermore, we define LU-sE-pseudo convexity and LU-sE-quasi convexity for interval-valued functions and study some of their properties.

Keywords: Fritz John optimality conditions, interval-valued functions, E-convexity, Multi objective programming, gH-symmetrically differentiation.
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1 Introduction

The main objective of multi criteria decision making is to find the best pareto optimal solutions. These solutions have great importance in a multi-objective programming problem from a theoretical point of view. In simple words, we can say that the pareto optimal solutions are those solutions which can not be dominated by the other solutions in the entire search space. Multi objective programming (MOP) for interval-valued objective functions was firstly studied by Ishibuchi and Tanaka [14] in 1990. They [14] proposed the ordering relation between the closed intervals for comparing them. After that Wu[27] had developed a new theory of derivatives called H-derivative or weak derivative and proved KKT optimality conditions under weak derivative concept for interval-valued optimization problems. Later Stefanini and Bede [23] expanded the notion of weak derivative to gH-derivative while Chalco-cano et.al. [8] discussed the optimality conditions of KKT type for gH-derivative. Afterwards, Guo et.al. [12] introduced the idea of gH-symmetrical derivative, which is more general than of weak derivative and gH-derivative. For more on symmetric differentiation, one can see: [16, 25].

Convexity plays an important role in optimization problems especially for interval-valued objective functions. In the recent past very useful efforts have been done to generalize the convexity hypothesis and thus to explore the

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Fritz John and KKT type optimality conditions. E-convexity is one of the generalizations of convexity, introduced by Youness [30]. He also discussed optimality criteria for E-convex programming problems [31].

Inspired by the above research work, we discuss the concept of gH-symmetrical derivative for multi objective interval-valued functions under E-convexity assumptions and derive the Fritz John sufficient optimality conditions.

We divide this paper in four sections. In Section 2, we recollect some basic definitions and discuss several properties of gH-symmetric derivative and E-convex functions for multi-objective interval-valued programming. Section 3 is devoted to derive the optimality conditions for multi-objective interval-valued optimization under E-convexity and gH-symmetrical derivative assumptions. We conclude this paper in Section 4.

2 Preliminaries

Assume that in the real line $\mathbb{R}$, $I_{(\mathbb{R})}$ be the set of all closed and bounded intervals. i.e

$$I_{\mathbb{R}} = [p^L, p^U] : p^L, p^U \in \mathbb{R} \; and \; p^L < p^U,$$

where, p^L and p^U represent the upper and lower limits of the interval.

If $P = [p^L, p^U]$ and $Q = [q^L, q^U]$, then $P + Q = [p^L + q^L, p^U + q^U]$.
$\alpha P = [\alpha p^L, \alpha p^U] \; if \; \alpha \geq 0$ and $[\alpha p^U, \alpha p^L] \; if \; \alpha < 0$. For more on interval analysis, see [[2], [17], [18]].

Definition 2.1. Order-relation Let $P = [p^L, p^U]$, then

$$P \preceq_{LU} Q \;\text{ iff } p^L \leq q^L \; and \; p^U \leq q^U$$

and

$$P \prec_{LU} Q \;\text{ iff } P \preceq_{LU} Q \;\text{ and }\; P \neq Q.$$

Equivalently,

$$P \prec_{LU} Q \text{ iff } p^L < q^L \text{ and } p^U \leq q^U \text{ OR } p^L \leq q^L \text{ and } p^U < q^U \text{ OR } p^L < q^L \text{ and } p^U < q^U.$$

Definition 2.2. A function $\phi_I : \mathbb{R}^n \to I_{\mathbb{R}}$ is said to be an interval-valued function (IVF) if it has a form

$$\phi_I(p) = [\phi^L(p), \phi^U(p)], \text{ such that } \phi^L(p) \leq \phi^U(p) \;\; \forall \, p \in \mathbb{R}^n,$$

where, $\phi^L(p)$ is the lower limit and $\phi^U(p)$ is the upper limit of $\phi_I(p)$.

The gH-difference (generalized Hukuhara difference) of two intervals P and Q is defined by Stefanini and Bede [23] as follows:

$$P \ominus_g Q = R \;\Leftrightarrow\; \begin{cases} (1) & P = Q + R \\ (2) & Q = P + (-1)R \end{cases}$$

For any two intervals in $I_{\mathbb{R}}$, this definition always exists and it also written as the following equivalent form:

$$P \ominus_g Q = \left[min\{p^L - q^L, p^U - q^U\}, max\{p^L - q^L, p^U - q^U\} \right].$$

To recall the concept of symmetric differentiation and its properties, one can see [18],[19].

Definition 2.3. [25] A real valued function $\phi : (p, q) \to \mathbb{R}$ is symmetrically differentiable (SD) at $p_0 \in (p, q)$ if $\exists$ a real number $A \in \mathbb{R}$, s.t.

$$\lim_{h \to 0} \frac{\phi(p_0 + h) - \phi(p_0 - h)}{2h} = A = \phi^s(p_0)$$

Theorem 2.4. [16] If ϕ is differentiable at p_0 then it is also SD and has same value.

Theorem 2.5. [16] Let ϕ be SD on $M \subset \mathbb{R}^n$. Then, ϕ convex on M iff

$$\nabla^s \phi(q)^T (p-q) \leq \phi(p) - \phi(q) \;\; \forall \;\; p, q \in M.$$

Definition 2.6. [23] Let $\phi_I : M \subset \mathbb{R}^n \to I_{\mathbb{R}}$, then ϕ_I is gH-differentiable at $p_0 \in M$, if $\exists \; \nabla_g \phi_I(p_0) \in I_R$, such that

$$\nabla_g \phi_I(p_0) \;\; = \;\; \lim_{h \to 0} \frac{[\phi_I(p_0 + h) \ominus_g \phi_I(p_0)]}{h}.$$

To generalize the concept of gH-differentiability, Guo et. al. [12] defined the gH-symmetric differentiabilty in $I_{\mathbb{R}}$.

Definition 2.7. Let $\phi_I : M \subset \mathbb{R}^n \to I_{\mathbb{R}}$. ϕ_I is gH-symmetrically differentiable (gH-SD) at p_0, if $\exists \; \nabla^s_g \phi_I(p_0) \in I_R$, such that

$$\nabla^s_g \phi_I(p_0) \;\; = \;\; \lim_{h \to 0} \frac{\phi_I(p_0 + h) \ominus_g \phi_I(p_0 - h)}{2h}.$$

Next two theorems were given by Guo et. al. [12], which generalized the idea of gH-derivative for interval-valued functions.

Theorem 2.8. [12] Let $\phi_I : M \subseteq \mathbb{R}^n \to I_{\mathbb{R}}$ be an IVF. If ϕ_I is gH-differentiable at p_0, then ϕ_I is gH-SD at p_0. But the converse need not be true.

Theorem 2.9. [12] The function $\phi_I : M \subseteq \mathbb{R}^n \to I_{\mathbb{R}}$ is gH-SD iff ϕ^L and ϕ^U are SD.

Youness [30] extended the concept of convexity as follows:

Definition 2.10. [30] Set $M \subset \mathbb{R}$ is called E-convex if there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$tE(p) + (1-t)E(q) \in M, \; for \; each \; p, q \in M, \; t \in [0,1]$$

Definition 2.11. [30] A function $\phi : M \subset \mathbb{R}^n \to \mathbb{R}$ is called E-convex on M, if there exist a map $E : \mathbb{R}^n \to \mathbb{R}^n$ s.t. M is E-convex and

$$\phi\big(tE(p) + (1-t)E(q)\big) \leq t\phi(E(p)) + (1-t)\phi(E(q)),$$

for each $p, q \in M, t \in [0,1]$.

Recently, Sachin et. al. [20] defined the concept of E-convexity for IVF as follows:

Definition 2.12. [20] Let ϕ_I be an IVF defined on an E-convex set $M \subset \mathbb{R}^n$ w.r.t. a map $E : \mathbb{R}^n \to \mathbb{R}^n$. We say that ϕ_I is LU-E-convex at p_0, if

$$\phi_I\big(tE(p_0) + (1-t)E(p)\big) \preceq_{LU} t\phi_I(E(p_0)) + (1-t)\phi_I(E(p)),$$

for each $p \in M$ and $t \in [0,1]$.

Proposition 2.13. [20] Let ϕ_I be an IVF defined on an E-convex set $M \subset \mathbb{R}^n$. ϕ_I is LU-E-convex at p_0, iff ϕ^L and ϕ^U are E-convex at p_0.

Theorem 2.14. [20] Suppose $\phi : M \subset \mathbb{R}^n \to \mathbb{R}$ and $E : M \subset \mathbb{R}^n \to \mathbb{R}^n$ be SD functions and M be an open E-convex set, then ϕ is E-convex iff

$$\phi(E(p)) - \phi(E(q)) \geq \nabla^s \phi(E(q))^T (E(p) - E(q)) \quad \forall p, q \in M.$$

3 Interval-valued Multi-objective E- convex Programming

In this section, we consider multi-valued interval function (MVIF):

$$\phi_I(p) = \big(\phi_{(I,1)}(p), \phi_{(I,2)}(p), ...\phi_{(I,r)}(p)\big)$$

defined on $M \subset \mathbb{R}^n$, where each $\phi_{(I,l)}(p) = [\phi_l^L, \phi_l^U], \quad l = 1, 2, ..r.$

The interval-valued multi-objective programming problem $(IVMP)$ for LR-convex and gH-SD functions is defined by Sachin et. al. [20] as follows:

$$min\phi_I(p) = \big(\phi_{(I,1)}(p), \phi_{(I,2)}(p), ...\phi_{(I,r)}(p)\big)$$

subject to

$$\zeta_{(I,i)}(p) \preceq_{LR} (0, 0), \quad i = 1, 2, .., s.$$

Now, we generalize the above convexity concept to E-convexity and define multiobjective interval-valued E-convex programming as follows

$$min(\phi_I oE)(p) = \big(\phi_{(I,1)}oE(p), \phi_{(I,2)}oE(p).......\phi_{(I,r)}oE(p)\big)$$

subject to

$$\zeta_{(I,i)}oE(p) \preceq_{LU} [0, 0], \; i = 1, 2, ..., s,$$

where each $\phi_{(I,l)} : M \subseteq \mathbb{R}^n \to I_{\mathbb{R}}$ and each $\zeta_{(I,i)} : M \subseteq \mathbb{R}^n \to I_R, \; l = 1, 2, ...r, i = 1, 2, ..s$ are interval-valued gH-SD and $LU - E - convex$ functions on an open E-convex set M w.r.t. a SD map $E : \mathbb{R}^n \to \mathbb{R}^n$.

We can convert interval-valued inequality constraints as follows.

$$\zeta_{(I,i)}oE(p) \preceq_{LU} [0, 0], \; i = 1, 2, ..s,$$

can be written as $\zeta_i^L oE(p) \leq 0$ and $\zeta_i^U oE(p) \leq 0, \; i = 1, 2, ..s$. Which can be combined as

$$\zeta_i oE(p) \leq 0, \quad i = i = 1, 2, ..2s.$$

Hence, the feasible set is

$$C^{'} = \{p : \zeta_i oE(p) \leq 0, \; i = 1, 2, ..2s\}.$$

Now, consider the E-convex problem $(IVMP)_E$:

$$min(\phi_I oE)(p)$$

$$subject\ to\ p \in C^{'},$$

where

$$C^{'} = \{p \in M : (\zeta_i oE)(p) \leq 0, \; i = 1, 2, ..2s\}.$$

If we take E as an identity map then $(IVMP)_E$ converts to (IVMP) as defined by Sachin et. al. [20].

Theorem 3.1. [20] Let $\phi_I : \mathbb{R}^n \to I_{\mathbb{R}}$ be MVIF, then ϕ_I is gH-SD at $p_0 \in M$ iff ϕ_l^L and ϕ_l^U are SD at $p_0 \in M$.

Definition 3.2. [21] A feasible solution p_0 is a pareto optimal solution (IVMP) if there exists no $p^{'} \in M$ s.t. $\phi_{(I,l)}(p^{'}) \preceq_{LU} \phi_{(I,l)}(p_0)$, for all $l = 1, 2, ..., r$ and $\phi_{(I,h)}(p^{'}) \prec_{LU} \phi_{(I,h)}(p_0)$ for at least one index $h \in (1, 2, 3...r)$.

Now, we define pareto optimality for $(IVMP)_E$.

Definition 3.3. A feasible solution p_0 is a pareto optimal solution $(IVMP)_E$, if there exists no $p^{'} \in M$ such that $\phi_{(I,l)}oE \preceq_{LU} \phi_{(I,l)}oE(p_0)$ for each $l = 1, 2, ..r$ and $\phi_{(I,h)}oE(p^{'}) \prec_{LU} \phi_{(I,h)}oE(p_0)$ for at least one index $h \in (1, 2, ...r)$.

LU- E- convexity for interval-valued multi-objective functions is defined as follows:

Definition 3.4. Let ϕ_I be a MVIF defined on an E-convex set $M \subset \mathbb{R}^n$ w.r.t. a map $E : \mathbb{R}^n \to \mathbb{R}^n$. We say ϕ_I is LU-E-convex at p_0 if each $\phi_{(I,l)}$ is LU-E-convex at p_0 for $l = 1, 2, ...r$, i.e.

$$\phi_{(I,l)}\big(tE(p_0) + (1-t)E(p)\big) \preceq_{LU} t\phi_{(I,l)}(E(p_0)) + (1-t)\phi_{(I,l)}(E(p))$$

for each $p \in M \quad and \quad t \in [0, 1]$.

Proposition 3.5. Let ϕ_I be an MVIF defined on an E-convex set $M \subset \mathbb{R}^n$, then ϕ_I is LU-E-convex at p_0 iff ϕ_l^L and ϕ_l^U are E-convex at p_0, where, $l = 1, 2, ...r$.

Proof . From Definition 3.3 and Proposition 2.13, the proof is obvious. □

Theorem 3.6. (Fritz john type sufficiency condition) Considering the same assumptions of $(\mathbf{IVMP})_E$ if there exists $p_0 \in C^{'}$ and real-valued multipliers $\alpha_l^L, \alpha_l^U > 0$ and $\gamma_i \geq 0$, for $l = 1, 2, ..., r, i = 1, 2, 3....2s$ respectively, such that the following conditions hold:
(1) $\sum_{l=1}^{r} \alpha_l^L \nabla^s \phi_l^L oE(p_0) \; + \; \sum_1^r \alpha_l^U \nabla^s \phi_l^U oE(p_0) \; + \; \sum_{i=1}^{2s} \gamma_i \nabla^s \zeta_i oE(p_0) = 0$,
(2) $\sum_{i=1}^{2s} \gamma_i \zeta_i oE(p_0) = 0$,
then, p_0 is a LU Pareto optimal solution of $(\mathbf{IVMP})_E$.

Proof . Let

$$\phi oE(p) = \sum_{l=1}^{r} \alpha_l^L \phi_l^L oE(p) + \sum_{l=1}^{r} \alpha_l^U \phi_l^U oE(p).$$

Since, ϕ_I is LU-E-convex and gH-SD at p_0, by Theorem (3.1) and Proposition (3.5), ϕ is also E-convex and SD at p_0, therefore

$$\nabla^s \phi oE(p_0) = \sum_{l=1}^{r} \alpha_l^L \nabla^s \phi_l^L oE(p_0) + \sum_{l=1}^{r} \alpha_l^U \nabla^s \phi_l^U oE(p_0),$$

so the given conditions become
(1) $\nabla^s \phi oE(p_0) + \sum_{i=1}^{2s} \gamma_i \nabla^s \zeta_i oE(p_0) = 0$
(2) $\sum_{i=1}^{2s} \gamma_i \zeta_i oE(p_0) = 0$.

Since ϕ is E-convex and SD, by Theorem 2.6,

$$\nabla^s \phi oE(p_0)^T (E(p) - E(p_0)) \leq \phi oE(p) - \phi oE(p_0) \quad \forall p \in C^{'}.$$

By the new condition (1), we get

$$-\sum_{i=1}^{2s} \gamma_i \nabla^s \zeta_i oE(p_0)^T (E(p) - E(p_0)) \leq \phi oE(p) - \phi oE(p_0) \; \forall \; p \in C^{'} \qquad (1).$$

Since, each ζ_i is E-convex and SD, again by Theorem 3.1, for $i = 1, 2, 3...2s$, we get

$$\nabla^s \zeta_i oE(p_0)^T (E(p) - E(p_0)) \leq \zeta_i oE(p) - \zeta_i oE(p_0) \quad \forall \; p \in C^{'}$$

or

$$\sum_{i=1}^{2s} \gamma_i \nabla^s \zeta_i oE(p_0)^T (E(p) - E(p_0)) \leq \sum_{i=1}^{2s} \gamma_i \Big(\zeta_i oE(p) - \zeta_i oE(p_0)\Big) \quad \forall p \in C^{'}.$$

Applying condition (2), we get

$$\sum_{i=1}^{2s} \gamma_i \nabla^s \zeta_l oE(p_0)^T (E(p) - E(p_0)) \leq \sum_{i=1}^{2s} \gamma_i (\zeta_i oE)(p) \; \forall \; p \in C^{'}. \qquad (2).$$

On adding (1) and (2), we get

$$-\sum_{i=1}^{2s} \gamma_i (\zeta_i oE)(p) \leq \phi oE(p) - \phi oE(p_0) \quad \forall p \in C^{'}.$$

But for each i, $\gamma_i \geq 0$ and $\zeta_i oE(p) \leq 0$, thus

$$\phi oE(p) \geq \phi oE(p_0) \quad \forall p \in C^{'}.$$

Hence, p_0 is an optimal solution of ϕ with respect to E.

Now, let p_0 be not a pareto optimal solution of the problem $(IVMP)_E$, by the Definition (3.2), $\exists\ p^{'} \in C^{'}$, such that

$$\phi_{(I,l)} oE(p^{'}) \preceq_{LU} \phi_{(I,l)} oE(p_0), \quad for\ each \ \ l = 1, 2, ...r,$$

and

$$\phi_{(I,h)} oE(p^{'}) \prec_{LU} \phi_{(I,h)} oE(p_0), \text{ for at least one index} h \in (1, 2, 3...r).$$

Therefore, by our first assumption

$$\phi oE(p) = \sum_{l=1}^{r} \left(\alpha_l^L \phi_l^L oE(p) + \alpha_l^U \phi_l^U oE(p)\right).$$

$\phi_l^L oE(p^{'}) \leq \phi_l^L oE(p_0)$ and $\phi_l^U oE(p^{'}) \leq \phi_l^U oE(p_0)$, for $l = 1, 2, ...r$,
and

$$\phi_h^L oE(p^{'}) < \phi_h^L oE(p_0) \quad , \quad \phi_h^U oE(p^{'}) < \phi_h^U oE(p_0),$$

for at least one index $h \in (1, 2, ...r)$ and since $\alpha_l^L, \alpha_l^U > 0, \quad \forall l = 1, 2, ...r$, we get

$$\phi oE(p^{'}) < \phi oE(p_0)$$

which is a contradiction by the fact that

$$\phi oE(p_0) \geq \phi oE(p), \forall p \in C^{'}.$$

Hence, p_0 is a pareto optimal solution of $(IVMP)_E$. □

Now, we construct an example to verify the above theorem.

Example 3.7. Contemplate the multi-objective E-convex interval-valued programming problem

$$Min\ \phi_{(I,l)} \quad = \quad \left(\phi_{(I,1)}, \phi_{(I,2)}\right),$$

subject to:

$$(p, q) \in M,$$

where

$$\phi_{(I,1)}(p, q) \quad = \quad [-|p| + q^2, |p| + 2q^2]$$
$$\phi_{(I,2)}(p, q) \quad = \quad [-p^2 - q, p^2 + 2q]$$

and

$$M = \{(p, q) \in \mathbb{R}^2 : p + q - 1 \leq 0, \ \ -q \leq 0\}.$$

The above problem is neither LU-convex nor differentiable, but it is gH-symmetric differentiable and E-convex w.r.t. a map $E(p, q) = (0, q)$. So, the E-convex version of above problem is as follows:

$$(\phi_{(I,l)} oE)(p, q) \quad = \quad \{[q^2, 2q^2], [-q, 2q]\}$$

subject to:

$$E(M) \quad = \quad \{(0, q) \in \mathbb{R}^2 : 0 \leq q \leq 1\}.$$

Now

$$\phi_1^L(p, q) \ = \ q^2 \ \ \& \ \ \phi_1^U(p, q) = 2q^2$$
$$\phi_2^L(p, q) = -q \ \ \& \ \ \phi^{U_2}(p, q) \ = \ 2q.$$

By Theorem (3.1) and Proposition(3.5), ϕ_1^L, ϕ_1^U *and* ϕ_2^L, ϕ_2^U are E-convex and symmetrically differentiable at (0,0). Thus, by the conditions (1) and (2) of Theorem (3.3), there exists multipliers $\lambda_1^L, \lambda_1^U, \lambda_2^L, \lambda_2^U, \mu_1$ *and* μ_2 such that

$$\begin{aligned}
\lambda_1^L[0,2q]^T + \lambda_1^U[0,4q]^T + \lambda_2^L[0,-1]^T + \lambda_2^U[0,2]^T + \mu_1[0,1]^T + \mu_2[0,-1]^T &= 0 \\
\mu_1(q-1) &= 0 \\
\mu_2(q) &= 0
\end{aligned}$$

$(p,q) = (0,0)$ is a feasible solution, so at (0,0), we have

$$\mu_1(-1) = 0 \quad \Longrightarrow \mu_1 = 0$$

$$-\lambda_2^L + 2\lambda_2^U - \mu_2 = 0.$$

Which satisfy

$$\lambda_2^L = \lambda_2^U = \mu_2,$$

for positive values of μ_2, we get

$$\lambda_2^L, \lambda_2^U > 0.$$

λ_1^L, λ_1^U is arbitrary, so if we take them positive then the conditions of Theorem (3.6) is satisfied and hence $(0,0)$ is the LU-Pareto solution of the given problem.

Using the Theorem (2.14), Sachin et. al. [20] generalized the E-convexity and introduced sE-pseudo convex and sE- quasi convex symmetrically differentiable real- valued functions. We extend the above concept of generalized E-convexity for IVF using gH-SD with the help of following Theorem.

Theorem 3.8. [20] Let ϕ_I be a gH-SD IVF on an open E-convex set $M \subset \mathbb{R}^n$ w.r.t. a map $E : \mathbb{R}^n \to \mathbb{R}^n$. Then, ϕ_I is LU-E-convex iff

$$\nabla_g^s \phi_I(E(p_0))^T (E(p) - E(p_0)) \preceq_{LU} \phi_I(E(p)) - \phi_I(E(p_0)), \quad \forall p \in M.$$

Definition 3.9. LU-sE-pseudo convex function
Let ϕ_I be an interval-valued gH-SD function defined on an E-convex set $M \subset \mathbb{R}^n$ w.rr.t. a map $E : \mathbb{R}^n \to \mathbb{R}^n$, then ϕ_I is said to be LU-sE-pseudoconvex at p_0 if

$$\nabla_g^s \phi_I(E(p_0))^T (E(p) - E(p_0)) \geq_{LU} 0 \implies \phi_I(E(p) \geq_{LU} \phi_I(E(p_0) \forall p \in M.$$

Proposition 3.10. Let ϕ_I be an IVF defined on an E-convex set $M \subset \mathbb{R}^n$, then ϕ_I is LU-sE-pseudo convex at p_0, iff ϕ^L and ϕ^U are sE-pseudo convex at $p_0 \in M$.

Proof . It is a direct consequnence of the Definition (3.4). □

Proposition 3.11. Let ϕ_I be a MVIF defined on an E-convex set $M \subset \mathbb{R}^n$, then ϕ_I is LU-sE-pseudo convex at p_0 iff ϕ_l^L and ϕ_l^U are sE-pseudo convex at $p_0 \in M$.

Proof . It follows from Definition (3.4) and Proposition (3.10). □

Definition 3.12. LU-sE-quasi convex function
Let ϕ_I be an interval-valued gH-SD function defined on an E-convex set $M \subset \mathbb{R}^n$ w.r.t. a map $E : \mathbb{R}^n \to \mathbb{R}^n$, then ϕ_I is said to be LU-sE-quasi convex at p_0 if

$$\phi_I(E(p)) \preceq_{LU} \phi_I(E(p_0)) \implies \nabla_g^s \phi_I(E(p_0))^T (E(p) - E(p_0)) \preceq_{LU} 0 \quad \forall p \in M.$$

Proposition 3.13. Let ϕ_I be an IVF defined on an E-convex set $M \subset \mathbb{R}^n$, then ϕ_I is LU-sE-quasi convex at p_0 iff ϕ^L and ϕ^U are sE-quasi convex at $p_0 \in M$

Proof . It is a direct consequence from the Theorem (3.8). □

Proposition 3.14. Let ϕ_I be a multi-valued interval function defined on an E-convex set $M \subset \mathbb{R}^n$, then ϕ_I is LU-sE-quasi convex at p_0 iff ϕ_l^L and ϕ_l^U are sE-quasi convex at $p_0 \in M$

Proof . From Definition (3.12) and Proposition (3.13), it can be proved easily. □

Theorem 3.15. Let $p_0 \in M$, $H = \{i : (\zeta_i oE)(p_0) = 0\}$, $\phi_{(I,l)}$ be a multi-valued LU-sE-pseudo convex at p_0 with respect to $E : \mathbb{R}^n \to \mathbb{R}^n$ and $\zeta_{I,i}$ be LU-sE- quasi convex at p_0 with respect to same E. If there exists real-valued multipliers $\gamma_i \geq 0, i = 1, 2, ...2s$, s.t. following conditions hold.
(1) $\nabla^s(\phi_l^L oE)(p_0) + \sum_{i=1}^{2s} \gamma_i \nabla^s(\zeta_i oE)(p_0) = 0$
(2) $\nabla^s(\phi_l^U oE)(p_0) + \sum_{i=1}^{2s} \gamma_i \nabla^s(\zeta_i oE)(p_0) = 0$
(3) $\sum_{i=1}^{2s} \gamma_i(\zeta_i oE)(p_0) = 0, \quad i = 1, 2, ...2s$
then p_0 is an LU-optimal solution of $(IVMP)_E$.

Proof . Since $\gamma_i \geq 0$ and each $\zeta_i oE(p_0) \leq 0$, by assumption (3)

$$\sum_{i=1}^{2s} \gamma_i(\zeta_i oE)(p_0) = 0.$$

This implies that $\gamma_i = 0$, for $i \notin H$. Now $\zeta_i(p_0)$ is sE-quasi convex for $i \in H$, then

$$(\zeta_i oE)(p) \leq (\zeta_i oE)(p_0) \implies \nabla^s(\zeta_i oE)(p_0)^T(E(p) - E(p_0)) \leq 0$$

or

$$\gamma_i \nabla^s(\zeta_i oE)(p_0)^T(E(p) - E(p_0)) \leq 0 \quad \forall p \in M$$

or

$$\sum_{i=1}^{2s} \gamma_i \nabla^s(\zeta_i oE)(p_0)^T(E(p) - E(p_0)) \leq 0 \quad \forall p \in M.$$

By assumption (1), we have

$$-\nabla^s(\phi_l^L oE)(p_0)^T(E(p) - E(p_0)) \leq 0 \quad \forall p \in M$$

$$\nabla^s(\phi_l^L oE)(p_0)^T(E(p) - E(p_0)) \geq 0 \quad \forall p \in M.$$

Since $\phi_{(I,l)}$ be LU-sE-pseudo convex function at p_0, by Proposition (3.13), ϕ_l^L *and* ϕ_l^U be sE-pseudo convex function at p_0
which implies

$$(\phi_l^L oE)(p) \geq (\phi_l^L oE)(p_0) \quad \forall p \in M.$$

Similarly

$$(\phi_l^U oE)(p) \geq (\phi_l^U oE)(p_0) \quad \forall p \in M$$

which can be written as

$$(\phi_{(I,l)} oE)(p_0) \preceq_{LU} (\phi_{(I,l)} oE)(p) \quad \forall p \in M$$

or

$$(\phi_I oE)(p_0) \preceq_{LU} (\phi_I oE)(p) \quad \forall p \in M.$$

Hence, p_0 is LU- optimal solution of $(IVMP)_E$ □

4 Conclusion

This paper shows the theory of multiobjective interval-valued E-convex programming. We derive Fritz John type sufficient conditions of optimality for E-convex programming problem with interval-valued objective and constraint functions, using gH-symmetrical derivative. We use LU ordering for comparing the intervals. We have also generalized E-convexity and defined LU-sE-pseudo convex and LU-sE-quasi convex functions for interval-valued functions.

In future, this work can be extended for the fractional programming problems. Also the equality constraints are not considered in this paper, which can be done by using the similar methodology. Moreover, the extension of our results to Hadamard manifolds would be desirable and which is an open problem for the future research.

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