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Relative uniquely remotal centers in normed linear spaces

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Abstract

In this paper, we consider "**uniquely remotal center**" and "**relative uniquely remotal center**" in normal spaces. We define relative sun sets and sunrise sets in normal spaces. We show that every uniquely remotal set has zero Chebyshev radius and every remotal set in strictly convex space has at most a singleton Chebyshev center.

Keywords: Uniquely remotal centers, Relative uniquely remotal centers, Chebyshev centers, Relative Chebyshev centers, Sun sets, Sunrise sets, Relative sunrise sets
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1 Introduction

Approximation theory, which many consists of theory nearest points (best approximation) and theory of farthest points (more approximation) is and old and rich branch of analysis, theory of relative Chebyshev centers starting in 1982, by D. Amir [2].

In this paper we define uniquely remotal centers and relative uniquely remotal centers about best approximation. At first, we express some definitions Amir [3].

Let A be a bounded set in the normed space X, and $G \subset X$ be an arbitrary set. For $x \in X$, we denote

$$r(x, A) = \inf\{r : \ A \subseteq B(x, r)\},$$

where

$$B(x, r) = \{y \in X : \ \|y - x\| \leq r\}.$$

and define the relative Chebyshev radius of A on G by

$$r(G, A) = \inf\{r(x, A), \ x \in G\}.$$

Denote finally the set of relative Chebyshev centers of A in G by $Z(G; A)$, i.e.,

$$Z(G; A) = \{C_A \in G, \ r(C_A, A) = r(G, A)\}.$$

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Lemma 1.1. Let $(X, \|\cdot\|)$ be an normed space, A a bounded set in X and $x \in X$. Then

$$r(x, A) = \sup_{y \in A} \|x - y\| := \delta(x, A).$$

Proof . We put $R = \{r : \ A \subseteq B(x, r)\}$, $a = \inf R$, and $S = \{\|x - y\|, \ y \in A\}$. If $r \in R$, $y \in A$, then $\|x - y\| \leq r$. therefore $\|x - y\| \leq \inf_{r \in R} r$. Therefore $a = \inf R$ is upper bound of S.

If b is an another upper bound of S. Then

$$\forall y \in A : \ \|x - y\| \leq b.$$

It follows $r(x, A) \leq b$. Therefore $r(x, A) = \sup_{y \in A} \|x - y\|$. □

In the special case, where $G = X$, we speak about the Chebyshev redius $r(A)$ and the Chebyshev center $Z(A)$.

If $A = \{x\}$ is a singleton, then $r(x, A) = \|x - y\|$, $r(G, A) = \operatorname{dist}(y, G)$ the distance from y to G, and $Z(G, A)$ in the metric projection (or set of best approximation) of y on to G.

We put

$$F_A(x) = \{y \in A : \ \|x - y\| = \delta(x, G)\}.$$

If for $x \in X$, the set $F_A(x)$ is non-empty, we say that A is remotal. If for $x \in X$, the set $F_A(x)$ is singleton, we say that A is uniquely remotal.

Definition 1.2. Let $(X, \|\cdot\|)$ be a normed space, A a bounded subset of X and G a subset of X. Put

$$P_A(x) = \left\{ y \in A : \ \|x - y\| = \inf_{y \in A} \|x - y\| \right\}.$$

We set $d(x, Al.) := \inf_{y \in A} \|x - y\|$.

If for $x \in X$, the set $P_A(x)$ is non-empty, we say that A is proximinal.

If for $x \in X$, the set $P_A(x)$ is a singulet on set, we say that A is Chebyshev. Suppose

$$d(G; A) = \sup_{x \in G} d(x, A).$$

We denote the set of relative uniquely remotal centers of A in G by $U(G; A)$, i.e.,

$$U(G; A) = \{f_A \in G : \ d(G, A) = d(f_A, A)\}.$$

Suppose $B^c(x, r) = \{y \in X : \ \|y - x\| \geq r\}$.

Lemma 1.3. Let $(X, \|\cdot\|)$ be a normed linear space, A a bounded subset of X and $x \in X \setminus \bar{A}$. Then

$$d(x, A) = \sup \{r : \ A \subseteq B^c(x, r)\}.$$

Proof . We know that for $y \in A$

$$\|y - x\| \geq d(x, A).$$

Therefore $d(x, A) = \{r : \ A \subseteq B^c(x, r)\}$.

If $r \in \{r, \ A \subseteq B^c(x, r)\}$, then $A \subseteq B^c(x, r)$.

Therefore for each $y \in A$,

$$\|y - x\| \geq r.$$

It follows that

$$d(x, A) = \inf_{y \in A} \|y - x\| \geq r$$

therefore $d(x, A)$ is an upper bound for $\{r : \ A \subseteq B^c(x, r)\}$. It follows that

$$d(x, A) = \sup\{r : \ A \subseteq B^c(x, r)\}.$$

□

Example 1.4. Suppose that $X = \mathbb{R}$ with Euclidean norm, $A = (0, 1)$ and $G = [3, 4]$. Then for $x \in G$

$$1 \le d(x, A) \le 3, \quad d(G, A) = 3, \quad f_A = 4$$
$$3 \le \delta(x, A) \le 4, \quad r(G, A) = 3, \quad c_A = 3$$

Example 1.5. Suppose that $X = \mathbb{R}^3$ with Euclidean norm,

$$A = \{(x, y, z) \in X : \ x^2 + y^2 \le 1, \ 0 \le z \le 2\},$$
$$G = \{(x, y, z) \in X : \ x^2 + y^2 \le 2, \ 0 \le z \le 2\}.$$

For each $(x, y, z) \in A$

$$0 \le d((x, y, z), A) \le 1, \quad d(G, A) = 2,$$
$$0 \le \delta((x, y, z), A) \le 2\sqrt{2}, \quad r(G, A) = 1,$$

and

$$Z(G, A) = \{(x, y, z) : \ x^2 + y^2 = 2, \ z = 0\},$$
$$U(G, A) = \{(x, y, z) : \ x^2 + y^2 = 2, \ 0 \le z \le 2\}.$$

In sequal we have two definition.

Definition 1.6. Let $(X, \|\cdot\|)$ be a normed linear space, A a Chebyshev bounded subset of X and G a convex subset of X, $A \subseteq G$, A is called relative sun in G, if $x \in G$ and $P_A(x) = g$, then for every $0 < \lambda < 1$

$$P_A(\lambda x + (1 - \lambda)g) = g.$$

Definition 1.7. Let $(X, \|\cdot\|)$ be a normed space, A uniquely remotal subset of X, G a convex subset of X and $A \subseteq G$, A is called a relative sunrise in G, if $x \in X$ and $F_A(x) = g$, then for every $0 < \lambda < 1$

$$F_A(\lambda x + (1 - \lambda)g) = g.$$

Example 1.8. Suppose $(X, \|\cdot\|)$ is a normed space $A = \{x : \ \|x\| = 1\}$, $G = \{x \in X : \ \|x\| \le 2\}$ and $x \in G - A$. Then

$$0 \le d(x, A) \le 1, \quad d(G, A) = 1,$$
$$2 \le \delta(x, A) \le 2, \quad r(G, A) = 2,$$

and

$$Z(G, A) = \{x \in G : \ \|x\| = 1\} = \{0\}, \quad \delta(x, A) = |1 + \|x\||,$$
$$U(G, A) = \{x \in G : \|x\| = 1\} = \{0\}, \quad d(x, A) = |1 - \|x\||.$$

Note that, if $x \in G \setminus A$, $g \in A$ and $P_A(x) = g$, then

$$P_A(\lambda x + (1 - \lambda)g) = g \quad \text{for } 0 \le \lambda \le 1.$$

Also, if $x \in G \setminus A$, $g \in A$ and $F_A(x) = g$, then $F_A(\lambda x + (1 - \lambda)g) = g$ for $\lambda \ge 0$.

Now, if $x \in G \setminus A$, $0 \le \lambda \le 1$ and $g = P_A(x)$, then

$$\frac{\lambda x + (1 - \lambda)g}{\|\lambda x + (1 - \lambda)g\|} = \frac{\lambda x + (1 - \lambda)\frac{x}{\|x\|}}{\left\|\lambda x + (1 - \lambda)\frac{x}{\|x\|}\right\|} = \frac{x}{\|x\|} = g.$$

Then $P_A(\lambda x + (1 - \lambda)g) = g$.

Also, if $x \in G \setminus A$, $g \in A$ and $\lambda \ge 0$, $g = F_A(x)$. Then

$$-\frac{\lambda x + (1 - \lambda)g}{\|\lambda x + (1 - \lambda)g\|} = -\frac{\lambda x + (1 - \lambda)\frac{x}{\|x\|}}{\left\|\lambda x + (1 - \lambda)\frac{x}{\|x\|}\right\|} = -\frac{x}{\|x\|} = -g.$$

Therefore

$$F_A(\lambda x + (1 - \lambda)g) = -g.$$

That A is a relative sun in G and an relative sunrise in G.

Definition 1.9. Let $(X, \|\cdot\|)$ be a normed space we say that X is strictly convex, if $x, y \in X$, $\|x\| < 1$, $\|y\| < 1$, then

$$\left\| \frac{x+y}{2} \right\| < 2.$$

2 Existence and uniqueness uniquely remotal cemter and Chebishev center

In this section, we consider existence and uniqueness of Chebyshev center and uniquely remotal center.

Theorem 2.1. Let $(X, \|\cdot\|)$ be a normed space and $A \subseteq X$ a bounded subset and $G \subset X$. Then for $x \in G$

$$d(x, A) - d(G, A) \leq \text{diam}(A).$$

Proof . For $x, y \in G$ and $z \in A$

$$d(x, z) \leq d(x, y) + d(y, z),$$

therefore

$$d(x, A) - d(y, z) \leq d(x, z) - d(y, z) \leq d(x, y).$$

It is follow that

$$d(x, A) \leq d(x, y) + d(y, A) \leq d(x, y) + d(G, A).$$

□

Lemma 2.2. Let $(X, \|\cdot\|)$ be a normed linear space, $A, G \subseteq A$ and $x, y \in G \setminus A$. Then

1. $|d(x, A) - d(y, A)| \leq d(x, y)$
2. $|\delta(x, A) - \delta(y, A)| \leq d(x, y)$.

Proof .

1. For $z \in A$, $d(x, z) \leq d(x, y) + d(y, z)$. Therefore

   $$d(x, A) \leq d(x, y) + d(y, z),$$

   and

   $$d(x, A) \leq d(x, y) + d(y, A).$$

   Also

   $$d(y, A) \leq d(x, y) + d(x, A),$$

   it is follows that

   $$|d(x, A) - d(y, A)| \leq d(x, y).$$

2. For $z \in A$,

   $$d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + \delta(y, A),$$

   and

   $$\delta(x, A) = \sup_{z \in A} d(x, z) \leq d(x, y) + d(y, A),$$

   to similar

   $$\delta(x, A) \leq d(x, y) + d(x, A),$$

   therefore

   $$|\delta(x, A) - \delta(y, A)| \leq d(x, y).$$

□

Corollary 2.3. Let $(X, \|\cdot\|)$ be a normed linear space and $A, G \subseteq X$. If A is compact, then $Z(G, A)$ and $U(G, A)$ are non-empty.

Proof . The maps $d(\cdot, A)$ and $\delta(\cdot, A)$ are continuous on G. □

Theorem 2.4. Let $(X, \|\cdot\|)$ be a normed space, A a bounded subset of X and G a convex subset of X, then

1. $Z(G, A)$ is convex.
2. $U(G, A)$ is convex.

Proof .

1. Suppose $x, y \in Z(G, A)$, $0 \leq \lambda \leq 1$. Then

$$\delta(y, A) = \delta(G, A) \quad \text{and} \quad \delta(x, A) = \delta(G, A).$$

We have

$$\begin{aligned}\delta(G, A) &\leq \delta(\lambda x + (1-\lambda)y, A) \\ &= \sup_{z \in A} \delta(\lambda x + (1-\lambda)y, z) \\ &\leq \lambda \sup_{z \in A} \delta(x, z) + (1-\lambda) \sup_{z \in A} \delta(y, z) \\ &\leq \lambda\delta(x, A) + (1-\lambda)\delta(y, A) \\ &= \lambda\delta(G, A) + (1-\lambda)\delta(G, A) \\ &= \delta(G, A).\end{aligned}$$

Therefore

$$\delta(\lambda x + (1-\lambda)y, A) = \delta(G, A),$$

and

$$\lambda x + (1-\lambda)y \in Z(G, A).$$

2. Suppose $x, y \in U(G, A)$, $0 \leq \lambda \leq 1$, then

$$d(x, A) = d(G, A) \quad \text{and} \quad d(y, A) = d(G, A).$$

Therefore

$$\begin{aligned}d(G, A) &\geq d(\lambda x + (1-\lambda)y, A) \\ &= \inf_{z \in A} d(\lambda x + (1-\lambda)y, z) \\ &\geq \inf_{z \in A} d(\lambda x, z) + \inf_{z \in A} d((1-\lambda)y, z) \\ &= \lambda d(x, A) + (1-\lambda)d(y, A) \\ &= \lambda d(G, A) + (1-\lambda)d(G, A) \\ &= d(G, A).\end{aligned}$$

Therefore

$$d(\lambda x + (1-\lambda)y, A) = d(G, A),$$

and

$$\lambda x + (1-\lambda)y \in U(G, A).$$

□

Example 2.5. Suppose $X = \{f \in C[0,1] : \ \|f\| \leq |f(0)|\}$ with sup norm is a normed linear space. Put $A = \{f \in X : \ f(0) = 0\}$ and $G = X$. Then A is a proximinal set and is not a Chebyshev set. Because for $f \in X$, we set $g = f \pm f(0)$, then $g \in P_A(f)$. For each $h \in A$,

$$\begin{aligned}\|f-h\| &= \sup_{x\in[0,1]} |f(x)-h(x)| \\ &\leq |f(0)-h(0)| \\ &= |f(0)|\end{aligned}$$

therefore

$$d(f,A) = |f(0)| = \|f-g\|$$

it is follow that

$$d(G,A) = \sup_{f\in X} |f(0)|.$$

For $n \geq 1$, we set $f = n$, then $f \in X$ and $d(G,A) \geq \sup_{n\geq 1} n$ and

$$d(G,A) = \infty.$$

Theorem 2.6. Let $(X, \|\cdot\|)$ be a normed space, A a Chebyshev subset of X and $G = X$. If G is a sun set, then for every $0 \leq \lambda \leq 1$ and $x \in X$,

$$d(G,A) \geq \lambda d(x,A).$$

Proof . Suppose $g \in A$, $x \in X \setminus A$ and $P_A(0) = g$. We know that

$$P_A(\lambda x + (1-\lambda)g) = g, \quad 0 \leq \lambda \leq 1.$$

It is follows that

$$\begin{aligned}d(\lambda x + (1-\lambda)g, A) &= \|\lambda x + (1-\lambda)g - g\| \\ &= \lambda\|x-g\| \\ &= \lambda d(x,A),\end{aligned}$$

and $d(G,A) \geq \lambda d(x,A)$. □

Theorem 2.7. Let $(X, \|\cdot\|)$ be a normed linear space, A a uniquely remotal subset of X and $G = X$, then for every $x \in G$, $r(G,A) = 0$.

Proof . Suppose $g \in A$, $x \in X \setminus A$, $F_A(x) = g$, we know that

$$F(\lambda x + (1-\lambda)g, A) = g.$$

It is follows that

$$\begin{aligned}\delta(\lambda x + (1-\lambda)g, A) &= \|\lambda x + (1-\lambda)g - g\| \\ &= \lambda\|x-g\| \\ &= \lambda\delta(x,G),\end{aligned}$$

therefore $r(G,A) \leq \lambda\delta(x,G)$, $(\lambda \geq 0)$ and $r(G,A) = 0$. □

Theorem 2.8. Let $(X, \|\cdot\|)$ be a normed strictly convex space, $A \subseteq X$ a bounded subset of X and G a convex subset of X.

1. If A is remotal, then $Z(G,A)$ is at most singleton.
2. If A is proximinal, then $U(G,A)$ is at most singleton.

Proof .

1. If $x, y \in Z(G, A)$, $x \neq y$. Since $Z(G, A)$ is convex, then
$$\frac{x+y}{2} \in Z(G, A).$$
It is follows that
$$\delta\left(\frac{x+y}{2}, A\right) = \inf_{x \in G} \delta(x, A) = r(G, A).$$
Since A is remotal, there exists $u \in A$ such that
$$\left\| u - \frac{x+y}{2} \right\| = r(G, A).$$
Also
$$\delta(x, A) = \sup_{x \in G} \delta(x, A) \leq r(G, A),$$
and
$$\delta(y, A) = \sup_{x \in G} \delta(x, A \leq r(G, A).$$
Since X is strictly convex, it follows that
$$\delta\left(\frac{x+y}{2}, A\right) < r(G, A).$$
That is a contradicts.
2. If $x, y \in U(G, A)$ and $x \neq y$, since $U(G, A)$ is convex, then $\frac{x+y}{2} \in U(G, A)$. It follows that
$$d\left(\frac{x+y}{2}, A\right) = \sup_{x \in G} d(x, A) = d(G; A).$$
Since A is proximinal, there exists $u \in A$ such that
$$\left\| u - \frac{x+y}{2} \right\| = d(G; A).$$
Also
$$d(x, A) = \sup_{x \in G} d(x, A) = d(G; A),$$
and
$$d(y, A) = \sup_{x \in G} d(x, A) = (G; A).$$
Since X is strictly convex, it is follows that
$$d\left(\frac{x+y}{2}, A\right) < d(G, A).$$
This is a contradicts. Therefore $U(G, A)$ is at most singleton.

□

Corollary 2.9. Let A be a remotal subset of a strictly convex normed space X and G is convex subset of X. If $\delta(\cdot, A) : G \to \mathbb{R}^+ \cup \{0\}$ attains its infimum on G. Then $Z(G; A)$ is exactly a singleton.

Corollary 2.10. Let A be a proximinal subset of a strictly convex normed space X and G is convex subset of X. If $d(\cdot, A) : G \to \mathbb{R}^+ \cup \{0\}$ attains its suprimum on G. Then $U(G; A)$ is exactly a singleton.

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