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Fixed points for weakly contractive mappings in rectangular b-metric spaces

Mohamed Rossafi[a,*], Abdelkarim Kari[b]

[a]LaSMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, P. O. Box 1796 Fez Atlas, Morocco

[b]Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco

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Abstract

In this paper, inspired by the concept of generalized weakly contractive mappings in metric spaces, we introduce the concept of generalized weakly contractive mappings in rectangular b-metric spaces to study the existence of fixed points for the mappings in these spaces.

Keywords: Fixed point, weakly contractive mapping, rectangular b-metric space
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1 Introduction

It is well known that the Banach contraction principle [3] is a fundamental result in the fixed point theory, several authors have obtained many interesting extensions and generalizations [5, 15, 18, 30]. The well known metric spaces have been generalized metric spaces introduced by Branciari [4]. Various fixed point results were established on such spaces [2, 13, 14, 20, 21, 33].

Recently, George *et al.* [11] announced the notion of b-rectangular metric space and formulated some fixed point theorems in the b-rectangular metric space. Many authors initiated and studied many existing fixed point theorems in such spaces [9, 10, 12, 16, 17, 22, 23, 24, 25, 32, 34, 35, 36].

Weak contraction principle is a generalization of the Banach contraction principle which was first given by Alber *et al.* in Hilbert spaces [1]. Coudhury *et al.* [8] proved some fixed point results for weakly contractive mappings in complete metric spaces. Several authors have studied weak contraction mapping in complete metric spaces [6, 19, 26, 27, 28, 29, 31, 37].

Very recently, Cho [7] introduced a special weakly contractive mappings called generalized weakly contractive mappings and proved some fixed point results for such mappings in complete metric spaces.

In this work, we introduce a new notion of generalized weakly contractive mappings and provide some fixed point results for such mappings in complete b-rectangular metric spaces. We also present some special examples of generalized weakly contractive mappings on b-rectangular metric spaces. Also, we derive some useful corollaries.

*Corresponding author
Email addresses: rossafimohamed@gmail.com; mohamed.rossafi@usmba.ac.ma (Mohamed Rossafi), abdkrimkariprofes@gmail.com (Abdelkarim Kari)

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2 Preliminaries

In the following, we collect background information needed in the presentation of our results.

Definition 2.1. [11] Let X be a nonempty set, $s \geq 1$ be a given real number and d: $X \times X \rightarrow [0,+\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$,

1. $d(x,y) = 0$ if only if $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ $(b - rectangular\ inequality)$.

Then (X,d) is called a b-rectangular metric space.

Example 2.2. [16]. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2,3,4,5,6,7\}\}$ and $B = [1,2]$. Define $d : X \times X \rightarrow [0,+\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ for\ all\ x, y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = d\left(\frac{1}{6},\frac{1}{7}\right) = 0,05 \\ d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{7}\right) = d\left(\frac{1}{5},\frac{1}{6}\right) = 0,08 \\ d\left(\frac{1}{2},\frac{1}{6}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{7}\right) = 0,4 \\ d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{6}\right) = d\left(\frac{1}{4},\frac{1}{7}\right) = 0,24 \\ d\left(\frac{1}{2},\frac{1}{7}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = d\left(\frac{1}{4},\frac{1}{6}\right) = 0,15 \\ \quad d(x,y) = (|x-y|)^2 \ otherwise. \end{cases}$$

Then (X,d) is a b-rectangular metric space with coefficient $s = 3$.

Lemma 2.3. [32] Let (X,d) be a b-rectangular metric space.

(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow +\infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{s}d(x,y) \leq \lim_{n\rightarrow+\infty} \inf d(x_n,y_n) \leq \lim_{n\rightarrow+\infty} \sup d(x_n,y_n) \leq sd(x,y).$$

(b) If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then

$$\frac{1}{s}d(x,y) \leq \lim_{n\rightarrow+\infty} \inf d(x_n,y) \leq \lim_{n\rightarrow+\infty} \sup d(x_n,y) \leq sd(x,y),$$

for all $x \in X$.

Lemma 2.4. [16] Let (X,d) be a b-rectangular metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\rightarrow+\infty} d(x_n,x_{n+1}) = \lim_{n\rightarrow+\infty} d(x_n,x_{n+2}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\varepsilon \leq \lim_{k\rightarrow+\infty} \inf d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leq \lim_{k\rightarrow+\infty} \sup d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon,$$

$$\frac{\varepsilon}{s} \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s^2\varepsilon.$$

Definition 2.5. A function $f : X \to \mathbb{R}^+$, where X is a b-rectangular metric space, is called lower semicontinuous if for all $x \in X$ and $x_n \in X$ with $lim_{n\to+\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n\to+\infty} f(x_n).$$

Definition 2.6. A function $g : X \to \mathbb{R}^+$, where X is a b-rectangular metric space, is called is a right upper semicontinuous function if for all $x \in X$ and $x_n \in X$ with $lim_{n\to+\infty} x_n = x$, we have

$$g(x) \geq \limsup_{n\to+\infty} f(x_n).$$

Definition 2.7. [19] A function $\psi : [0, +\infty[\to [0, +\infty[$ is said to be an altering distance function if it satisfies the following conditions:

(a) is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.8. Define ψ_1; ψ_2; ψ_3: $[0, +\infty[\to [0, +\infty[$ by $\psi_1(t) = t$, $\psi_t(t) = 2t$ and $\psi_3(t) = t^2$. Then they are altering distance functions.

Definition 2.9. [7] Let X be a complete metric space with metric d, and $T : X \to X$. Also let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi\left(d\left(Tx, Ty\right) + \varphi\left(Tx\right) + \varphi\left(Ty\right)\right) \leq \psi\left(m\left(x, y, d, T, \varphi\right)\right) - \phi\left(l\left(x, y, d, T, \varphi\right)\right),$$

where

$$m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty),$$
$$\frac{1}{2}\{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(Tx) + \varphi(y)\}\}$$

and

$$l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}$$

for all $x, y \in X$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\psi(t) = 0$ if and only if $t = 0$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.10. [7] Let X be complete. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

3 Main results

Aspired by idea of the generalized weakly contractive mapping on metric space introduced by Cho [7], we introduce the notion of generalized weakly contractive mapping on rectangular b-metric space and establish some fixed point on such mapping.

Definition 3.1. Let X be a complete b-rectangular metric space with metric d and parameter s and $T : X \to X$. Also let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi\left(s^2 d\left(Tx, Ty\right) + \varphi\left(Tx\right) + \varphi\left(Ty\right)\right) \leq \psi\left(M\left(x, y, d, T, \varphi\right)\right) - \phi\left(M\left(x, y, d, T, \varphi\right)\right), \tag{3.1}$$

where

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,Tx)+\varphi(x)+\varphi(Tx),d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

for all $x,y\in X$, and ψ is an altering distance function and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function with $\phi(t)=0$ if and only if $t=0$.

Theorem 3.2. Let X be a complete b-rectangular metric space with parameter s ≥ 1. If T is a generalized weakly contractive mapping, then T has a unique fixed point $z\in X$ such that $z=Tz$ and $\varphi(z)=0$.

Proof . Let $x_0\in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1}=Tx_n$, for all $n\in\mathbb{N}$.

If there exists $n_0\in\mathbb{N}$ such that $x_{n_0}=x_{n_0+1}=0$, then x_{n_0} is a fixed point of T.

Next, we assume that $x_n\neq x_{n+1}$.
We claim that

$$\lim_{n\to+\infty} d\left(x_n,x_{n+1}\right)=0$$

and

$$\lim_{n\to+\infty} d\left(x_n,x_{n+2}\right)=0.$$

Letting $x=x_{n-1}$ and $y=x_n$ in (3.1) for all $n\in\mathbb{N}$, we have

$$\begin{aligned}&\psi\left(s^2 d\left(Tx_{n-1},Tx_n\right)+\varphi\left(Tx_{n-1}\right)+\varphi\left(Tx_n\right)\right)\\&\quad\leq\psi\left(M\left(x_{n-1},x_n,d,T,\varphi\right)\right)-\phi\left(M\left(x_{n-1},x_n,d,T,\varphi\right)\right),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}M\left(x_{n-1},x_n,d,T,\varphi\right)=\max\{&d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),d(x_{n-1},x_n)+\varphi(x_{n-1})\\&+\varphi(x_n),d(x_n,Tx_n)+\varphi(x_n)+\varphi(Tx_n)\}\\&=\max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),d(x_n,x_{n+1})+\varphi(x_n)+\varphi(Tx_{n+1})\}.\end{aligned}$$

If $M\left(x_{n-1},x_n,d,T,\varphi\right)=d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$, then we have

$$\begin{aligned}\psi\left(d\left(Tx_{n-1},Tx_n\right)+\varphi\left(Tx_{n-1}\right)+\varphi\left(x_{n+1}\right)\right)&=\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\&\leq\psi\left(s^2 d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\&\leq\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\&-\phi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right),\end{aligned}$$

which implies

$$\phi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)=0,$$

and so

$$d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)=0.$$

Hence

$$d\left(x_n,x_{n+1}\right)=0 \ and\ \varphi\left(x_n\right)=\varphi\left(x_{n+1}\right)=0,$$

which is a contradiction. Thus we have

$$d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\leq d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right),\ for\ all\ n=1,2,3,...,\tag{3.3}$$

and

$$M\left(x_{n-1},x_n,d,T,\varphi\right)=d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),\ for\ all\ n=1,2,3,....\tag{3.4}$$

for all $n=1,2,3,....$ It follows from (3.2) that

$$\begin{aligned}\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)&\leq\psi\left(d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right)\\&-\phi\left(d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right).\end{aligned}\tag{3.5}$$

It follows from (3.3) that the sequence $\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})\}_{n\in\mathbb{N}}$ is nonincreasing.
Hence $d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \to r$ as $n \to +\infty$ for some $r \geq 0$. Assume $r > 0$ and letting $n \to +\infty$ in (3.5) and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned}\psi\left(s^2 r\right) &\leq \psi(r) - \liminf_{n\to\infty} \phi\left(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})\right)\\ &\leq \psi(r) - \phi(r).\end{aligned}$$

It follows that $\psi(r) \leq \psi\left(s^2 r\right) \leq \psi(r) - \phi(r) < \psi(r)$, which is a contradiction and hence we have $r = 0$ and consequently, $\lim_{n\to+\infty} d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = 0$. So

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0, \tag{3.6}$$

$$\lim_{n\to+\infty} \varphi(x_n) = \lim_{n\to+\infty} \varphi(x_{n+1}) = 0. \tag{3.7}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case. Then $x_n \neq x_m$ for all $n, m \in \mathbb{N}$, $n \neq m$.

In (3.1), letting $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\begin{aligned}&\psi\left(s^2 d(Tx_{n-1}, Tx_{n+1}) + \varphi(Tx_{n-1}) + \varphi(Tx_{n+1})\right)\\ &\quad\leq \psi\left(M(x_{n-1}, x_{n+1}, d, T, \varphi)\right) - \phi\left(M(x_{n-1}, x_{n+1}, d, T, \varphi)\right),\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_{n+1}, d, T, \varphi) &= \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}),\\ &\qquad d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}\\ &= \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}.\end{aligned}$$

So we get

$$\begin{aligned}&\psi\left(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2})\right) \leq \psi\left(s^2 d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2})\right)\\ &\leq \psi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right)\\ &-\phi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right).\end{aligned} \tag{3.8}$$

Take $a_n = d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2})$ and $b_n = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$.
Then by (3.8), one can write

$$\begin{aligned}\psi(a_n) &\leq \psi\left(\max(a_{n-1}, b_{n-1})\right) - \phi\left(\max(a_{n-1}, b_{n-1})\right)\\ &\leq \psi\left(\max(a_{n-1}, b_{n-1})\right).\end{aligned}$$

Since ψ is increasing, we get

$$a_n \leq \max\{a_{n-1}, b_{n-1}\}.$$

By (3.3), we have

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\},$$

which implies that

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\lambda \geq 0$ such that

$$\lim_{n\to+\infty} \max\{a_n, b_n\} = \lambda.$$

Assume that $\lambda > 0$. By (3.6), it is obvious that

$$\lambda = \lim_{n\to+\infty} \sup a_n = \lim_{n\to+\infty} \sup \max\{a_n, b_n\} = \lim_{n\to+\infty} \max\{a_n, b_n\}. \tag{3.9}$$

Taking $\limsup_n \to +\infty$ in (3.8), using (3.9) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}\psi(\lambda) &= \psi\left(\limsup_{n\to+\infty} a_n\right) \\ &= \limsup_{n\to+\infty} \psi\left(a_n\right) \\ &\leq \limsup_{n\to+\infty} \psi\left(\max\left\{a_n, b_n\right\}\right) - \liminf_{n\to+\infty} \phi\left(\max\left\{a_n, b_n\right\}\right) \\ &\leq \psi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right) - \phi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right) \\ &= \psi(\lambda) - \phi(\lambda),\end{aligned}$$

which implies that $\phi(\lambda) = 0$, a contradiction. Thus, from (3.9),

$$\limsup_{n\to+\infty} a_n = 0$$

and hence

$$\lim_{n\to+\infty} d\left(x_n, x_{n+2}\right) = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n,m\to+\infty} d\left(x_n, x_m\right) = 0$ for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 2.4, there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$ such that

i) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon$,

ii) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \leq s\varepsilon$,

iii) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \leq s\varepsilon$,

vi) $\frac{\varepsilon}{s} \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) \leq s^2\varepsilon$.

From (3.1) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned}M(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi) = \max\{&d(x_{m_{(k)}}, x_{n_{(k)}}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}}), \\ &d(x_{m_{(k)}}, x_{m_{(k)}+1}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}+1}), d(x_{n_{(k)}}, x_{n_{(k)}+1}) + \varphi(x_{n_{(k)}}) + \varphi(x_{n_{(k)}+1})\}.\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.6), (3.7) and (iii) of Lemma 2.4, we have

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right) \leq s\varepsilon. \tag{3.10}$$

Now letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we have

$$\begin{aligned}&\psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)}+1\right) + \varphi\left(n_{(k)}+1\right)\right] \\ &\qquad \leq \psi\left[d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)}+1\right) + \varphi\left(n_{(k)}+1\right)\right] \\ &\qquad\qquad - \phi\left[d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)}\right) + \varphi\left(n_{(k)}\right)\right].\end{aligned}$$

Letting $k \to +\infty$, using (3.6), (3.7), (3.10), and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\lim_{k\to+\infty} \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \leq \psi(s\varepsilon) - \phi(s\varepsilon).$$

Using (3.10) and (iv) of Lemma 2.4, we obtain

$$\psi(s\varepsilon) = \psi\left(s^2 \frac{\varepsilon}{s}\right) \leq \lim_{k\to+\infty} \sup \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \leq \psi(s\varepsilon) - \phi(s\varepsilon).$$

This is a contradiction. Thus

$$\lim_{n,m\to+\infty} d\left(x_m, x_n\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d), there exists $z \in X$ such that

$$\lim_{n\to+\infty} d(x_n, z) = 0.$$

Since φ is lower semicontinuous, we get

$$\varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) \leq \lim_{n\to+\infty} \varphi(x_n) = 0,$$

which implies

$$\varphi(z) = 0. \tag{3.11}$$

Now, putting $x = x_n$ and $y = z$ in (3.1), we have

$$\begin{aligned} M(x_n, z, d, T, \varphi) \quad = \quad & \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), \\ & d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(z, Tz) + \varphi(z) + \varphi(Tz)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$ and using (3.6), (3.7) and (3.11), we have

$$\lim_{n\to+\infty} M(x_n, z, d, T, \varphi) = d(z, Tz) + \varphi(Tz).$$

Since $x_n \to z$ as $n \to +\infty$, from Lemma 2.3 , we conclude that

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\to+\infty} \sup d(Tx_n, Tz) \leq s d(z, Tz).$$

Hence

$$sd(z, Tz) = s^2 \frac{1}{s} d(z, Tz) \leq \lim_{n\to+\infty} \sup s^2 d(Tx_n, Tz),$$

which implies

$$\lim_{n\to+\infty} \sup [sd(z, Tz) + \varphi(x_{n+1}) + \varphi(Tz)] \leq \lim_{n\to+\infty} \sup \left[s^2 d(Tx_n, Tz) + (x_{n+1}) + \varphi(Tz)\right].$$

Then using (3.1), we have

$$\begin{aligned} \psi\left[s^2 d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz)\right] &= \psi\left[s^2 d(x_{n+1}, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right] \\ &\leq \psi[M(x_n, z, d, T, \varphi)] - \phi[M(x_n, z, d, T, \varphi)]. \end{aligned}$$

Letting $n \to +\infty$ and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned} &\psi\left[\lim_{n\to+\infty} \sup (sd(z, Tz) + \varphi(x_{n+1}) + \varphi(Tz))\right] \\ &\qquad \leq \psi\left[\lim_{n\to+\infty} \sup \left(s^2 d(Tx_n, Tz) + (x_{n+1}) + \varphi(Tz)\right)\right] \\ &\qquad\qquad \leq \psi\left[\lim_{n\to+\infty} \sup M(x_n, z, d, T, \varphi)\right] - \lim_{n\to+\infty} \phi[M(x_n, z, d, T, \varphi)], \end{aligned}$$

which implies

$$\psi[sd(z, Tz) + \varphi(Tz)] \leq \psi[d(z, Tz) + \varphi(Tz)] - \phi[d(z, Tz) + \varphi(Tz)].$$

This holds if and only if $\phi(d(z, Tz) + \varphi(Tz)) = 0$ and from the property of ϕ, we have

$$d(z, Tz) + \varphi(Tz) = 0.$$

Hence $d(z, Tz) = 0$ and so $z = Tz$ and $\varphi(Tz) = 0$. It is a contradiction to the assumption: that T does not have a periodic point. Thus T has a periodic point, say, z of period n. Suppose that the set of fixed points of T is empty. Then we have

$$q > 0 \textit{ and } d(z, Tz) > 0.$$

Since T has a periodic point, $z=T^nz$. Letting $x=T^{n-1}z$ and $y=T^nz$, we obtain

$$\begin{aligned}M\left(T^nz,T^{n-1}z,d,T,\varphi\right)=\max\{&d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),\\ &d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),d\left(T^nz,TT^nz\right)+\varphi\left(T^nz\right)+\varphi\left(TT^nz\right)\}.\end{aligned}$$

By a similar method to (3.4), we conclude that

$$M\left(T^nz,T^{n-1}z,d,T,\varphi\right)=d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right).$$

From (3.1), we have

$$\begin{aligned}\psi\left[s^2d\left(z,Tz\right)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]&=\psi\left[s^2d\left(T^nz,T^{n+1}z\right)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]\\ &\leq\psi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\ &-\phi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\ &\leq\psi\left[s^2d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\ &\vdots\\ &\leq\psi\left[d\left(z,Tz\right)+\varphi\left(z\right)+\varphi\left(Tz\right)\right]\\ &-\phi\left[d\left(z,Tz\right)+\varphi\left(z\right)+\varphi\left(Tz\right)\right]\end{aligned}$$

Taking the limit as $n\to+\infty$ and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\psi\left[s^2d\left(z,Tz\right)\right]\leq\psi\left[d\left(z,Tz\right)\right]-\phi\left[d\left(z,Tz\right)\right].$$

Hence $d(z,Tz)=0$, which is a contradiction. Thus the set of fixed points of T is non-empty, that is, T has at least one fixed point.

Suppose that $z,u\in X$ are two fixed points of T such that $u\neq z$. Then $Tz=z$ and $Tu=u$.

Letting $x=z$ and $y=u$ in (3.1), we have

$$\psi\left(s^2d\left(Tz,Tu\right)+\varphi\left(Tz\right)+\varphi\left(Tu\right)\right)=\psi\left(s^2d\left(z,u\right)\right)\leq\psi\left(M\left(z,u,d,T,\varphi\right)\right)-\phi\left(M\left(z,u,d,T,\varphi\right)\right),$$

where

$$\begin{aligned}M(z,u,d,T,\varphi)&=\max\{d(z,u)+\varphi(z)+\varphi(u),d(z,Tz)+\varphi(z)+\varphi(Tz),d(u,Tu)+\varphi(u)+\varphi(Tu)\}\\ &=d(z,u).\end{aligned}$$

So

$$\psi\left(s^2d\left(z,u\right)\right)\leq\psi\left(d\left(z,u\right)\right)-\phi\left(d\left(z,u\right)\right).$$

This holds if $\phi\left(d\left(z,u\right)\right)=0$ and so we have $(d\left(z,u\right)=0$. Hence $z=u$ and T has a unique fixed point. □

Corollary 3.3. Let (X,d) be a complete b-rectangular metric space and $T:X\to X$ be a mapping. Suppose that there exists $k\in\,]0,1[$ such that for all $x,y\in X$,

$$\begin{aligned}&s^2d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right)\leq\\ &\qquad k\max\{d\left(x,y\right)+\varphi\left(x\right)+\varphi\left(y\right),d\left(x,Tx\right)+\varphi\left(x\right)+\varphi\left(Tx\right),d\left(y,Ty\right)+\varphi\left(y\right)+\varphi\left(Ty\right)\},\end{aligned}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof . It suffices to take $\psi(t)=t$ and $\phi(t)=(1-k)t$ in Theorem 3.2. □

Corollary 3.4. Let (X,d) be a complete b-rectangular metric space and $T:X\to X$ be a mapping. Suppose that there exists $\alpha\in\left]0,\frac{1}{2}\right[$ such that for all $x,y\in X$,

$$s^2d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right)\leq\alpha\left[\left(d\left(Tx,x\right)+\varphi\left(x\right)+\varphi\left(Tx\right)+d+\varphi\left(y\right)+\varphi\left(Ty\right)+\left(Ty,y\right)\right)\right],\tag{3.12}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof . Let $k = 2\alpha$. Then $k \in]0,1[$. Also, if (3.12) holds, then

$$\begin{aligned} s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) &\le \alpha\left[d(Tx,x) + \varphi(x) + \varphi(Tx) + d + \varphi(y) + \varphi(Ty) + (Ty,y)\right] \\ &= k\frac{\left[d(Tx,x) + \varphi(x) + \varphi(Tx) + d + \varphi(y) + \varphi(Ty) + (Ty,y)\right]}{2} \\ &\le k\max\{d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty)\} \\ &\le k\max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty)\}. \end{aligned}$$

Thus it suffices to apply Corollary 3.3. □

Corollary 3.5. Let (X,d) be a complete b-rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in \left]0, \frac{1}{3}\right[$ such that for all $x, y \in X$,

$$\begin{aligned} &s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \le \\ &\qquad \lambda\left[d(x,y) + \varphi(x) + \varphi(y) + d(Tx,x) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(Ty) + d(Ty,y)\right], \end{aligned} \tag{3.13}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof . Let $k = 3\lambda$. Then $k \in]0,1[$. Also, if (3.13) holds, then

$$\begin{aligned} &s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \\ &\le \lambda\left[d(x,y) + \varphi(x) + \varphi(y) + d(Tx,x) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(Ty) + d(Ty,y)\right] \\ &= k\frac{\left[d(x,y) + \varphi(x) + \varphi(y) + d(Tx,x) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(Ty) + d(Ty,y)\right]}{3} \\ &\le k\max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty)\}. \end{aligned}$$

Thus it suffices to apply Corollary 3.3. □

Corollary 3.6. Let $d(X,d)$ be a complete b-rectangular metric space with parameter $s > 1$ and T be a self mapping on X. If there exists $k \in]0,1[$ such that for all $x, y \in X$,

$$\begin{aligned} &s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Tx) \le \\ &\qquad k\left[\beta_1\left(d(x,y) + \varphi(x) + \varphi(y)\right) + \beta_2\left(d(Tx,x) + \varphi(x) + \varphi(Tx)\right) + \beta_3\left(d(Ty,y) + \varphi(y) + \varphi(Ty)\right)\right], \end{aligned}$$

where $\beta_i \ge 0$ for $i \in \{1,2,3\}$, $\sum_{i=0}^{i=3} \beta_i \le 1$, φ is a lower semicontinuous function. Then T has a unique fixed point.

Proof . Take $\psi(t) = t$ and $\phi(t) = (1-k)t$. Then it suffices to apply Corollary 3.3. □

Example 3.7. Let $X = A \cup B$, where $A = \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $B = \left[\frac{1}{2}, 1\right]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ \textit{for all}\ x,y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(0, \frac{1}{9}\right) = d\left(\frac{1}{5}, \frac{1}{16}\right) = 0,1 \\ d\left(0, \frac{1}{5}\right) = d\left(\frac{1}{5}, \frac{1}{9}\right) = 0,5 \\ d\left(0, \frac{1}{16}\right) = d\left(\frac{1}{9}, \frac{1}{16}\right) = 0,05 \\ d(x,y) = (|x-y|)^2 \ \textit{otherwise}. \end{cases}$$

Then (X,d) is a b-rectangular metric space with coefficient $s = 3$. However we have the following:

1) (X,d) is not a metric space, since $d\left(\frac{1}{5},\frac{1}{9}\right)=0.5>0.15=d\left(\frac{1}{5},\frac{1}{16}\right)+d\left(\frac{1}{16},\frac{1}{9}\right)$.

2) (X,d) is not a b-metric space for s=3, since $d\left(\frac{1}{5},\frac{1}{9}\right)=0.5>0.45=3\left[d\left(\frac{1}{5},\frac{1}{16}\right)+d\left(\frac{1}{16},\frac{1}{9}\right)\right]$.

3) (X,d) is not a rectangular metric space, since $d\left(\frac{1}{5},\frac{1}{9}\right)=0.5>0.25=d\left(\frac{1}{5},\frac{1}{16}\right)+d\left(\frac{1}{16},0\right)+d\left(0,\frac{1}{9}\right)$.

Define a mapping $T:X\to X$ by

$$T(x)=\begin{cases}\dfrac{1}{16}\ if\ x\in\left[\dfrac{1}{2},1\right]\\ 0\ if\ x\in A.\end{cases}$$

Then $T(x)\in X$ for all $x\in X$. Let

$$\varphi(t)=\begin{cases}t\ if\ t\in[0,1]\\ 2t\ if\ t>1\end{cases}$$

$$\varphi(t)=\begin{cases}\dfrac{t}{16}\ if\ t\in[0,1]\\ \dfrac{t}{8}\ if\ t>1\end{cases}$$

and

$$\psi(t)=\frac{3t}{2}.$$

Then ψ is an altering distance function and φ is a lower semicontinuous function and ϕ is a lower semicontinuous function such that $\psi(t)=0\Leftrightarrow t=0$, $\phi(t)=0\Leftrightarrow t=0$ and $\varphi(t)=0\Leftrightarrow t=0$.

Consider the following possibilities:

Case I: $x,y\in\left\{0,\frac{1}{5},\frac{1}{9},\frac{1}{16}\right\}$.

Assume that $x\geq y$. Then

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)=\psi\left(9.d(0,0)+\varphi(0)+\varphi()\right)=\psi(0)=0.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y)=d(x,y)+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d(x,0)+x,$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d(y,0)+y$$

and

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(x,0)+x,d(y,0)+y\}.$$

Since $x\geq y$, we have

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(x,0)+x\}.$$

If

$$M(x,y,d,T,\varphi)=d(x,y)+x+y\geq\frac{1}{20},$$

then

$$\begin{aligned}\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right)&=\psi\left(d(x,y)+x+y\right)-\phi\left(d(x,y)+x+y\right)\right)\\&=\frac{23}{16}\left(d(x,y)+x+y\right)\geq 0\end{aligned}$$

and so

$$0=\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi)=d(x,0)+x\geq\frac{1}{20},$$

then

$$\begin{aligned}\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)&\leq\frac{3}{2}\cdot\frac{1}{20}-\frac{1}{20}\cdot\frac{1}{16}=\frac{23}{320}\\&\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).\end{aligned}$$

Assume that $x < y$. Then

$$M(x,y,d,T,\varphi) = \max\{d(x,y)+x+y, d(y,0)+y\}.$$

If

$$M(x,y,d,T,\varphi) = d(x,y)+x+y \geq \frac{1}{20}+\frac{1}{16} = \frac{9}{80},$$

then

$$\begin{aligned}\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) &\leq \frac{3}{2}\cdot\frac{9}{80}-\frac{1}{16}\cdot\frac{9}{80} = \frac{207}{1280} \\ &\leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).\end{aligned}$$

If

$$M(x,y,d,T,\varphi) = d(y,0)+y,$$

then

$$d(y,0)+y \geq \frac{9}{80},$$

since $x < y$ and $0 < y$. Thus

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case II: $x \in \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $y \in \left[\frac{1}{2}, 1\right]$. This implies $x < y$. Then

$$\psi\left(s^2\left(d\left(Tx,Ty\right)+\varphi(Tx)+\varphi(Ty)\right)\right) = \frac{3}{2}\left[9d\left(\frac{1}{16},0\right)+\frac{1}{16}\right] = \frac{123}{160}.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y) = (x-y)^2+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx) = d(x,0)+x$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty) = d\left(y,\frac{1}{16}\right)+y+\frac{1}{16},$$

and

$$M(x,y,d,T,\varphi) = \max\{(x-y)^2+x+y, d(x,0)+x, d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}$$

Since $x < y$, we have

$$M(x,y,d,T,\varphi) = \max\{(x-y)^2+x+y, d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}.$$

If

$$M(x,y,d,T,\varphi) = d(x-y)^2+x+y \geq \frac{1}{2}+\left(\frac{1}{2}-\frac{1}{5}\right)^2 = \frac{59}{100},$$

then

$$\psi\left(M(x,y,d,T,\varphi)\right)-\phi\left(M(x,y,d,T,\varphi)\right) = \frac{23}{16}\left(d(x,y)^2+x+y\right) \geq \frac{23}{16}\cdot\frac{59}{100} = \frac{1357}{1600} \geq \frac{123}{160}.$$

Then

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi) = d\left(y,\frac{1}{16}\right)+y+\frac{1}{16} \geq \frac{1}{2}+\frac{1}{16}+\left(\frac{1}{2}-\frac{1}{16}\right)^2 = \frac{193}{256},$$

then

$$\psi\left(M(x,y,d,T,\varphi)\right)-\phi\left(M(x,y,d,T,\varphi)\right) = \frac{23}{16}d\left(y,\frac{1}{16}\right)+y+\frac{1}{16} \geq \frac{23}{16}\cdot\frac{193}{256} = \frac{4439}{4096} \geq \frac{123}{160}.$$

Then

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case III: $y \in \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $x \in \left[\frac{1}{2}, 1\right]$.

By a similar method to Case II, we deduce that

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi) - \phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case IV: $x, y \in \left[\frac{1}{2}, 1\right]$.

If $x \geq y$, then

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) = \psi\left(9d\left(\frac{1}{16}, \frac{1}{16}\right) + \varphi\left(\frac{1}{16}\right) + \varphi\left(\frac{1}{16}\right)\right) = \frac{3}{16}.$$

Also

$$d(x,y) + \varphi(x) + \varphi(y) = (x-y)^2 + x + y,$$
$$d(x,Tx) + \varphi(x) + \varphi(Tx) = d\left(x, \frac{1}{16}\right) + x + \frac{1}{16},$$
$$d(y,Ty) + \varphi(y) + \varphi(Ty) = d\left(y, \frac{1}{16}\right) + y + \frac{1}{16}$$

and

$$M(x,y,d,T,\varphi) = \max\{(x-y)^2 + x + y, d\left(x, \frac{1}{16}\right) + x + \frac{1}{16}, d\left(y, \frac{1}{16}\right) + y + \frac{1}{16}\}.$$

Since $x \geq y$, we have

$$M(x,y,d,T,\varphi) = \max\{(x-y)^2 + x + y, d\left(x, \frac{1}{16}\right) + x + \frac{1}{16}\}.$$

If

$$M(x,y,d,T,\varphi) = (x-y)^2 + x + y \geq 1,$$

then

$$\frac{3}{16} \leq \frac{23}{16} \leq \psi\left(M(x,y,d,T,\varphi) - \phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi) = d\left(x, \frac{1}{16}\right) + x + \frac{1}{16} \geq d\left(\frac{1}{2}, \frac{1}{16}\right) + \frac{1}{2} + \frac{1}{16} = \frac{193}{256},$$

then

$$\frac{3}{16} \leq \frac{23}{16} \cdot \frac{193}{256} = \frac{4439}{4096} \leq \psi\left(M(x,y,d,T,\varphi) - \phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If $x, y \in A$ and $x < y$, then

$$M(x,y,d,T,\varphi) = \max\{(x-y)^2 + x + y, d\left(y, \frac{1}{16}\right) + y\}.$$

By a similar method to the condition $x \geq y$, we have

$$\frac{3}{16} \leq \frac{4439}{4096} \leq \psi\left(M(x,y,d,T,\varphi) - \phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Hence

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi) - \phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Thus all the conditions of Theorem 3.2 are satisfied and 0 is the unique fixed point of T.

Definition 3.8. Let X be a complete b-rectangular metric space with metric d and parameter s, and $T : X \to X$ be a mapping. Also let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized (ψ, φ, ϕ) contractive mapping if it satisfies the following condition:

$$\psi\left(s^2 d\left(Tx, Ty\right) + \varphi\left(Tx\right) + \varphi\left(Ty\right)\right) \leq \phi\left(M\left(x,y,d,T,\varphi\right)\right), \tag{3.14}$$

where

$$M(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty)\}$$

for all $x, y \in X$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an altering distance function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a right upper semi-continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$.

Theorem 3.9. Let X be a complete b-rectangular metric space with parameter s ≥ 1 and $T : X \to X$ be a mapping. If T is a generalized (ψ, φ, ϕ) contractive mapping then T has a unique fixed point $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

Proof . Let $x_0 \in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = 0$, then x_{n_0} is a fixed point of T.

Now we assume that $x_n \neq x_{n+1}$.
We claim that

$$\lim_{n\to+\infty} d\left(x_n,x_{n+1}\right) = 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.14) for all $n \in \mathbb{N}$, we have

$$\psi\left(s^2 d\left(Tx_{n-1}, Tx_n\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(Tx_n\right)\right) \leq \phi\left(M\left(x_{n-1}, x_n, d, T, \varphi\right)\right),$$

where

$$\begin{aligned} M\left(x_{n-1}, x_n, d, T, \varphi\right) &= \max\{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n-1}, x_n) + \varphi(x_{n-1}) \\ &\quad + \varphi(x_n), d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})\}. \end{aligned}$$

If $M\left(x_{n-1}, x_n, d, T, \varphi\right) = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$, then we have

$$\begin{aligned} \psi\left(d\left(Tx_{n-1}, Tx_n\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(x_{n+1}\right)\right) &= \psi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\ &\leq \psi\left(s^2 d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\ &\leq \phi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\ &< \psi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right). \end{aligned}$$

This is a contradiction. Thus

$$M\left(x_{n-1}, x_n, d, T, \varphi\right) = d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n).$$

Therefore,

$$\psi\left(s^2 d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) < \psi\left(d\left(x_{n-1}, x_n\right) + \varphi\left(x_{n-1}\right) + \varphi\left(x_n\right)\right). \tag{3.15}$$

Since ψ is increasing,

$$d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) < d\left(x_{n-1}, x_n\right) + \varphi\left(x_{n-1}\right) + \varphi\left(x_n\right). \tag{3.16}$$

From (3.16), the sequence $\{d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\}$ is decreasing and bounded below. Hence $d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) \to r$ as $n \to +\infty$ for some $\lambda \geq 0$. Assume $\lambda > 0$. Letting $n \to +\infty$ in (3.15) and using the lower continuity of ψ and the upper semi-continuous of ϕ, we have

$$\begin{aligned} \psi\left(\lambda\right) &\leq \psi\left(s^2\lambda\right) \\ &= \limsup_{n\to+\infty} \psi\left(s^2 d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\ &\leq \limsup_{n\to+\infty} \phi\left(d\left(x_{n-1}, x_n\right) + \varphi\left(x_{n-1}\right) + \varphi\left(x_n\right)\right) \\ &\leq \phi(\lambda) \\ &< \psi\left(\lambda\right). \end{aligned}$$

It follows that $\psi\left(\lambda\right) \leq \psi\left(s^2 r\right) < \psi\left(\lambda\right)$, which is a contradiction and hence we have $\lambda = 0$ and consequently, $\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) = 0$, which implies

$$\lim_{n\to+\infty} d\left(x_n, x_{n+1}\right) = 0, \tag{3.17}$$

$$\lim_{n\to+\infty} \varphi\left(x_n\right) = \lim_{n\to+\infty} \varphi\left(x_{n+1}\right) = 0. \tag{3.18}$$

Next, we shall prove that

$$\lim_{n\to+\infty} d\left(x_n, x_{n+2}\right) = 0.$$

Assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N},\ n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$. Using (3.16), we have

$$\begin{aligned} d\left(x_m, x_{m+1}\right) + \varphi\left(x_m\right) + \varphi\left(x_{m+1}\right) &= d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) \\ &< d\left(x_{n-1}, x_n\right) + \varphi\left(x_{n-1}\right) + \varphi\left(x_n\right). \end{aligned}$$

Continuing this process, we can that

$$\begin{aligned} d\left(x_m, x_{m+1}\right) + \varphi\left(x_m\right) + \varphi\left(x_{m+1}\right) &= d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) \\ &< d\left(x_m, x_{m+1}\right) + \varphi\left(x_m\right) + \varphi\left(x_{m+1}\right), \end{aligned}$$

which implies that

$$d\left(x_m, x_{m+1}\right) < d\left(x_m, x_{m+1}\right).$$

This is a contradiction. Therefore, $d\left(x_n, x_m\right) > 0$ for all $n, m \in \mathbb{N}$, $n \neq m$.
Letting $x = x_{n-1}$ and $y = x_{n+1}$ in (3.14), we have

$$\psi\left(s^2 d\left(Tx_{n-1}, Tx_{n+1}\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(Tx_{n+1}\right)\right) \leq \phi\left(M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right)\right),$$

where

$$\begin{aligned} M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right) &= \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), \\ &\quad d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\} \\ &= \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}. \end{aligned}$$

So we get

$$\begin{aligned} \psi\left(d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) &\leq \psi\left(s^2 d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) \\ &\leq \phi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} &\psi\left(d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) \\ &\quad \leq \phi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right). \end{aligned} \tag{3.19}$$

Take $a_n = d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)$ and $b_n = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$. Then, by (3.19), one can write

$$\psi\left(a_n\right) \leq \psi\left(\max\left(a_{n-1}, b_{n-1}\right)\right).$$

Since ψ is increasing, we get

$$a_n \leq \max\left\{a_{n-1}, b_{n-1}\right\}.$$

By (3.16), we have

$$b_n \leq b_{n-1} \leq \max\left\{a_{n-1}, b_{n-1}\right\}.$$

This implies that

$$\max\left\{a_n, b_n\right\} \leq \max\left\{a_{n-1}, b_{n-1}\right\},\ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\left\{a_{n-1}, b_{n-1}\right\}_{n\in\mathbb{N}}$ is nonnegative decreasing sequence of real numbers. Thus there exists $\beta \geq 0$ such that

$$\lim_{n\to+\infty} \max\left\{a_n, b_n\right\} = \beta.$$

Assume that $\beta > 0$. Now, by (3.17), it is obvious that

$$\beta = \lim_{n\to+\infty} \sup a_n = \lim_{n\to+\infty} \sup \max\left\{a_n, b_n\right\} = \lim_{n\to+\infty} \max\left\{a_n, b_n\right\}. \tag{3.20}$$

Taking $\limsup_n \to +\infty$ in (3.19) and using (3.20) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}\psi(\beta) &= \limsup_{n\to+\infty} \psi\left(a_n\right)\\ &\leq \limsup_{n\to+\infty} \phi\left(\max\left\{a_n, b_n\right\}\right)\\ &\leq \phi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right)\\ &= \phi(\beta)\\ &< \psi(\beta),\end{aligned}$$

which implies that $\phi(\beta) = 0$, a contradiction. Thus

$$\limsup_{n\to+\infty} a_n = 0$$

and hence

$$\lim_{n\to+\infty} d\left(x_n, x_{n+2}\right) = 0. \tag{3.21}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n,m\to+\infty} d\left(x_n, x_m\right) = 0$ for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 2.4, there is a $\varepsilon > 0$ such that for an integer k there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$ such that

i) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon$,

ii) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \leq s\varepsilon$,

iii) $\varepsilon \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \leq s\varepsilon$,

vi) $\frac{\varepsilon}{s} \leq \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) \leq \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) \leq s^2\varepsilon$.

Setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.14), we have

$$\begin{aligned}7M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right) = \max\{&d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + \varphi\left(x_{m_{(k)}}\right) + \varphi\left(x_{m_{(k)}}\right),\\ &d\left(x_{m_{(k)}}, x_{m_{(k)}+1}\right) + \varphi\left(x_{m_{(k)}}\right) + \varphi\left(x_{m_{(k)}+1}\right), d\left(x_{n_{(k)}}, x_{n_{(k)}+1}\right) + \varphi\left(x_{n_{(k)}}\right) + \varphi\left(x_{n_{(k)}+1}\right)\}.\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.15), (3.16) and (iii) of Lemma 2.4, we have

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right) \leq s\varepsilon. \tag{3.22}$$

Now, taking the upper limit as $k \to +\infty$ in (3.14), using (3.17), (3.18), (3.22) and using the properties of ψ and ϕ, we have

$$\begin{aligned}\psi(s\varepsilon) = \psi\left(s^2\frac{\varepsilon}{s}\right) &\leq \limsup_{n\to+\infty} \psi\left[s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right]\\ &\leq \limsup_{n\to+\infty} \psi\left[s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + \varphi\left(x_{m_{(k)+1}}\right) + \varphi\left(x_{n_{(k)+1}}\right)\right]\\ &\leq \limsup_{n\to+\infty} \phi\left[M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right)\right]\\ &\leq \phi(s\varepsilon)\\ &< \psi(s\varepsilon),\end{aligned}$$

which is a contradiction. Thus

$$\lim_{n,m\to+\infty} d\left(x_m, x_n\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to+\infty} d\left(x_n, z\right) = 0.$$

Since φ is lower semicontinuous, we get

$$\varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) \leq \lim_{n\to+\infty} \varphi(x_n) = 0,$$

which implies

$$\varphi(z) = 0.$$

Now, putting $x = x_n$ and $y = z$ in (3.14), we have

$$\begin{aligned} M(x_n, z, d, T, \varphi) &= \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), \\ &\quad d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(z, Tz) + \varphi(z) + \varphi(Tz)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$\lim_{n\to+\infty} M(x_n, z, d, T, \varphi) = d(z, Tz) + \varphi(Tz). \tag{3.23}$$

Since $x_n \to z$ as $n \to +\infty$, from Lemma 2.3, we conclude that

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\to+\infty} \sup d(Tx_n, Tz) \leq sd(z, Tz),$$

which implies that

$$sd(z, Tz) + \varphi(Tz) \leq \lim_{n\to+\infty} \sup \left(s^2 d(Tx_n, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right).$$

Letting $n \to +\infty$ in (3.14), using (3.23) and the property of ψ and the supper semicontinuity of ϕ, we have

$$\begin{aligned} \psi(sd(z, Tz) + \varphi(Tz)) &\leq \psi\left(\lim_{n\to+\infty} \sup \left(s^2 d(Tx_n, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right)\right) \\ &= \lim_{n\to+\infty} \sup \psi\left(\left(s^2 d(Tx_n, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right)\right) \\ &\leq \lim_{n\to+\infty} \sup \phi[M(x_n, z, d, T, \varphi)] \\ &\leq \phi\left[\lim_{n\to+\infty} M(x_n, z, d, T, \varphi)\right] \\ &< \psi\left[\lim_{n\to+\infty} M(x_n, z, d, T, \varphi)\right] \\ &= \psi(d(z, Tz) + \varphi(Tz)), \end{aligned}$$

which implies

$$\psi[sd(z, Tz) + \varphi(Tz)] < \psi[d(z, Tz) + \varphi(Tz)].$$

This holds if and only if $\psi(d(z, Tz) + \varphi(Tz)) = 0$ and from the property of ϕ, we have

$$d(z, Tz) + \varphi(Tz) = 0.$$

Hence $d(z, Tz) = 0$ and so $z = Tz$ and $\varphi(Tz) = 0$.

Suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Then $Tz = z$ and $Tu = u$.

Letting $x = z$ and $y = u$ in (3.1), we get

$$\psi\left(s^2 d(Tz, Tu) + \varphi(Tz) + \varphi(Tu)\right) = \psi\left(s^2 d(z, u)\right) \leq \phi(M(z, u, d, T, \varphi)),$$

where

$$\begin{aligned} M(z, u, d, T, \varphi) &= \max\{d(z, u) + \varphi(z) + \varphi(u), d(z, Tz) + \varphi(z) + \varphi(Tz), d(u, Tu) + \varphi(u) + \varphi(Tu)\} \\ &= d(z, u). \end{aligned}$$

So

$$\psi\left(s^2 d(z, u)\right) \leq \phi(d(z, u)) < \psi(d(z, u)).$$

This is a contradiction. Hence $z = u$, T has a unique fixed point. □

Example 3.10. Let (X,d) be the rectangular b-metric space such that $X = A \cup B$, where $A = \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $B = \left[\frac{1}{2}, 1\right]$ and $d : X \times X \to [0, +\infty[$ is defined by

$$\begin{cases} d(x,y) = d(y,x) \ for\ all\ x,y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(0, \frac{1}{9}\right) = d\left(\frac{1}{5}, \frac{1}{16}\right) = 0,1 \\ d\left(0, \frac{1}{5}\right) = d\left(\frac{1}{5}, \frac{1}{9}\right) = 0,5 \\ d\left(0, \frac{1}{16}\right) = d\left(\frac{1}{9}, \frac{1}{16}\right) = 0,05 \\ d(x,y) = (|x-y|)^2 \ otherwise. \end{cases}$$

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{16} \ if\ x \in \left[\frac{1}{2}, 1\right] \\ 0 \ if\ x \in A. \end{cases}$$

Then $T(x) \in X$ for all $x \in X$. Let

$$\varphi(t) = \begin{cases} t \ if\ t \in [0,1] \\ 3t \ if\ t > 1 \end{cases}$$

$$\phi(t) = \frac{4t}{5}$$

and

$$\psi(t) = \frac{5t}{6}.$$

Then ψ is an altering distance function and φ is a lower semicontinuous function and ϕ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$.
Consider the following possibilities:

Case I: $x, y \in \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$.
Assume that $x \geq y$. Then

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) = \psi\left(9.d(0,0) + \varphi(0) + \varphi()\right) = \psi(0) = 0.$$

Also

$$d(x,y) + \varphi(x) + \varphi(y) = d(x,y) + x + y,$$
$$d(x,Tx) + \varphi(x) + \varphi(Tx) = d(x,0) + x,$$
$$d(y,Ty) + \varphi(y) + \varphi(Ty) = d(y,0) + y$$

and

$$M(x,y,d,T,\varphi) = \max\{d(x,y) + x + y, d(x,0) + x, d(y,0) + y\}.$$

Since $x \geq y$, we have

$$M(x,y,d,T,\varphi) = \max\{d(x,y) + x + y, d(x,0) + x\}.$$

If

$$M(x,y,d,T,\varphi) = d(x,y) + x + y \geq \frac{1}{20},$$

then

$$\phi\left(M(x,y,d,T,\varphi)\right) = \phi\left(d(x,y) + x + y\right) \geq \frac{4}{5}\left(d(x,y) + x + y\right) \geq \frac{4}{5} \cdot \frac{1}{20} \geq 0$$

and so

$$0 = \psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) \leq \phi\left(M(x,y,d,T,\varphi)\right)).$$

If
$$M(x,y,d,T,\varphi)=d(x,0)+x\geq\frac{1}{20},$$
then
$$0=\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\phi\left(M(x,y,d,T,\varphi)\right)).$$
If $x<y$, then
$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(y,0)+y\}.$$
If
$$M(x,y,d,T,\varphi)=d(x,y)+x+y\geq\frac{1}{20}+\frac{1}{16}=\frac{9}{80},$$
then
$$\phi\left(M(x,y,d,T,\varphi)\right)=\phi\left(d(x,y)+x+y\right)\geq\frac{9}{80}\cdot\frac{4}{5}=\frac{9}{100}$$
and so
$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\phi\left(M(x,y,d,T,\varphi)\right).$$
If
$$M(x,y,d,T,\varphi)=d(y,0)+y,$$
then
$$d(y,0)+y\geq\frac{9}{80}$$
since $x<y$ and $0<y$. So
$$\phi\left(M(x,y,d,T,\varphi)\right)=\phi\left(d(y,0)+y\right)\geq\frac{9}{100}.$$
Thus
$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\phi\left(M(x,y,d,T,\varphi)\right).$$

Case II: $x\in\left\{0,\frac{1}{5},\frac{1}{9},\frac{1}{16}\right\}$ and $y\in\left[\frac{1}{2},1\right]$. This implies $x<y$. Then
$$\psi\left(s^2\left(d\left(Tx,Ty\right)+\varphi(Tx)+\varphi(Ty)\right)\right)=\frac{5}{6}\left[9d\left(\frac{1}{16},0\right)+\frac{1}{16}\right]=\frac{41}{91}.$$
Also
$$d(x,y)+\varphi(x)+\varphi(y)=(x-y)^2+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d(x,0)+x,$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}$$
and
$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d(x,0)+x,d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}$$
Since $x<y$, we have
$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}.$$
If
$$M(x,y,d,T,\varphi)=d(x-y)^2+x+y\geq\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{5}\right)^2=\frac{59}{100},$$
then
$$\phi\left(M(x,y,d,T,\varphi)\right)\geq\frac{4}{5}\left(d(x,y)^2+x+y\right)\geq\frac{4}{5}\cdot\frac{59}{100}=\frac{59}{125}$$
and so
$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\phi\left(M(x,y,d,T,\varphi)\right).$$
If
$$M(x,y,d,T,\varphi)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\geq\frac{1}{2}+\frac{1}{16}+\left(\frac{1}{2}-\frac{1}{16}\right)^2=\frac{193}{256},$$

then

$$\phi\left(M(x,y,d,T,\varphi)\right) \geq \frac{4}{5}\left(d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\right) \geq \frac{4}{5}\cdot\frac{193}{256}=\frac{193}{320}$$

and so

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \phi\left(M(x,y,d,T,\varphi)\right).$$

Case III: $y \in \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $x \in \left[\frac{1}{2}, 1\right]$. By a similar method to Case II, we deduce that

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case IV: $x, y \in \left[\frac{1}{2}, 1\right]$.

If $x \geq y$, then

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)=\psi\left(9d\left(\frac{1}{16},\frac{1}{16}\right)+\varphi\left(\frac{1}{16}\right)+\varphi\left(\frac{1}{16}\right)\right)=\frac{5}{48}.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y)=(x-y)^2+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d\left(x,\frac{1}{16}\right)+x+\frac{1}{16},$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}$$

and

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y, d\left(x,\frac{1}{16}\right)+x+\frac{1}{16}, d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}.$$

Since $x \geq y$, we have

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y, d\left(x,\frac{1}{16}\right)+x+\frac{1}{16}\}.$$

If

$$M(x,y,d,T,\varphi)=(x-y)^2+x+y \geq 1,$$

then

$$\frac{5}{48} \leq \frac{4}{5} \leq \phi\left(M(x,y,d,T,\varphi)\right).$$

If

$$M(x,y,d,T,\varphi)=d\left(x,\frac{1}{16}\right)+x+\frac{1}{16} \geq d\left(\frac{1}{2},\frac{1}{16}\right)+\frac{1}{2}+\frac{1}{16}=\frac{193}{256},$$

then

$$\frac{5}{48} \leq \frac{4}{5}\cdot\frac{193}{256}=\frac{193}{320} \leq \phi\left(M(x,y,d,T,\varphi)\right).$$

If $x, y \in A$ and $x < y$, then

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y, d\left(y,\frac{1}{16}\right)+y\}.$$

By a similar method to the condition $x \leq y$, we have

$$\frac{5}{48} \leq \frac{4}{5}\cdot\frac{193}{256}=\frac{193}{320} \leq \phi\left(M(x,y,d,T,\varphi)\right).$$

Hence

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \leq \phi\left(M(x,y,d,T,\varphi)\right).$$

Thus all the conditions of Theorem 3.9 are satisfied and 0 is the unique fixed point of T.

4 Conclusion

In this paper, inspired by the concept of generalized weakly contractive mappings in metric spaces, we introduced the concept of generalized weakly contractive mappings in rectangular b-metric spaces to study the existence of fixed point for the mappings in this spaces. Furthermore, we provided some useful examples.

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