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Generalization of the Titchmarsh's theorem for the second Hankel-Clifford transformation

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Abstract

Using a generalized translation operator, we obtain an analogue of Titchmarsh's theorem for the second Hankel-Clifford transformation for functions satisfying the second Hankel-Clifford Lipschitz condition in the space $L^2_\mu((0,+\infty), x^\mu)$.

Keywords: Generalized translation operator, Second Hankel-Clifford transformation, Second Hankel-Clifford Lipschitz class
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1 Introduction and preliminaries

The theorem 85 in [15], Titchmarsh characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz class by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. ([15], Theorem 85) Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents

1. $\|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O\left(h^\alpha\right)$ as $h \longrightarrow 0$
2. $\displaystyle\int_{|\lambda|\geq s} |\hat{f}(\lambda)|^2 d\lambda = O\left(s^{-2\alpha}\right)$,

where $\hat{f}$ stands for the Fourier transform of f.

In this paper we obtain an analogue of this theorem 1.1 for the Second Hankel-Clifford transformation. There are many analogues of this result: for the Fourier transform, for the Jacobi transform, for the Fourier transform on the group of p-Adic Numbers, For the Fourier-Walsh transform, for the generalized Dunkl transform, for the generalized Bessel transform etc (see, for exemple [3, 4, 5, 6, 12, 13]).

We briefly overview the theory of second Hankel-Clifford transformation and related harmonic analysis (see [10, 11, 14]).

We define the space $L^p_\mu = L^p_\mu((0,+\infty))$, $1 \leq p < \infty$ and $\mu \geq 0$, as the space of all those real-valued measurable functions f on $(0,+\infty)$, such that

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$$\|f\|_{L^p_\mu} = \left(\int_0^\infty |f(x)|^p x^\mu dx\right)^{1/p} < \infty.$$

The Bessel-Clifford function of the first kind of order $\mu \geq 0$ (See [7]).

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)},$$

is a solution of the differential equation

$$xy'' + (\mu+1)y' + y = 0,$$

and we have

$$c_\mu(x) = x^{-\frac{\mu}{2}} J_\mu(2\sqrt{x}), \tag{1.1}$$

where J_μ the Bessel function of first kind.

For $f \in L^1_\mu$. Hayek [9] introduced the second Hankel-Clifford transformation by

$$h_{2,\mu}(f)(\lambda) = \int_0^{+\infty} c_\mu(\lambda x) f(x) x^\mu dx,$$

and its inversion formula is defined by

$$f(x) = \int_0^{+\infty} c_\mu(\lambda x) h_{2,\mu}(f)(\lambda) \lambda^\mu d\lambda.$$

The corresponding Parseval's equality now takes the form [11]

$$\int_0^{+\infty} f(x)g(x)x^\mu dx = \int_0^{+\infty} F_2(\lambda)G_2(\lambda)\lambda^\mu d\lambda,$$

where $F_2(\lambda) = h_{2,\mu}(f)(\lambda)$ and $G_2(\lambda) = h_{2,\mu}(g)(\lambda)$.
i.e,. For $f \in L^2_\mu$, we have

$$\|f\|_{L^2_\mu} = \|h_{2,\mu}(f)\|_{L^2_\mu}.$$

Let $\Delta = \Delta(x, y, z)$ be the area of triangle with sides x, y, z (see [8, 16]). Set

$$D_\mu(x, y, z) = \frac{\Delta^{2\mu+1}}{2^{2\mu}(xyz)^\mu \Gamma(\mu + \frac{1}{2})\sqrt{\pi}}.$$

If Δ exists and zero otherwise. We note that $D_\mu(x, y, z) \geq 0$ and it is symmetric in x, y, z.

The generalized translation operator value of $f \in L^2_\mu$ is defined by

$$T_h(f)(x) = \int_0^{+\infty} f(z) D_\mu(h, x, z) z^\mu dz, \;\; 0 < x, \;\; h < \infty$$

From lemma 1.3 in [14], we have

$$h_{2,\mu}(T_h(f))(\lambda) = c_\mu(\lambda h) h_{2,\mu}(f)(\lambda), \tag{1.2}$$

where $f \in L^2_\mu$.

For $\mu \geq -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_μ defined by

$$j_\mu(x) = \frac{2^\mu \Gamma(\mu+1) J_\mu(x)}{x^\mu}. \tag{1.3}$$

From [1], we have the following lemma:

Lemma 1.2. Let $\mu \geq -\frac{1}{2}$. The following inequalities hold

1. $|j_\mu(x)| \leq 1$
2. $1 - j_\mu(x) = O(x^2); \quad 0 \leq x \leq 1$
3. $\sqrt{x} J_\mu(x) = O(1)$.

Lemma 1.3. The following inequality is true

$$|1 - j_\mu(x)| \geq c,$$

with $|x| \geq 1$, where $c > 0$ is certain constant.

Proof . Analog of lemma 2.9 in [2]. □

It follows from (1.1) and (1.3) that

$$c_\mu(x) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{x}).$$

2 Main result

In this section we give the main result of this paper. We need first to define the second Hankel-Clifford Lipschitz class.

Definition 2.1. Let $\alpha \in (0,1)$. A function $f \in L^2_\mu$ is said to be in the second Hankel-Clifford Lipschitz class, denoted by $Lip(\alpha, 2, \mu)$, If

$$\left\| T_h f(x) - \frac{1}{\Gamma(\mu+1)} f(x) \right\|_{L^2_\mu} = O\left(h^\alpha\right) \quad \text{as} \;\; h \longrightarrow 0.$$

Our main result is the next theorem

Theorem 2.2. Let $f \in L^2_\mu$. Then the following are equivalent:

1. $f \in Lip(\alpha, 2, \mu)$
2. $\displaystyle\int_N^{+\infty} \left| h_{2,\mu}(f)(\lambda) \right|^2 \lambda^\mu d\lambda = O\left(N^{-2\alpha}\right) \quad \text{as } N \longrightarrow +\infty$

Proof . 1) $\Longrightarrow$ 2) Let $f \in L^2_\mu$. It follows from (1.2) and (1.4) that

$$\begin{aligned} h_{2,\mu}\left(T_h f - \frac{1}{\Gamma(\mu+1)} f\right)(\lambda) &= \left(C_\mu(\lambda h) - \frac{1}{\Gamma(\mu+1)}\right) h_{2,\mu}(f)(\lambda) \\ &= \frac{1}{\Gamma(\mu+1)} \left(j_\mu(2\sqrt{\lambda h}) - 1\right) h_{2,\mu}(f)(\lambda), \end{aligned}$$

then, using the Parseval's identity, we have

$$\left\| T_h f - \frac{1}{\Gamma(\mu+1)} f \right\|^2_{L^2_\mu} = \frac{1}{\Gamma(\mu+1)} \int_0^{+\infty} \left| 1 - j_\mu(2\sqrt{\lambda h}) \right|^2 \left| h_{2,\mu}(f)(\lambda) \right|^2 \lambda^\mu d\lambda.$$

Assume that $f \in Lip(\alpha, 2, \mu)$. Then we have

$$\left\| T_h f - \frac{1}{\Gamma(\mu+1)} f \right\|_{L^2_\mu} = O\left(h^\alpha\right) \quad \text{as} \;\; h \longrightarrow 0.$$

If $\lambda \in \left[\frac{1}{4h}, \frac{2}{4h}\right]$, then $2\sqrt{\lambda h} \geq 1$. From lemme 1.3 we obtain

$$1 \leq \frac{1}{c^2}\left|1 - j_\mu(2\sqrt{\lambda h})\right|^2,$$

i.e.,

$$\frac{1}{\Gamma(\mu+1)} \leq \frac{1}{c^2\Gamma(\mu+1)}\left|1 - j_\mu(2\sqrt{\lambda h})\right|^2.$$

Then

$$\begin{aligned}
\frac{1}{\Gamma(\mu+1)}\int_{\frac{1}{4h}}^{\frac{2}{4h}} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda &\leq \frac{1}{c^2\Gamma(\mu+1)}\int_{\frac{1}{4h}}^{\frac{2}{4h}} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\
&\leq \frac{1}{c^2\Gamma(\mu+1)}\int_{0}^{+\infty} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\
&= O\left(h^{2\alpha}\right).
\end{aligned}$$

we conclude that

$$\int_N^{2N} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(N^{-2\alpha}\right) \quad \text{as } N \longrightarrow +\infty$$

Thus there exists $C_1 > 0$ such that

$$\int_N^{2N} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \leq C_1 N^{-2\alpha}.$$

So that

$$\begin{aligned}
\int_N^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda &= \left(\int_N^{2N} + \int_{2N}^{4N} + \int_{4N}^{8N} + ...\right) |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\
&\leq C_1\left(N^{-2\alpha} + (2N)^{-2\alpha} + (4N)^{-2\alpha} + ...\right) \\
&\leq C_1 N^{-2\alpha}\left(1 + 2^{-2\alpha} + (2^{-2\alpha})^2 + (2^{-2\alpha})^3 + ...\right) \\
&\leq C_1 K_\alpha N^{-2\alpha},
\end{aligned}$$

where $K_\alpha = \left(1 - 2^{-2\alpha}\right)^{-1}$ since $2^{-2\alpha} < 1$.
This proves that

$$\int_N^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(N^{-2\alpha}\right) \quad \text{as } N \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_N^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(N^{-2\alpha}\right) \quad \text{as } N \longrightarrow +\infty.$$

we have to show that

$$\frac{1}{\Gamma(\mu+1)}\int_0^{+\infty} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O(h^{2\alpha}) \text{ as } h \longrightarrow 0.$$

We write

$$\int_0^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = I_1 + I_2,$$

where

$$I_1 = \int_0^{\frac{1}{4h}} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda,$$

and

$$I_2 = \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda.$$

From (1) of lemma 1.2, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\leq 4\int_{\frac{1}{4h}}^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &= O\left(h^{2\alpha}\right) \ as\ h \longrightarrow 0. \end{aligned}$$

Then

$$\frac{1}{\Gamma(\mu+1)} \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{h\lambda})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(h^{2\alpha}\right).$$

Set

$$\psi(x) = \int_x^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda$$

We know from (2) of lemma 1.2 that $1 - j_\mu(2\sqrt{h\lambda}) = O(\lambda h)$ for $0 \leq 2\sqrt{\lambda h} \leq 1$. Thus there exists $C_2 > 0$ such that $|1 - j_\mu(2\sqrt{h\lambda})| \leq C_2\lambda h$ for $0 \leq 2\sqrt{\lambda h} \leq 1$. Then

$$I_1 \leq -C_2 h^2 \int_0^{\frac{1}{4h}} x^2 \psi'(x) dx.$$

An integration by parts yields

$$\begin{aligned} I_1 &\leq -C_2 h^2 \int_0^{\frac{1}{4h}} x^2 \psi'(x) dx \\ &\leq -C_2 \psi\left(\frac{1}{4h}\right) + 2C_2 h^2 \int_0^{\frac{1}{4h}} x^2 \psi(x) dx \\ &\leq 2C_2 h^2 \int_0^{\frac{1}{4h}} x\psi(x) dx \\ &\leq 2C_2 h^2 \int_0^{\frac{1}{4h}} x x^{-2\alpha} dx \\ &\leq 2C_2 h^2 \int_0^{\frac{1}{4h}} x^{1-2\alpha} dx \ \ (\textit{the integral exists since } \alpha < 1) \\ &\leq C_2 K h^{2\alpha}. \end{aligned}$$

where K is a positive constant. Then

$$\frac{1}{\Gamma(\mu+1)}\int_0^{\frac{1}{4h}}\left|1-j_\mu(2\sqrt{h\lambda})\right|^2|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda=O\left(h^{2\alpha}\right),$$

and this ends the proof. □

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