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Some iterative algorithms for Reich-Suzuki nonexpansive mappings and relaxed (α, k)-cocoercive mapping with applications to a fixed point and optimization problems

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Abstract

In this paper, we propose an iterative method for finding the common element of the set of fixed points of Reich-Suzuki nonexpansive mappings and the set of solutions of the variational inequalities problems in the framework of Hilbert spaces. In addition, we establish convergence results for these proposed iterative methods under some mild conditions. Furthermore, we establish analytically and numerically that our newly proposed iterative method converges to a common element of the set of fixed points of a Reich-Suzuki nonexpansive mapping and the set of solutions of the variational inequalities problems faster compared to some well-known iterative methods in the literature. Finally, we apply our proposed iterative method to approximate the solution of a convex minimization problem. The results obtained in this paper improve, extend and unify some related results in the literature.

Keywords: Variational inequality problem; inertial iterative scheme; fixed point problem; Reich-Suzuki nonexpansive mappings
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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C a nonempty closed convex subset of H and $A : H \to H$ a nonlinear operator. The classical Variational Inequality Problem (VIP) is formulated as: Find $x^* \in C$ such that for any $x \in C$

$$\langle Ax^*, x - x^* \rangle \geq 0. \tag{1.1}$$

The notion of VIP (1.1) was introduced independently by Stampacchia [20] and Fichera [11, 12] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, pure and applied sciences can be formulated as VIP (1.1). We denote the solution set of a VIP (1.1) by $\Omega(C, A) = \{x^* \in$

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$C : \langle Ax^*, x^* - x\rangle \geq 0 \ \ \forall \ \ x \in C\}$. Thereafter, a lot of researchers in this area of mathematics have explore the notion of VIP (1.1) for detail work on VIP (1.1) the reader should (see [7, 13, 20, 24] and the references therein). It has been established over the years that the existence and approximation of a VIP (1.1) is equivalent to finding the fixed point problem: Find

$$x^* \in C \ \text{ such that } \ x^* = P_C(I - \eta A)x^*, \tag{1.2}$$

where $\eta > 0$ and P_C is called the metric projection of H onto C. We recall that the metric projection $(P_C x)$ is such that

$$||x - P_C x|| \leq ||x - y||, \ \ \forall \ y \in C.$$

It has been established in the literature that if A is L-Lipschitzian and v-strongly monotone, then the operator $P_C(I - \eta A)$ is a contraction on C provided thet $0 < \eta < \frac{2v}{L^2}$. In the light of this fact, the Banach contraction principle clearly guarantees the existence and uniqueness of an approximate solution for a VIP (1.1). The well-known Picard iterative process takes the form:

$$x_{n+1} = P_C(I - \eta A)x_n. \tag{1.3}$$

This approach of approximating the solution of a VIP (1.1) is called projected gradient method. It is well-known that

$$x^* \in \Omega(C, A) \ \text{ if and only if } \ \ x^* = P_C(x^* - \eta A x^*).$$

Definition 1.1. Let C be an arbitrary space with self mapping $T : C \to C$, a point $x \in C$ is called a fixed point of a mapping T if

$$Tx = x. \tag{1.4}$$

The set of all fixed points of T is denoted by $F(T)$. Many problems in mathematics, engineering, physics, economics, game theory, and other fields can be formulated into fixed point problems, making fixed point theory a useful field of study. In general, it is nearly impossible to solve fixed point problems analytically, necessitating the consideration of iterative methods of solutions for fixed point problems. Researchers have created multiple iterative methods for solving fixed point problems for various operators (nonlinear) over the years, but the search for quicker and more efficient iterative algorithms continues. The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{1.5}$$

is one of the earliest iterative process used to approximate the solution of Equation (1.4), where T is a contraction mapping. It is well-known that the Picard iterative method fails to approach the solution of Equation (1.4) when T is a nonexpansive mapping and the initial point picked for the iteration is not the fixed point of T. However, Browder [6] shown that a fixed point exists for the class of nonexpansive self mappings on a closed and bounded subset of a uniformly convex Banach space. Following that, researchers in this field devised many iterative procedures to approximate the fixed points of nonexpansive mappings and a variety of other nonlinear mappings. Developing faster and more effective iterative techniques for approximating fixed points of nonlinear mappings is still an open problem in this area of research. The following are some well-known iterative methods for approximating fixed points of nonlinear mappings that have been published in the literature. Among many others, are; Mann [18], Ishikawa [16], Krasnosel'skii [17] and so on. For detail work on iterative processes, the reader should (see [1, 2, 3, 4, 8] and the references therein). The following iterative methods are referred to as Noor [19], S-iterative method [4], Picard-S [15] and Thakur-New iterative method [21], respectively:

$$\begin{cases} c_0 \in C, \\ a_n = (1 - \alpha_n)c_n + \alpha_n T c_n, \\ b_n = (1 - \beta_n)c_n + \beta_n T a_n \\ c_{n+1} = (1 - \gamma_n)c_n + \beta_n T b_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and C be a nonempty, closed and convex subset of a convex subset of a normed space X. It is easy to see that if $\beta_n = \alpha_n = 0$ for all $n \in \mathbb{N}$, we obtain the well-known Mann iterative method [18]. In addition, if $\alpha_n = 0$, we obtain the Ishikawa iterative method [16].

$$\begin{cases} p_0 \in C, \\ s_n = (1-\alpha_n)p_n + \alpha_n T p_n, \\ p_{n+1} = (1-\beta_n)Tp_n + \beta_n T s_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and C be a nonempty, closed and convex subset of a convex subset of a normed space X and T a nonlinear mapping.

$$\begin{cases} u_0 \in C, \\ w_n = (1-\alpha_n)u_n + \alpha_n T u_n, \\ v_n = (1-\beta_n)Tu_n + \beta_n T w_n \\ u_{n+1} = Tv_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, C be a nonempty, closed and convex subset of a convex subset of a normed space X and T a nonlinear mapping.

$$\begin{cases} v_0 \in C, \\ u_n = (1-\alpha_n)v_n + \alpha_n T v_n, \\ y_n = T((1-\beta_n)v_n + \beta_n T u_n), \\ v_{n+1} = Ty_n, \quad n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and C be a nonempty, closed and convex subset of a convex subset of a normed space X and T a nonlinear mapping.

Remark 1.2. We note that the iterative method (1.9) can be expressed in the form:

$$\begin{cases} v_0 \in C, \\ u_n = (1-\alpha_n)v_n + \alpha_n T v_n, \\ w_n = (1-\beta_n)v_n + \beta_n T u_n \\ y_n = Tw_n, \\ v_{n+1} = Ty_n, \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and C be a nonempty, closed and convex subset of a convex subset of a normed space X and T a nonlinear mapping.

Remark 1.3. We claim that iterative method (1.8) and (1.9) have the same rate of convergence. This claim will be justified in Theorem 3.3

Question 1. Is it possible to define a new iterative method whose rate of convergence is better than the above listed iterative methods for a Reich-Suzuki nonexpansive mapping?

Question 2. Is it possible to modify the above iterative methods and then use to approximate the common element of the set of fixed points of a Reich-Suzuki nonexpansive mappings and the set of solutions of the variational inequalities problems in the frame work of Hilbert spaces?

Motivated by Remark 1.3, Question 1, Question 2, the research works described above and the recent research interests in this direction, we provide an affirmative answer to the above questions raised in this work by introducing an iterative method for finding the common element of the set of fixed points of a Reich-Suzuki nonexpansive mappings and the set of solution of the variational inequalities problems in the framework of Hilbert spaces. In addition, we establish convergence results for this proposed iterative method under some mild conditions. Furthermore, we establish analytically and numerically that our newly proposed iterative method converges to a fixed point of Reich-Suzuki nonexpansive mappings faster compared to some well-known iterative methods in the literature. Finally, we apply our proposed iterative method to approximate the solution of a convex minimization problem. The results obtained in this paper improve, extend and unify some related results in the literature.

The rest of this paper is organized as follows: In Section 2, we shall recall some useful definitions and Lemmas. In Section 3, we present our proposed method, strong convergence analysis of our method is investigated and the rate of convergence of our iterative method in comparison with other existing methods are investigated. In Section 4, we present an application and some numerical experiments to show the efficiency and implementation of our method (in comparison with other methods in the literature) are also discussed in the framework of infinite dimensional Hilbert spaces. Lastly, in Section 5 we give a conclusion of the paper.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H.

Definition 2.1. Let $A : H \to H$ be an operator. Then the operator A is called

1. L-Lipschitz continuous if

$$\|Ax - Ay\| \leq L\|x - y\|,$$

where $L > 0$ and $x, y \in H$. If $L = 1$, Then, the operator A is called nonexpansive. Also, if $y \in F(A)$ and $L = 1$, Then A is called quasi-nonexpansive.

2. monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0 \;\; \forall x, y \in H.$$

3. k-inverse strongly monotone (k-ism) if there exists $k > 0$, such that

$$\langle Ax - Ay, x - y\rangle \geq k\|Ax - Ay\|^2 \;\; \forall\, x, y \in H.$$

4. v-strongly monotone (v-sm) if there exists $v > 0$, such that

$$\langle Ax - Ay, x - y\rangle \geq v\|x - y\|^2 \;\; \forall\, x, y \in H.$$

5. relaxed (α, k)-cocoercive if there exist $\alpha, k > 0$, such that

$$\langle Ax - Ay, x - y\rangle \geq -\alpha\|Ax - Ay\|^2 + k\|x - y\|^2 \;\; \forall\, x, y \in H.$$

6. condition (C) mapping if there exist an $\alpha \in (0, 1)$ and for all $x, y \in H$,

$$\frac{1}{2}\|Ax - x\| \leq \|x - y\| \Rightarrow \|Ax - Ay\| \leq \|x - y\|.$$

7. Reich-Suzuki nonexpansive mapping if there exists an $\alpha \in (0, 1)$ and for all $x, y \in H$,

$$\begin{aligned}&\frac{1}{2}\|Ax - x\| \leq \|x - y\| \\ \Rightarrow &\|Ax - Ay\| \leq \alpha\|Ax - x\| + \alpha\|Ay - y\| + (1 - 2\alpha)\|x - y\|.\end{aligned}$$

Remark 2.2. It is easy to see that if $\alpha = 0$, Reich-Suzuki nonexpansive mapping becomes a mapping satisfying condition (C).

Lemma 2.3. Let $A : C \to C$ be a Reich-Suzuki nonexpansive mapping with a fixed point, then A is quasi-nonexpansive.

Proof . Let $x \in F(A), \alpha \in (0, 1)$ and $y \in C$,

$$\frac{1}{2}\|Ax - x\| = \frac{1}{2}\|x - x\| = 0 \leq \|x - y\|.$$

So, we have

$$\begin{aligned}\|x - Ay\| = \|Ax - Ay\| &\le \alpha\|Ax - x\| + \alpha\|Ay - y\| + (1-2\alpha)\|x - y\| \\ &= \alpha\|Ay - y\| + (1-2\alpha)\|x - y\| \\ &\le \alpha\|Ay - x\| + \alpha\|x - y\| + (1-2\alpha)\|x - y\| \\ &= \alpha\|x - Ay\| + (1-\alpha)\|x - y\| \\ \Rightarrow (1-\alpha)\|x - Ay\| &\le (1-\alpha)\|x - y\| \\ \Rightarrow \|x - Ay\| &\le \|x - y\|.\end{aligned}$$

Hence, A is quasi-nonexpanisve. □

Remark 2.4. Let T be Reich Suzuki-nonexpansive nonexpansive mapping, if $x^* \in F(T) \cap \Omega(C, A)$, we have the following assertion. Since $x^* \in F(T) \cap \Omega(C, A)$, we have that $x^* \in F(T)$ and $x^* \in \Omega(C, A)$, which implies that

$$x^* \in F(T) \Rightarrow x^* = Tx^*, \tag{2.1}$$

also

$$x^* \in \Omega(C, A) \Rightarrow x^* = P_C(x^* - \eta Ax^*). \tag{2.2}$$

It follows from (2.1) and (2.2), we have

$$x^* = Tx^* = TP_C(x^* - \eta Ax^*). \tag{2.3}$$

P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping of H onto C and that P_C satisfies

$$\langle x - y, P_Cx - P_Cy\rangle \ge \|P_Cx - P_Cy\|^2,$$

for all $x, y \in H$. Furthermore, P_Cx is characterized by the properties $P_Cx \in C$,

$$\langle x - P_Cx, P_Cx - y\rangle \ge 0$$

for all $y \in C$ and

$$\|x - y\|^2 \ge \|x - P_Cx\|^2 + \|y - P_Cx\|^2$$

for all $x \in H$ and $y \in C$.

Lemma 2.5. [5] Supose that $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers converging to the same fixed point x_0, with the following error estimate:

$$\begin{aligned}\|x_n - x_0\| &\le \tau_n \\ \|y_n - x_0\| &\le \eta_n,\end{aligned}$$

for all $n \in \mathbb{N}$, where τ_n and η_n are two sequences of positive numbers converging to zero (0). Then, $\{x_n\}$ converges faster than $\{y_n\}$ to x_0 if

$$\lim_{n\to\infty} \frac{\tau_n}{\eta_n} = 0.$$

If $\lim_{n\to\infty} \frac{\tau_n}{\eta_n} = k$, where $k \in (0, \infty)$, then $\{x_n\}$ and $\{y_n\}$ are said to have the same rate of convergence.

3 Main Results

In this section, we introduce some iterative algorithms for finding the common element of the set of fixed point of a Reich-Suzuki nonexpansive mappings and the set of solution of the variational inequalities. In addition, we establish convergence results for these proposed iterative algorithms under some mild conditions. In view of Remark 2.4, we obtain the following equivalent iterative methods for (1.6), (1.7), (1.8) and (1.10).

$$\begin{cases} c_0 \in C, \\ a_n = (1-\alpha_n)c_n + \alpha_n TP_C(I-\eta A)c_n, \\ b_n = (1-\beta_n)c_n + \beta_n TP_C(I-\eta A)a_n \\ c_{n+1} = (1-\gamma_n)c_n + \beta_n TP_C(I-\eta A)b_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

$$\begin{cases} p_0 \in C, \\ s_n = (1-\alpha_n)p_n + \alpha_n TP_C(I-\eta A)p_n, \\ p_{n+1} = (1-\beta_n)TP_C(I-\eta A)p_n + \beta_n TP_C(I-\eta A)s_n, \quad n \geq 1, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

$$\begin{cases} u_0 \in C, \\ w_n = (1-\alpha_n)u_n + \alpha_n TP_C(I-\eta A)u_n, \\ v_n = (1-\beta_n)TP_C(I-\eta A)u_n + \beta_n TP_C(I-\eta A)w_n \\ u_{n+1} = TP_C(I-\eta A)v_n, \quad n \geq 1, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

$$\begin{cases} v_0 \in C, \\ u_n = (1-\alpha_n)v_n + \alpha_n TP_C(I-\eta A)v_n, \\ w_n = (1-\beta_n)v_n + \beta_n TP_C(I-\eta A)u_n \\ y_n = TP_C(I-\eta A)w_n, \\ v_{n+1} = TP_C(I-\eta A)y_n, \quad n \geq 1, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

In the light of providing an affirmative answer to the above questions, we introduce the following iterative method.

$$\begin{cases} x_0 \in C, \\ u_n = (1-\alpha_n)x_n + \alpha_n TP_C(I-\eta A)x_n, \\ v_n = (1-\beta_n)u_n + \beta_n TP_C(I-\eta A)u_n \\ w_n = TP_C(I-\eta A)v_n, \\ y_n = TP_C(I-\eta A)w_n, \\ x_{n+1} = TP_C(I-\eta A)y_n, \quad n \geq 1, \end{cases} \tag{3.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H and A be a relaxed (α,k)-cocoercive and L-Lipschitzian mapping of C onto H and T Reich Suzuki-nonexpansive nonexpansive mapping on C such that $F(T) \cap \Omega(C,A) \neq \emptyset$ and

$$0 < \eta < \left(\frac{2(k-\alpha L^2)}{L^2}, \alpha L^2\right) < k,$$

holds. Then, the iterative sequences $\{x_n\}$ defined by (3.5), with sequences $\{\alpha_n\}$, and $\{\beta_n\}$ in $[0,1]$ converges strongly to $x^* \in F(T) \cap \Omega(C,A)$.

Proof . Let $x^* \in F(T) \cap \Omega(C, A)$. Using (3.5) and Lemma 2.3, we have

$$\begin{aligned}
\|u_n - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T P_C(I-\eta A)x_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|TP_C(I-\eta A)x_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|P_C(I-\eta A)x_n - x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|P_C(I-\eta A)x_n - P_C(I-\eta A)x^*\| \\
&\leq 1-\alpha_n)\|x_n - x^*\| + \alpha_n\|(I-\eta A)x_n - (I-\eta A)x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|(x_n - x^*) - \eta(A_n - Ax^*)\|. \qquad (3.6)
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\|(x_n - x^*) - \eta(Ax_n - Ax^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle Ax_n - Ax^*, x_n - x^*\rangle + \eta^2\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\eta\alpha\|Ax_n - Ax^*\|^2 - 2\eta k\|x_n - x^*\| + \eta^2\|Ax_n - Ax^*\|^2 \\
&\leq (1 - 2\eta k + 2\eta\alpha L^2 + \eta^2 L^2)\|x_n - x^*\|^2, \qquad (3.7)
\end{aligned}$$

which implies that

$$\|(x_n - x^*) - \eta(A_n - Ax^*)\| \leq \sqrt{(1 - 2\eta k + 2\eta\alpha L^2 + \eta^2 L^2)}\|x_n - x^*\|, \qquad (3.8)$$

where

$$\sqrt{(1 - 2\eta k + 2\eta\alpha L^2 + \eta^2 L^2)} = \delta \in (0, 1). \qquad (3.9)$$

Thus by (3.6), we have

$$\begin{aligned}
\|u_n - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\delta\|x_n - x^*\| \\
&= (1 - \alpha_n(1-\delta))\|x_n - x^*\|. \qquad (3.10)
\end{aligned}$$

Also, using Algorithm 3.5, Lemma 2.3, (3.6) and similar approach as in (3.8), we have

$$\begin{aligned}
\|v_n - x^*\| &= \|(1-\beta_n)u_n + \beta_n TP_C(I-\eta A)u_n - x^*\| \\
&\leq (1-\beta_n)\|u_n - x^*\| + \beta_n\|TP_C(I-\eta A)u_n - x^*\| \\
&\leq (1-\beta_n)\|u_n - x^*\| + \beta_n\delta\|u_n - x^*\| \\
&= (1 - \beta_n(1-\delta))\|u_n - x^*\| \\
&\leq (1 - \beta_n(1-\delta))(1 - \alpha_n(1-\delta))\|x_n - x^*\|. \qquad (3.11)
\end{aligned}$$

Similarly, using Algorithm 3.5, Lemma 2.3, (3.11) and similar approach as in (3.8), we have

$$\begin{aligned}
\|w_n - x^*\| &= \|TP_C(I-\eta A)v_n - x^*\| \\
&\leq \delta\|v_n - x^*\| \\
&\leq \delta(1 - \beta_n(1-\delta))(1 - \alpha_n(1-\delta))\|x_n - x^*\|. \qquad (3.12)
\end{aligned}$$

In addition, using Algorithm 3.5, Lemma 2.3, (3.12) and similar approach as in (3.8), we have

$$\begin{aligned}
\|y_n - x^*\| &= \|TP_C(I-\eta A)w_n - x^*\| \\
&\leq \delta\|w_n - x^*\| \\
&\leq \delta^2(1 - \beta_n(1-\delta))(1 - \alpha_n(1-\delta))\|x_n - x^*\|. \qquad (3.13)
\end{aligned}$$

Finally, using Algorithm 3.5, Lemma 2.3, (3.13) and similar approach as in (3.8), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|TP_C(I-\eta A)y_n - x^*\| \\
&\leq \delta\|y_n - x^*\| \\
&\leq \delta^3(1 - \beta_n(1-\delta))(1 - \alpha_n(1-\delta))\|x_n - x^*\|, \qquad (3.14)
\end{aligned}$$

which implies that

$$\|x_n - x^*\| \leq \delta^{3(n+1)}\|x_0 - x^*\| \prod_{k=0}^{m}(1 - \beta_k(1-\delta))(1 - \alpha_k(1-\delta)). \qquad (3.15)$$

Since $(1-\beta_k(1-\delta)) \in (0,1), (1-\alpha_k(1-\delta)) \in (0,1)$, we have $(1-\beta_k(1-\delta))(1-\alpha_k(1-\delta)) \in (0,1)$ and $\delta \in (0,1)$. Passing the limit in (3.15), we obtain

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{3.16}$$

□

Theorem 3.2. Let C, H, T, A and δ be as defined in Theorem 3.1. Suppose that $F(T) \cap \Omega(C, A) \neq \emptyset$ and

$$0 < \eta < \left(\frac{2(k-\alpha L^2)}{L^2}, \alpha L^2\right) < k,$$

holds. Then, the iterative sequences $\{v_n\}, \{u_n\}, \{p_n\}$ and $\{c_n\}$ defined by (3.4), (3.3), (3.2) and (3.1) respectively with sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$ converges strongly to $x^* \in F(T) \cap \Omega(C, A)$.

Proof . Using similar approach as in Theorem 3.1, we obtain

$$\|v_n - x^*\| \leq \delta^{2(n+1)} \|v_0 - x^*\| \prod_{k=0}^{m} (1 - \alpha_k \beta_k (1-\delta)), \tag{3.17}$$

$$\|u_n - x^*\| \leq \delta^{2(n+1)} \|u_0 - x^*\| \prod_{k=0}^{m} (1 - \alpha_k \beta_k (1-\delta)), \tag{3.18}$$

$$\|p_n - x^*\| \leq \delta^{(n+1)} \|p_0 - x^*\| \prod_{k=0}^{m} (1 - \alpha_k \beta_k (1-\delta)), \tag{3.19}$$

$$\|c_n - x^*\| \leq \|c_0 - x^*\| \prod_{k=0}^{m} \left[1 - \alpha_k\left(1 - \delta\left\{1 - \beta_k(1 - \delta[1 - \gamma_k(1-\delta)])\right\}\right)\right]. \tag{3.20}$$

As in Theorem 3.1, we obtain

$$\lim_{n\to\infty} \|v_n - x^*\| = 0. \tag{3.21}$$

$$\lim_{n\to\infty} \|u_n - x^*\| = 0. \tag{3.22}$$

$$\lim_{n\to\infty} \|p_n - x^*\| = 0. \tag{3.23}$$

$$\lim_{n\to\infty} \|c_n - x^*\| = 0. \tag{3.24}$$

□

Theorem 3.3. Let C, H, T, A and δ be as defined in Theorem 3.1 and $\{x_n\}, \{v_n\}, \{u_n\}, \{p_n\}$ and $\{c_n\}$ be iterative methods defined by (3.5), (3.4), (3.3), (3.2) and (3.1) respectively with sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $[0,1]$ such that

1. $0 < \alpha < \alpha_n < 1$,
2. $0 < \beta < \beta_n < 1$ and
3. $0 < \gamma < \gamma_n < 1$.

Suppose that $F(T)\cap\Omega(C,A)\neq\emptyset$ and

$$0<\eta<\left(\frac{2(k-\alpha L^2)}{L^2},\alpha L^2\right)<k \tag{3.25}$$

holds. Then, $\{x_n\}$ converges faster than all of $\{v_n\},\{u_n\},\{p_n\}$ and $\{c_n\}$ to $x^*\in F(T)\cap\Omega(C,A)$ provided that $x_0=c_0=p_0=u_0=v_0$. Furthermore, the iterative method (3.3) and (3.4) have the same rate of convergence.

Proof . From (3.15) in Theorem 3.1, and using the assumption, we obtain

$$\begin{aligned}\|x_n-x^*\| &\le \delta^{3(n+1)}\|x_0-x^*\|(1-\beta(1-\delta))^{n+1}(1-\alpha(1-\delta))^{n+1}\\ &=\Big(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}\|x_0-x^*\|.\end{aligned} \tag{3.26}$$

Similarly, we obtain

$$\begin{aligned}\|v_n-x^*\| &\le \delta^{2(n+1)}\|v_0-x^*\|(1-\alpha\beta(1-\delta))^{n+1}\\ &=\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|v_0-x^*\|,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\|u_n-x^*\| &\le \delta^{2(n+1)}\|u_0-x^*\|(1-\alpha\beta(1-\delta))^{n+1}\\ &=\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|u_0-x^*\|,\end{aligned} \tag{3.28}$$

$$\begin{aligned}\|p_n-x^*\| &\le \delta^{(n+1)}\|p_0-x^*\|(1-\alpha\beta(1-\delta))^{n+1}\\ &=\Big(\delta(1-\alpha\beta(1-\delta))\Big)^{n+1}\|p_0-x^*\|,\end{aligned} \tag{3.29}$$

$$\|c_n-x^*\|\le\|c_0-x^*\|\left[1-\alpha\left(1-\delta\left\{1-\beta(1-\delta[1-\gamma(1-\delta)])\right\}\right)\right]^{n+1}. \tag{3.30}$$

It has been established that the sequences $\{x_n\},\{c_n\},\{p_n\},\{v_n\}$ and $\{u_n\}$ converges strongly to zero. Now, before applying Lemma 2.5, we claim that

$$\frac{(1-\beta(1-\delta))(1-\alpha(1-\delta))}{(1-\alpha\beta(1-\delta))}<1 \tag{3.31}$$

$$\frac{(1-\beta(1-\delta))(1-\alpha(1-\delta))}{1-\alpha\left(1-\delta\left\{1-\beta(1-\delta[1-\gamma(1-\delta)])\right\}\right)}<1. \tag{3.32}$$

To see this, since α,β and γ in $(0,1)$ and the fact that $\delta\in(0,1)$, we obtain

$$\begin{aligned}&\alpha\beta+\alpha\beta-\alpha\beta\delta<\alpha+\beta\\ \Rightarrow&-\alpha-\beta+\alpha\beta(1-\delta)<-\alpha\beta\\ \Rightarrow&-\alpha(1-\delta)-\beta(1-\delta)+\alpha\beta(1-\delta)^2<-\alpha\beta(1-\delta)\qquad\text{since }(1-\delta)>0\\ \Rightarrow&1-\alpha(1-\delta)-\beta(1-\delta)+\alpha\beta(1-\delta)^2<1-\alpha\beta(1-\delta)\\ \Rightarrow&(1-\beta(1-\delta))(1-\alpha(1-\delta))<(1-\alpha\beta(1-\delta))\\ \Rightarrow&\frac{(1-\beta(1-\delta))(1-\alpha(1-\delta))}{(1-\alpha\beta(1-\delta))}<1.\end{aligned} \tag{3.33}$$

In addition, we have

$$
\begin{aligned}
&\alpha + \alpha\gamma\delta^3 < 1\\
\Rightarrow & -\alpha\beta - \alpha\beta\gamma\delta^3 > -\beta\\
\Rightarrow & -\beta + \alpha\beta < -\alpha\beta\gamma\delta^3\\
\Rightarrow & -\beta(1-\delta) + \alpha\beta(1-\delta) < \alpha\beta\gamma\delta^3 - \alpha\beta\gamma\delta^2\\
\Rightarrow & 1 - \alpha + \alpha\delta - \alpha\beta\delta + \alpha\beta\delta^2 - \beta(1-\delta) + \alpha\beta(1-\delta) < 1 - \alpha + \alpha\delta - \alpha\beta\delta + \alpha\beta\delta^2\alpha\beta\gamma\delta^3 - \alpha\beta\gamma\delta^2\\
\Rightarrow & (1-\beta(1-\delta))(1-\alpha(1-\delta)) < 1 - \alpha\left(1 - \delta\left\{1 - \beta(1 - \delta[1 - \gamma(1-\delta)])\right\}\right)\\
\Rightarrow & \frac{(1-\beta(1-\delta))(1-\alpha(1-\delta))}{1 - \alpha\left(1 - \delta\left\{1 - \beta(1 - \delta[1 - \gamma(1-\delta)])\right\}\right)} < 1.
\end{aligned}
\tag{3.34}
$$

Now, let

$$\Psi_n = \left(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\right)^{n+1} \|x_0 - x^*\|.$$

$$\Phi_n = \left(\delta^2(1-\alpha\beta(1-\delta))\right)^{n+1} \|v_0 - x^*\|.$$

$$\Omega_n = \left(\delta^2(1-\alpha\beta(1-\delta))\right)^{n+1} \|u_0 - x^*\|.$$

$$\Gamma_n = \left(\delta(1-\alpha\beta(1-\delta))\right)^{n+1} \|p_0 - x^*\|.$$

$$\phi_n = \|c_0 - x^*\| \left[1 - \alpha\left(1 - \delta\left\{1 - \beta(1 - \delta[1 - \gamma(1-\delta)])\right\}\right)\right]^{n+1}.$$

It is easy to see that $\lim_{n\to\infty} \Psi_n = 0, \lim_{n\to\infty} \Phi_n = 0, \lim_{n\to\infty} \Omega_n = 0, \lim_{n\to\infty} \Gamma_n = 0$ and $\lim_{n\to\infty} \phi_n = 0$. Thus, using our hypothesis $x_0 = u_0 = v_0 = p_0 = c_0$ (3.33) and (3.34) we have that

$$
\begin{aligned}
\mu_n = \frac{\Psi_n}{\Phi_n} &= \frac{\left(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\right)^{n+1} \|x_0 - x^*\|}{\left(\delta^2(1-\alpha\beta(1-\delta))\right)^{n+1} \|v_0 - x^*\|}\\
&= \frac{\left(\delta(1-\beta(1-\delta))(1-\alpha(1-\delta))\right)^{n+1} \|x_0 - x^*\|}{\left((1-\alpha\beta(1-\delta))\right)^{n+1} \|x_0 - x^*\|}\\
&= \left(\frac{\delta(1-\beta(1-\delta))(1-\alpha(1-\delta))}{(1-\alpha\beta(1-\delta))}\right)^{n+1} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

$$\nu_n = \frac{\Psi_n}{\Omega_n} = \frac{\Big(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}\|x_0-x^*\|}{\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|u_0-x^*\|}$$

$$= \frac{\Big(\delta(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}\|x_0-x^*\|}{\Big((1-\alpha\beta(1-\delta))\Big)^{n+1}\|x_0-x^*\|}$$

$$= \left(\frac{\delta(1-\beta(1-\delta))(1-\alpha(1-\delta))}{(1-\alpha\beta(1-\delta))}\right)^{n+1} \to 0 \text{ as } n\to\infty.$$

$$\eta_n = \frac{\Psi_n}{\Gamma_n} = \frac{\Big(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}\|x_0-x^*\|}{\Big(\delta(1-\alpha\beta(1-\delta))\Big)^{n+1}\|p_0-x^*\|}$$

$$= \frac{\Big(\delta^2(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}\|x_0-x^*\|}{\Big((1-\alpha\beta(1-\delta))\Big)^{n+1}\|x_0-x^*\|}$$

$$= \left(\frac{\delta^2(1-\beta(1-\delta))(1-\alpha(1-\delta))}{(1-\alpha\beta(1-\delta))}\right)^{n+1} \to 0 \text{ as } n\to\infty.$$

$$\theta_n = \frac{\Psi_n}{\phi_n} = \frac{\|x_0-x^*\|\Big(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}}{\|c_0-x^*\|\left[1-\alpha\Big(1-\delta\Big\{1-\beta(1-\delta[1-\gamma(1-\delta)])\Big\}\Big)\right]^{n+1}}$$

$$= \frac{\|x_0-x^*\|\Big(\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))\Big)^{n+1}}{\|x_0-x^*\|\left[1-\alpha\Big(1-\delta\Big\{1-\beta(1-\delta[1-\gamma(1-\delta)])\Big\}\Big)\right]^{n+1}}$$

$$= \left(\frac{\delta^3(1-\beta(1-\delta))(1-\alpha(1-\delta))}{1-\alpha\Big(1-\delta\Big\{1-\beta(1-\delta[1-\gamma(1-\delta)])\Big\}\Big)}\right)^{n+1} \to 0 \text{ as } n\to\infty.$$

It follows from Lemma 2.5, $\{x_n\}$ converges faster than $\{c_n\},\{p_n\},\{v_n\}$ and $\{u_n\}$ to $x^*\in F(T)\cap\Omega(C,A)$. In addition, we have

$$\mu_n = \frac{\Phi_n}{\Omega_n} = \frac{\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|v_0-x^*\|}{\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|u_0-x^*\|} \tag{3.35}$$

$$= \frac{\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|u_0-x^*\|}{\Big(\delta^2(1-\alpha\beta(1-\delta))\Big)^{n+1}\|u_0-x^*\|} \to 1 \text{ as } n\to\infty. \tag{3.36}$$

By Lemma 2.5, it is easy to see that $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence. □ We now provide some numerical example to justify our analytical proof.

Example 3.4. Let $\mathbb{H}=\mathbb{R}$ and $C=[0,1]$. Define a mapping $T:[0,1]\to[0,1]$ and $A:[0,1]\subset\mathbb{H}\to\mathbb{H}$ as

$$Tx=\begin{cases}1-x & \text{if } x\in[0,\frac{1}{5}),\\ \frac{x+4}{5} & \text{if } x\in[\frac{1}{5},1]\end{cases} \tag{3.37}$$

and

$$A(x)=3x.$$

It is easy to see that T satisfy condition (C), (see Remark 2.2 and [22, 23]) thus it is a Reich-Suzuki nonexpansive mapping. Now, observe that

$$|Ax-Ay|=|3x-3y|=3|x-y|,$$

clearly, A is 3-Lipschitzian mapping. Furthermore, we have

$$\begin{aligned}\langle Ax-Ay,x-y\rangle &= \langle 3x-3y,x-y\rangle\\ &=(4-1)\langle x-y,x-y\rangle\\ &=-1|x-y|^2+4|x-y|^2\\ &=-\frac{1}{9}|3x-3y|^2+4|x-y|^2.\end{aligned}$$

It is clear that A is a relaxed $(\frac{1}{9},4)$-cocoercive. In addition, we have $L=3,\alpha=\frac{1}{9}$ and $k=4$, thus condition (3.25) takes the form

$$0<\eta<\left(\frac{2}{3},1\right)<4$$

and the metric projection

$$P_C(x)=\begin{cases}0, & \text{if } x\in(-\infty,0),\\ x, & \text{if } x\in[0,1],\\ 1 & \text{if } x\in(1,\infty).\end{cases} \tag{3.38}$$

Thus, we obtain

$$TP_C(x)=\begin{cases}0, & \text{if } x\in(-\infty,0),\\ T(x), & \text{if } x\in[0,1],\\ T(1) & \text{if } x\in(1,\infty).\end{cases} \tag{3.39}$$

With respect to Algorithm 3.5, Algorithm 3.4, Algorithm 3.3, Algorithm 3.2 and Algorithm 3.1, we randomly choose $x_0\in[0,1]$. We choose $\eta=0.2,\beta_n=\frac{7n}{76n+60},\alpha_n=\frac{7n}{800n+26}$ and $\gamma_n=\frac{76n+87}{190n+78}$ We consider the following cases for our numerical experiment.

Case 1: Take $x_0=u_0=v_0=c_0=p_0=0.6$.
Case 2: Take $x_0=u_0=v_0=c_0=p_0=0.4$.
Case 3: Take $x_0=u_0=v_0=c_0=p_0=0.35$.

The report of this experiment is presented in Figure 1.

Example 3.5. Let $H=\ell_2(\mathbb{R})$, where

$$\ell_2(\mathbb{R}):=\{x=(x_1,x_2,...,x_i...),\quad x_i\in\mathbb{R}\quad\text{and}\quad\sum_{i=1}^{\infty}|x_i|^2<\infty\},$$

with inner product $\langle .,.\rangle:\ell_2\times\ell_2\to\mathbb{R}$ defined by $\langle x,y\rangle:=\sum_{i=1}^{\infty}x_iy_i$ and the norm $\|.\|:\ell_2\to\mathbb{R}$ by $\|x\|:=\sqrt{\sum_{i=1}^{\infty}|x_i|^2}$, where $x=\{x_i\}_{i=1}^{\infty}$ and $y=\{y_i\}_{i=1}^{\infty}$. Define the mapping $A:\ell_2\to\ell_2$ by $Ax=\left(\frac{x_1+|x_1|}{3},\frac{x_2+|x_2|}{3},...,\frac{x_i+|x_i|}{3},...\right)$, $\forall\ x=\{x_i\}_{i=1}^{\infty}\in\ell_2$.

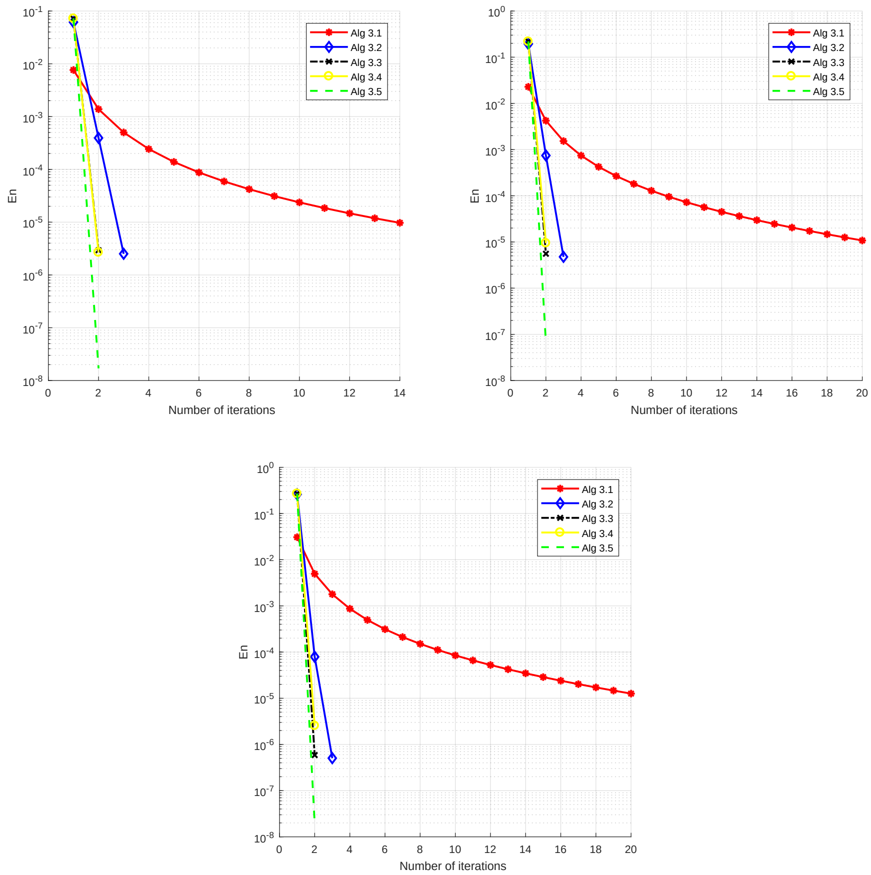

Figure 1: Example 3.4. Top left: Case 1, Top right: Case 2, Bottom Case 3.

Let $T : \ell_2 \to \ell_2$ be defined by $Tx = \left(\frac{x_1}{6}, \frac{x_2}{6}, ..., \frac{x_i}{6}, ...\right)$, for all $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$. Furthermore, let $C := \{x \in \ell_2 : \|x\| \leq 1\}$ be the unit ball. Then, we define the metric projection P_C as:

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{\ell_2}}, & \text{if } \|x\|_{\ell_2} > 1, \\ x, & \text{if } \|x\|_{\ell_2} \leq 1. \end{cases} \tag{3.40}$$

With respect to Algorithm 3.5, Algorithm 3.4, Algorithm 3.3, Algorithm 3.2 and Algorithm 3.1, we randomly choose $x_0 \in H$. We choose $\eta = 0.2$, $\beta_n = \frac{7n}{76n+60}, \alpha_n = \frac{7n}{800n+26}$ and $\gamma_n = \frac{76n+87}{190n+78}$. We consider the following cases for our numerical experiment.

Case 1: Take $x_0 = p_0 = v_0 = c_0 = u_0 = (7.2108, -5.1081, 0, ..., 0, ...)^T$.

Case 2: Take $x_0 = p_0 = v_0 = c_0 = u_0 = (4.6507, -6.5670, 0, ..., 0, ...)^T$.

Case 3: Take $x_0 = p_0 = v_0 = c_0 = u_0 = (7.5647, -11.1256, 0, ..., 0, ...)^T$.

The report of this experiment is presented in Figure 2.

It is easy to see from Figure 1 and Figure 2 that our propose iterative method converges faster than the existing ones. In addition, our claim that the iterative method (3.3) and (3.4) has been justified both analytical and with examples.

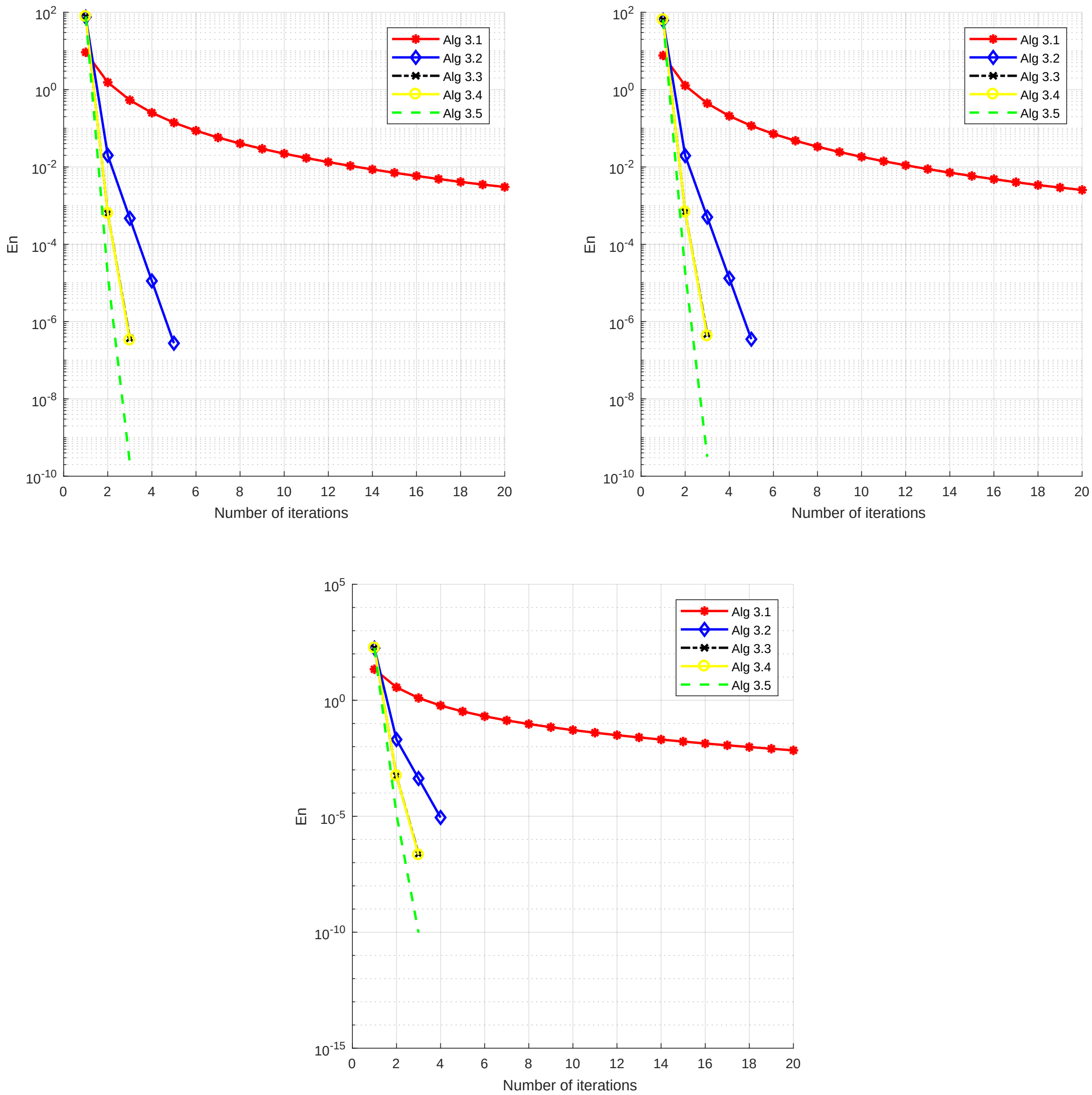

Figure 2: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom).

4 Application

In this section, the application presented was inspired by the works of the authors in [9, 10, 14]. In addition, we give a numerical experiment to compare Algorithm 3.5, Algorithm 3.4, Algorithm 3.3, Algorithm 3.2 and Algorithm 3.1.

4.1 Application

Let $f : C \to \mathbb{R}$ be a convex mapping where C is a closed and convex subset of a Hilbert space H. Considering the convex minimization problem

$$\min_{x\in C} f(x). \tag{4.1}$$

Let $P_C : H \to C$ be a projection map and f be the Frêchet differentiable. Denote the gradient of f by ∇f. It is well-known that x^* solves (4.1) if and only if the following variational inequality holds:

$$x^* \in C, \langle \nabla f x^*, x - x^* \rangle \geq 0 \;\; \forall x \in C, \tag{4.2}$$

that is $x^* \in \Omega(C, A)$. In addition x^* solves (4.1) if and only if $x^* = P_C(x^* - \eta\nabla f(x^*))$, where $\eta > 0$. In order to solve (4.1) the gradient projection algorithm (GPA) is usually used and its defined as

$$x_{n+1} = P_C(x_n - \eta\nabla f(x_n)),$$

where $x_0 \in C$ and η is step size. Now, suppose that $T = I$ (identity mapping) and A is taken as the gradient of a convex function f in the iterative process (3.5), then we get the following iterative process which converges to a solution of a convex minimization problem (4.1),

$$\begin{cases} x_0 \in C, \\ u_n = (1-\alpha_n)x_n + \alpha_n P_C(I - \eta\nabla f)x_n, \\ v_n = (1-\beta_n)u_n + \beta_n P_C(I - \eta\nabla f)u_n \\ w_n = P_C(I - \eta\nabla f)v_n, \\ y_n = P_C(I - \eta\nabla f)w_n, \\ x_{n+1} = P_C(I - \eta\nabla f)y_n, \quad n \geq 1 \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Theorem 4.1. Suppose that problem (4.1) has a solution. Let $f : C \to \mathbb{R}$ be a convex mapping such that its gradient *bigtriangledownf* is a relaxed (α, k)-cocorcive and L-Lipschitzian mapping of C onton H. Let $\{x_n\}$ be sequence defined by (4.3) for any $x_0 \in C$, such that condition (3.26) and δ is defined as in (3.9) hold, then $\{x_n\}$ obtained from (4.3) converges strongly to x^* to the solution of (4.1).

Proof . The prove follow similar approach as in Theorem 3.1, by taking $T = I$, which is clearly a nonexpansive mapping, $A = \nabla f$. We obtain in Theorem 3.1 that $x^* \in F(T) \cap \Omega(C, A) = \Omega(C, \nabla f) = \{x \in C : \langle \nabla f x, y - x\rangle \geq 0 \;\; \forall y \in C\}$. It follows that x^* is a solution of (4.1). □

4.2 Numerical Example

[9] Let $H = L^2([0, 1])$, H is a Hilbert space with the induced inner product

$$\|x(t)\|_2 = \sqrt{\langle x(t), x(t)\rangle} = \left(\int_0^1 x^2(t)dt\right)^{\frac{1}{2}} \;\forall\; x \in L^2([0, 1]).$$

It is well-known that the set $C = \{x \in L^2([0, 1]) : \|x(t)\|_2 \leq 1\}$ is closed and convex subset of H. We define $f : C \to H$ as $f(x) = \|x(t)\|_2^2$, f is a convex function and $x(0) = 0$ a unique minimum of f. In addition, f is Frêchet differentiable at x and its gradient $\nabla f : C \to H$ is defined as $\nabla f(x) = 2x$. Now observe that

$$
\begin{aligned}
\|\nabla f(x(t)) - \nabla f(y(t))\|_2 &= \Big(\int_0^1 (2x(t) - 2y(t))^2 dt\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^1 (2(x(t) - y(t)))^2 dt\Big)^{\frac{1}{2}} \\
&= 2\Big(\int_0^1 (x(t) - y(t))^2 dt\Big)^{\frac{1}{2}} \\
&= 2\|x(t) - y(t)\|_2,
\end{aligned}
$$

clearly $\nabla f(x)$ is 2-Lipschitzian mapping. In addition, we have that

$$
\begin{aligned}
\langle \nabla f(x(t)) - \nabla f(y(t)), x(t) - y(t)\rangle &= \int_0^1 (2x(t) - 2y(t)(x(t) - y(t)))dt \\
&= 2\int_0^1 (x(t) - y(t))^2 dt \\
&= (3-1)\int_0^1 (x(t) - y(t))^2 dt \\
&= -\int_0^1 (x(t) - y(t))^2 dt + 3\int_0^1 (x(t) - y(t))^2 dt \\
&= -\|x(t) - y(t)\|^2 + 3\|x(t) - y(t)\|^2 \\
&= -\frac{1}{4}\|2x(t) - 2y(t)\|^2 + 3\|x(t) - y(t)\|^2.
\end{aligned}
$$

clearly $\nabla f(x)$ is a relaxed $(\frac{1}{4}, 3)$-cocorcive. we have that $L = 2, \alpha = \frac{1}{4}$ and $k = 3$, then condition (**??**) takes the form

$$
0 < \eta < 1 < 3. \tag{4.4}
$$

Let us choose $\eta = \frac{1}{6}, \beta_n = \gamma_n = \alpha_n = \frac{1}{6n+15}$, then iterative scheme (4.3), (3.1), (3.2),(3.3) and becomes

$$
\begin{cases}
x_0 \in C, \\
u_n = (1 - \frac{1}{6n+15})x_n + \frac{1}{6n+15}P_C(\frac{2}{3}x_n), \\
v_n = (1 - \frac{1}{6n+15})u_n + \frac{1}{6n+15}P_C(\frac{2}{3}u_n) \\
w_n = P_C(\frac{2}{3}v_n), \\
y_n = P_C(\frac{2}{3}w_n), \\
x_{n+1} = P_C(\frac{2}{3}y_n), \quad n \geq 1
\end{cases} \tag{4.5}
$$

$$
\begin{cases}
c_0 \in C, \\
a_n = (1 - \frac{1}{6n+15})c_n + \frac{1}{6n+15}P_C(\frac{2}{3}c_n), \\
b_n = (1 - \frac{1}{6n+15})c_n + \frac{1}{6n+15}P_C(\frac{2}{3}a_n) \\
c_{n+1} = (1 - \frac{1}{6n+15})c_n + \frac{1}{6n+15}P_C(\frac{2}{3}b_n), \quad n \geq 1,
\end{cases} \tag{4.6}
$$

$$
\begin{cases}
p_0 \in C, \\
s_n = (1 - \frac{1}{6n+15})p_n + \frac{1}{6n+15}P_C(\frac{2}{3}p_n), \\
p_{n+1} = (1 - \frac{1}{6n+15})P_C(\frac{2}{3}p_n) + \frac{1}{6n+15}P_C(\frac{2}{3}s_n), \quad n \geq 1,
\end{cases} \tag{4.7}
$$

$$
\begin{cases}
u_0 \in C, \\
w_n = (1 - \frac{1}{6n+15})u_n + \frac{1}{6n+15}P_C(\frac{2}{3}u_n), \\
v_n = (1 - \frac{1}{6n+15})P_C(\frac{2}{3}u_n) + \frac{1}{6n+15}P_C(\frac{2}{3}w_n) \\
u_{n+1} = P_C(\frac{2}{3}v_n), \quad n \geq 1,
\end{cases} \tag{4.8}
$$

$$\begin{cases} v_0 \in C, \\ u_n = (1 - \frac{1}{6n+15})v_n + \frac{1}{6n+15}P_C(\frac{2}{3}v_n), \\ w_n = (1 - \frac{1}{6n+15})v_n + \frac{1}{6n+15}P_C(\frac{2}{3}u_n) \\ y_n = P_C(\frac{2}{3}w_n), \\ v_{n+1} = P_C(\frac{2}{3}y_n), \quad n \geq 1. \end{cases} \tag{4.9}$$

where

$$P_C = \begin{cases} x(t) & \text{if } \ x(t) \in C, \\ \frac{x(t)}{\|x(t)\|} & \text{if } \ x(t) \notin C. \end{cases} \tag{4.10}$$

We plot the graph of error against number of iterations with tolerance level($||x_{n+1} - x_n|| = 10 \times e^{-5}$) and varying values of $x_0 = c_0 = v_0 = u_0 = p_0$. For case 1 $x_0 = 3t + t^2$, case 2, $x_0 = t^4 + 5t^3 - 100t^2 - t + 7$ and case 3, $x_0 = e^{-7t^3} + 4t^2$. The report of this experiment is presented in Figure 3.

5 Conclusion

A new iterative method for finding the common element of the set of fixed points of a Reich-Suzuki nonexpansive mappings and the set of solutions of the variational inequalities problems in the framework of Hilbert spaces was introduced. In addition, we established that our proposed iterative method converges strongly to the solution of the aforementioned problems. Finally, we considered some numerical examples of our proposed method in comparison with other existing iterative methods in the literature. In all our comparisons, the numerical and analytical results shows that our method performs better than these other methods.

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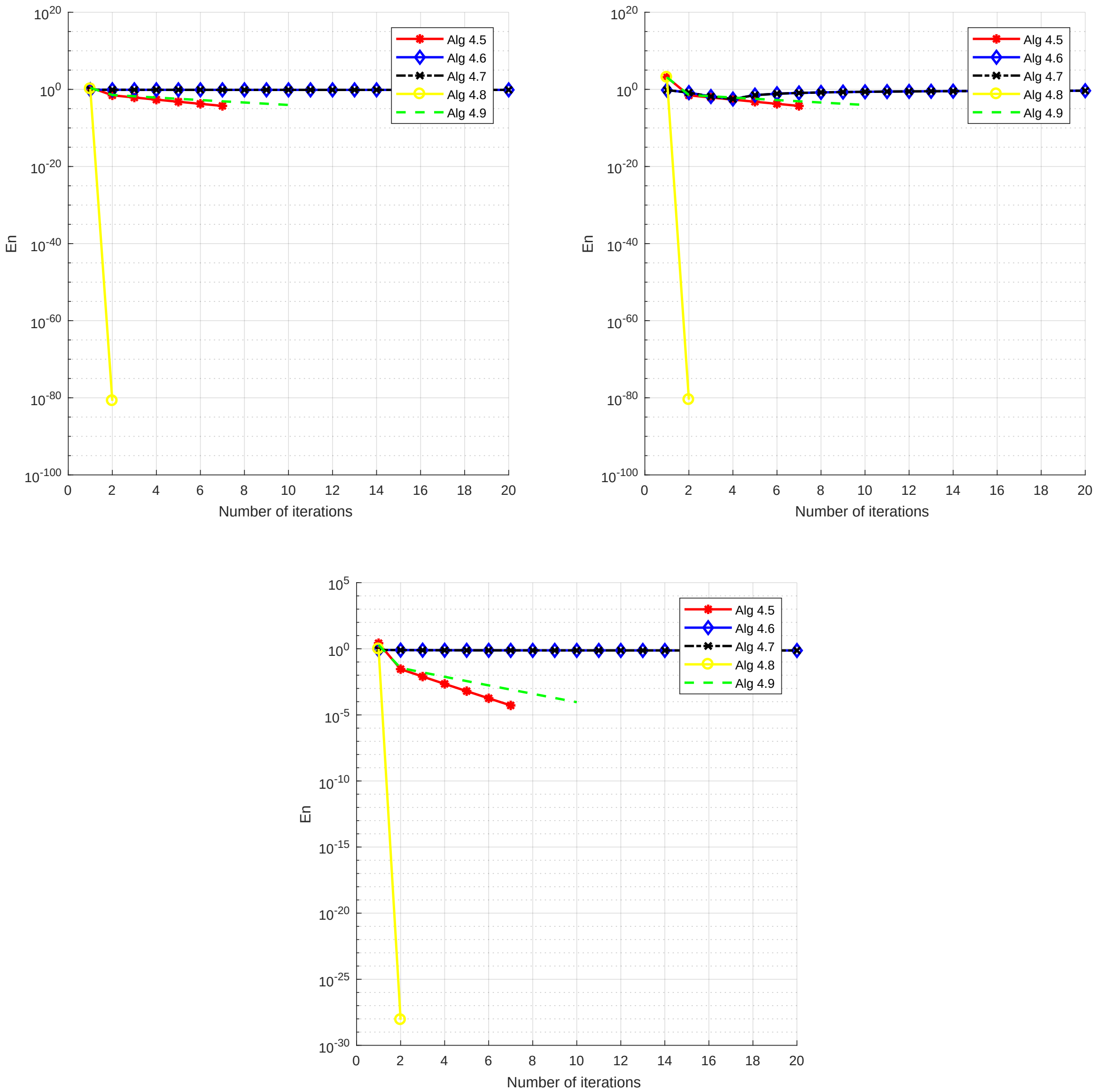

Figure 3: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom).

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