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Convergence analysis and approximation of fixed point of multivalued generalized α-nonexpansive mapping in uniformly convex Banach space

Unwana E. Udofia*, Donatus I. Igbokwe

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

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Abstract

Recently, the authors introduce a four-step iterative algorithm called the UD-iteration scheme (Udofia and Igbokwe [35]). Here we introduce the multivalued version of the UD-iteration scheme and show that it can be used to approximate the fixed points of multivalued contraction and multivalued generalized α-nonexpansive mappings. we prove strong and weak convergence of the iteration scheme to the fixed point of multivalued generalized α-nonexpansive mapping. We also prove that the scheme is Υ-stable and Data dependent. Convergence analysis shows that the multivalued UD-iteration scheme has a faster rate of convergence for multivalued contraction and multivalued generalized α-nonexpansive mappings than some well-known existing iteration schemes in the literature.

Keywords: Uniformly convex Banach space, Multivalued generalized α-nonexpansive Mapping, Convergence, Data dependence, Stability
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1 Introduction

Over the past four decades, researchers have developed various iteration schemes to approximate the fixed points of different evolving classes of single-valued and multivalued operators. The existence, stability and convergence results of such research findings abound in literature.

In 2008, Suzuki [32] introduced the class of nonexpansive mappings known as generalized nonexpansive mapping (i.e, a mapping satisfying condition (C) defined in (2.5)) for single-valued mappings and proved some existence and convergence theorems for this class of operator in a Banach space.

In 2011, Eslamain and Abkar [2] introduced the multivalued Suzuki's generalized nonexpansive mapping defined in (2.10) and proved a fixed point theorem satisfying the modified Suzuki condition amongst other results in a uniformly convex Banach space.

Also, Aoyama and Kohsaka [5] introduced the class of α-nonexpansive mapping defined in (2.4), and obtained a fixed point theorem for this class of operator in a uniformly convex Banach space.

*Corresponding author
Email addresses: unwanaudofia.aksu@yahoo.com (Unwana E. Udofia), diigbokwe@yahoo.com (Donatus I. Igbokwe)

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In 2017, Pant and Shukla [25] introduced the class of generalized α-nonexpansive mapping defined in (2.6) which contains the Suzuki type generalized mappings. (If $\alpha = 0$, generalized α-nonexpansive mapping reduces to Suzuki type generalized mappings), and proved some existence and convergence results for this class of operator in a uniformly convex Banach space.

In 2109, motivated by the work of [2, 25], Iqbal *et al* [17] introduced the multivalued generalized α-nonexpansive mapping defined in (2.11) and obtained some convergence and stability results in a uniformly convex Banach space. They also proposed a multivalued modified Piri *et al.* iteration process defined by:
Let $\Upsilon : \zeta \to \zeta$, and ζ a non empty, closed and convex subset of a real Banach space Ω.

$$\begin{cases} \xi_1 \in \zeta, \\ \xi_{n+1} = (1-\alpha_n)v_n + \alpha_n t_n \\ y_n \in P_\Upsilon(z_n), \\ z_n = (1-\beta_n)\xi_n + \beta_n s_n, \end{cases} \quad \forall n \geq 1. \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, and $s_n \in P_\Upsilon(\xi_n),\ t_n \in P_\Upsilon(y_n),\ v_n \in P_\Upsilon(z_n)$. They proved that the multivalued version of the modified Piri et al. iteration process (1.1) converges faster than iteration processes of Mann [21], Ishikawa [18], and Thakur et al. [33], (see definition in Table 1).

Interestingly, solutions of nonlinear equations can be rigorous and sometimes very difficult to obtain. One of the easiest ways to solve such problems is to transform into a fixed point problem. Let $\Upsilon : \zeta \to \zeta$, and ζ a non empty, closed and convex subset of a real Banach space Ω, then

$$\Upsilon\xi = \xi, \tag{1.2}$$

where $\xi \in \zeta$, is a fixed point problem. The solution of the fixed point equation (1.2) of the mapping Υ is considered as the solution to the nonlinear problem.

In furtherance of intense research in this regard, a number of iterative processes have been introduced to study the fixed point of various single-valued and multivalued operators, such as:
Mann [21], Ishikawa [18], Noor [23], Agarwal et al. [3] (S-iteration), Abbas and Nazir [1], Thakur et al. [33], Piri *et al.* and many others.

Recently, Udofia and Igbokwe [35] introduced the UD-iteration process defined as follows: Let $\Upsilon : \zeta \to \zeta$, and ζ a non empty, closed and convex subset of a real Banach space Ω. Define the four step UD-iteration process by:

$$\begin{cases} \xi_1 \in \zeta, \\ \xi_{n+1} = (1-\gamma_n)\Upsilon y_n + \gamma_n \Upsilon z_n \\ y_n = (1-\beta_n)\Upsilon\xi_n + \beta_n z_n, \\ z_n = \Upsilon w_n, \\ w_n = (1-\alpha_n)\xi_n + \alpha_n \Upsilon\xi_n \end{cases} \quad \forall n \in \mathbb{N}. \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$. They proved that UD-iteration process converges faster than some well known iteration process including Mann [21], Ishikawa [18], Noor [23], Agarwal et al. [3] (S-iteration), Abbas and Nazir [1], Thakur et al. [33] for contraction and generalized α-nonexpansive mappings.

Motivated by the work of Iqbal et al [17], we introduce the multivalued version of the UD-iteration process and show that the scheme can be used to approximate the fixed points of multivalued generalized α-nonexpansive mapping in the framework of uniformly convex Banach space. We propose some existence and convergence theorems. Furthermore, we show that the proposed multivalued UD-iteration scheme is data dependent and Υ-stable. Finally, analytically and with concrete numerical examples, we show that the multivalued UD-iteration scheme converges faster than the multivalued modified Piri et al. iteration process introduced by Iqbal et al [17] and many other multivalued iteration processes in literature for multivalued contraction mappings and multivalued generalized α-nonexpansive mapping.

Shahzad and Zegeye [28] presented the set $P_\Upsilon = \{v \in \Upsilon u : d(u, \Upsilon u) = \|u - v\|\}$ for a multivalued mapping, $\Upsilon : \zeta \to P(\zeta)$ and showed that Mann and the Ishikawa iteration processes for multivalued mappings are well defined. Outlined below in Table (1) is the single-valued and multivalued versions of the schemes adopted for the purpose of this research work. Let $\Upsilon : \zeta \to \zeta$, and ζ a non empty, closed and convex subset of a uniformly convex Banach space Ω, then we have the following: where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, and $s_n \in P_\Upsilon(\phi_n),\ t_n \in P_\Upsilon(y_n),\ v_n \in P_\Upsilon(z_n)$.

Table 1: Table showing Single-valued and Multivalued Iteration Schemes

Iteration	Single Valued Version	Multivalued Version
MANN	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n s_n$
ISHIKAWA	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Upsilon y_n$ $y_n = (1-\beta_n)\xi_n + \beta_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n t_n$ $y_n = (1-\beta_n)\xi_n + \beta_n s_n$
NOOR	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Upsilon y_n$ $y_n = (1-\beta_n)\xi_n + \beta_n \Upsilon z_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n t_n$ $y_n = (1-\beta_n)\xi_n + \beta_n v_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n s_n$
ARGAWAL (S-ITERATION)	$\xi_{n+1} = (1-\alpha_n)\Upsilon \xi_n + \alpha_n \Upsilon y_n$ $y_n = (1-\beta_n)\xi_n + \beta_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)\Upsilon \xi_n + \alpha_n t_n$ $y_n = (1-\beta_n)\xi_n + \beta_n s_n$
ABBAS-NAZIR	$\xi_{n+1} = (1-\alpha_n)\Upsilon z_n + \alpha_n \Upsilon y_n$ $y_n = (1-\beta_n)\Upsilon \xi_n + \beta_n \Upsilon z_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)v_n + \alpha_n t_n$ $y_n = (1-\beta_n)s_n + \beta_n v_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n s_n$
THAKUR	$\xi_{n+1} = (1-\alpha_n)\Upsilon z_n + \alpha_n \Upsilon y_n$ $y_n = (1-\beta_n)z_n + \beta_n \Upsilon z_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)v_n + \alpha_n t_n$ $y_n = (1-\beta_n)z_n + \beta_n v_n$ $z_n = (1-\gamma_n)\xi_n + \gamma_n s_n$
Modified PIRI-et-al	$\xi_{n+1} = (1-\alpha_n)\Upsilon z_n + \alpha_n \Upsilon y_n$ $y_n = \Upsilon z_n$ $z_n = (1-\beta_n)\xi_n + \beta_n \Upsilon \xi_n$	$\xi_{n+1} = (1-\alpha_n)v_n + \alpha_n t_n$ $y_n \in P_{\Upsilon}(z_n)$ $z_n = (1-\beta_n)\xi_n + \beta_n s_n$
.		

2 Preliminaries

Throughout this paper, let ζ be a nonempty, closed and convex subset of a uniformly convex Banach space Ω and $\Upsilon : \zeta \to \zeta$. Let $F(\Upsilon) = \{u \in \Omega : \Upsilon u = u\}$ denote the set of all fixed points of the mapping Υ.

A mapping Υ is called;

(1) *contraction* if there exists a constant $\delta \in [0, 1)$ such that

$$\|\Upsilon u - \Upsilon v\| \le \delta \|u - v\|, \ \forall \, u, v \in \zeta. \tag{2.1}$$

(2) *nonexpansive* if

$$\|\Upsilon u - \Upsilon v\| \le \|u - v\|, \ \forall \, u, v \in \zeta. \tag{2.2}$$

(3) *quasi-nonexpansive* if there exists $F(\Upsilon) \neq \emptyset$ such that

$$\|\Upsilon u - p\| \le \|u - p\|, \tag{2.3}$$

for all $p \in F(\Upsilon)$ and $u \in \zeta$.

(4) α*-nonexpansive* if for some $\alpha < 1$, and for all $u, v \in \zeta$

$$\|\Upsilon u - \Upsilon v\|^2 \le \alpha \|\Upsilon u - v\|^2 + \alpha \|\Upsilon v - u\|^2 + (1 - 2\alpha)\|u - v\|^2. \tag{2.4}$$

(5) *Suzuki's generalized nonexpansive* [32] if Υ satisfy condition (C), that is $\frac{1}{2}\|u - \Upsilon u\| \le \|u - v\|$, implies that

$$\|\Upsilon u - \Upsilon v\| \quad \le \quad \|u - v\|, \ \forall \, u, v \in \zeta. \tag{2.5}$$

(6) *generalized* α*-nonexpansive* [25], if there exists $\alpha \in [0, 1)$, such that $\frac{1}{2}\|u - \Upsilon u\| \le \|u - v\|$ implies that

$$\|\Upsilon u - \Upsilon v\|^2 \quad \le \quad \alpha \|\Upsilon u - v\|^2 + \alpha \|\Upsilon v - u\|^2 + (1 - 2\alpha)\|u - v\|^2, \tag{2.6}$$

for all $u, v \in \zeta$. If $\alpha = 0$, then (2.6) reduces to (2.5).

A set ζ is called proximinal if for each $u \in \Omega$ there exists an element $v \in \zeta$ such that

$$d(u, v) = d(u, \zeta) = \inf\{d(u, v) : v \in \zeta\}.$$

A point $p \in \zeta$ is called a fixed point of $\Upsilon : \zeta \to P(\zeta)$ if $p \in \Upsilon p$, where $P(\zeta)$ is the set of all subsets of ζ. The set of all fixed points of Υ is denoted by $F(\Upsilon)$. Let $P_{cb}(\zeta)$, $P_{cp}(\zeta)$, $P_{px}(\zeta)$ and $P(\zeta)$ denote the families of nonempty closed and bounded subsets, compact subsets, proximinal subsets, and all subsets of ζ, respectively.

Let $X, Y \in P_{cb}(\zeta)$. A mapping $\mathcal{H} : P_{cb}(\zeta) \times P_{cb}(\zeta) \to \Re^+$ defined by

$$\mathcal{H}(X,Y) = \max\left\{ \sup_{u\in X} d(u,Y),\ \sup_{v\in Y} d(v,X) \right\},$$

is called the Hausdorff-Pompeiu metric on $P_{cb}(\zeta)$ (or on$P_{cp}(\zeta)$) induced by d. A multivalued mapping $\Upsilon : \zeta \to P(\zeta)$ is said to be;

(1) *contraction* if there exists a constant $\delta \in [0,1)$ such that for any $u, v \in \zeta$;

$$\mathcal{H}(\Upsilon u, \Upsilon v) \le \delta \|u - v\|. \tag{2.7}$$

(2) *nonexpansive* if for any $u, v \in \zeta$

$$\mathcal{H}(\Upsilon u, \Upsilon v) \le \|u - v\|. \tag{2.8}$$

(3) *quasi-nonexpansive* if there exists $F(\Upsilon) \neq \emptyset$ such that

$$\mathcal{H}(\Upsilon u, \Upsilon p) \le \|u - p\|, \tag{2.9}$$

for all $p \in F(\Upsilon)$ and $u \in \zeta$.

(4) *Suzuki's generalized nonexpansive* [2] if Υ satisfy condition (C), that is $\frac{1}{2} d(u, \Upsilon u) \le \|u - v\|$, implies that

$$\mathcal{H}(\Upsilon u, \Upsilon v) \quad \le \quad \|u - v\|, \quad \text{for any } u, v \in \zeta \tag{2.10}$$

(5) *generalized α-nonexpansive* [25], if there exists $\alpha \in [0,1)$, such that for any $u, v \in \zeta$, $\frac{1}{2} d(u, \Upsilon u) \le \|u - v\|$ implies that

$$\mathcal{H}(\Upsilon u, \Upsilon v) \quad \le \quad \alpha d(u, \Upsilon v) + \alpha d(v, \Upsilon u) + (1 - 2\alpha)\|u - v\|, \tag{2.11}$$

The following definitions and lemmas will be useful in proving our main results.

Definition 2.1. A Banach space Ω is said to be:

(i) *Strictly convex* if $\frac{1}{2}\|u + v\| < 1$ for all $u, v \in \Omega$ with $\|u\| = \|v\| = 1$ and $u \neq v$.
(ii) *Uniformly convex* if, for all $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{1}{2}\|u + v\| \le 1 - \delta$, for all $u, v \in \Omega$ with $\|u\| \le 1$, $\|v\| \le 1$ and $\|u - v\| \ge \epsilon$.

Definition 2.2. Let ζ be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space Ω and $\{\xi_n\}$ be a bounded sequence in Ω. For each $\xi \in \Omega$, we define the following:

(i) *Asymptotic radius* of $\{\xi_n\}$ at ξ by $r(\xi, \{\xi_n\}) := \limsup_{n\to\infty} \|\xi_n - \xi\|$.
(ii) *Asymptotic radius* of $\{\xi_n\}$ relative to ζ by $r(\zeta, \{\xi_n\}) := \inf\{r(\xi, \xi_n) : \xi \in \zeta\}$.
(iii) *Asymptotic center* of $\{\xi_n\}$ relative to ζ by $A(\zeta, \{\xi_n\}) := \{\xi \in \zeta : r(\xi, \{\xi_n\}) = r(\zeta, \{\xi_n\})\}$.

Definition 2.3. [24]: A Banach space Ω is said to satisfy the Opial's condition if for any sequence $\{u_k\}$ in Ω, $u_k \rightharpoonup u$, implies that

$$\limsup_{k\to\infty} \|u_k - u\| < \limsup_{k\to\infty} \|u_k - v\|, \quad \text{for all } v \in \Omega \text{ with } v \neq u,$$

or by contrapositive,

$$\limsup_{k\to\infty} \|u_k - v\| \le \limsup_{k\to\infty} \|u_k - u\|.$$

Example 2.4. Examples of spaces that satisfy the Opial's property include:

(1) Every Hilbert space has the Opial property.
(2) The sequence space l_p, $1 \le p < \infty$ have the Opial property.
(3) For uniformly convex Banach spaces, Opial property holds if and only if Delta-convergence coincides with weak convergence.
(4) For every separable Banach space, there exists an equivalent norm that endows it with the Opial property.

However, $L^p[0, \pi]$ space with $1 < p \neq 2$ fails to satisfy the Opial's property.

Definition 2.5. [9] Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers that converge to u and v respectively, and assume that there exists

$$\ell = \lim_{n\to\infty} \frac{\|u_n - u\|}{\|v_n - v\|}. \tag{2.12}$$

Then,

(R_1) if $\ell = 0$, we say that $\{u_n\}$ converges faster to u than $\{v_n\}$ does to v.
(R_2) If $0 < \ell < \infty$, we say that $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence.

Definition 2.6. [9] Let Υ, $\tilde{\Upsilon} : \zeta \to \zeta$ be two operators. We say that $\tilde{\Upsilon}$ is an approximate operator for Υ if for some $\epsilon > 0$, we have

$$\|\Upsilon u - \tilde{\Upsilon} u\| \leq \epsilon, \ \forall\, u \in \zeta. \tag{2.13}$$

Definition 2.7. [14] Let $\{t_n\}$ be any sequence in ζ. Then, an iteration process $\xi_{n+1} = f(\Upsilon, \xi_s)$, converging to fixed point q, is said to be Υ-stable with respect to Υ, if for $\varepsilon_n = \|t_{n+1} - f(\Upsilon, t_n)\|$, for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \varepsilon_n = 0 \ \Leftrightarrow \ \lim_{n\to\infty} t_n = q. \tag{2.14}$$

Lemma 2.8. [27] Let Ω be a uniformly convex Banach space and $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$, for all $n \geq 1$. If $\{f_n\}$ and $\{g_n\}$ are two sequences in Ω such that $\limsup_{n\to\infty} \|f_n\| \leq r$, $\limsup_{n\to\infty} \|g_n\| \leq r$ and $\lim_{n\to\infty} \|\lambda_n f_n + (1-\lambda_n) g_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|f_n - g_n\| = 0$.

Lemma 2.9. [11] Let $\Upsilon : \zeta \to P_{cb}(\zeta)$. A sequence $\{\xi_n\}$ in ζ is called an approximate fixed point sequence (or a.f.p.s) for Υ provided that $d(\xi_n, \Upsilon(\xi_n)) \to 0$ as $n \to \infty$. If $\mathcal{H}(\{\xi_n\}, \Upsilon(\xi_n)) \to 0$, we say that (ξ_n) is a strong a.f.p.s. for Υ.

Lemma 2.10. [20] A multivalued mapping $\Upsilon : \zeta \to P(\zeta)$ is called demiclosed at $v \in \zeta$ if for any sequence $\{f_n\}$ in ζ weakly converges to u and a sequence $g_n \in \Upsilon f_n$ strongly converges to v, then we have $v \in \Upsilon u$.

Lemma 2.11. [17] A multivalued mapping $\Upsilon : \zeta \to P(\zeta)$ is said to satisfy Condition (I) if there exists a continuous nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(u, \Upsilon u) \geq f(d(u, F(\Upsilon)))$ for all $u \in \zeta$.

Lemma 2.12. [30] Let $\{\psi_n\}$ and $\{\phi_n\}$ be nonnegative real sequences satisfying the following inequalities:

$$\psi_{n+1} \leq (1-\sigma_n)\psi_n + \sigma_n \phi_n, \tag{2.15}$$

where $\sigma_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\phi_n \geq 0$ for all $n \in \mathbb{N}$, then

$$0 \leq \limsup_{n\to\infty} \psi_n \leq \limsup_{n\to\infty} \phi_n. \tag{2.16}$$

Lemma 2.13. [39] Let $\{\psi_n\}$ and $\{\phi_n\}$ be nonnegative real sequences satisfying the following inequalities:

$$\psi_{n+1} \leq (1-\sigma_n)\psi_n + \phi_n, \tag{2.17}$$

where $\sigma_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\lim_{n\to\infty} \frac{\phi_n}{\sigma_n} = 0$, then $\lim_{n\to\infty} \psi_n = 0$.

Lemma 2.14. (See Goebel and Reich [13], Theorem 5.1) Let ζ be a bounded closed convex subset of a uniformly convex Banach space Ω. If $\Upsilon : \zeta \to \zeta$ is nonexpansive, then Υ has a fixed point.

Lemma 2.15. (See Goebel and Kirk [12], Lemma 3.4) If ζ is a closed and convex subset of a uniformly convex Banach space Ω and if $\Upsilon : \zeta \to \zeta$ is nonexpansive, then the set $F(\Upsilon))$ of fixed points of Υ is closed and convex.

Lemma 2.16. [40] For any real numbers $q > 1$ and $r > 0$, a Banach space Ω is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with $g(0) = 0$ such that

$$\|tu + (1-t)v\|^q = t\|u\|^q + (1-t)\|v\|^q - \omega(q,t)g(\|u-v\|), \tag{2.18}$$

for all $u, v \in B_r(0) = \{u \in \Omega : \|u\| \le r\}$ and $t \in [0,1]$, where $\omega(q,t) = t^q(1-t) + t(1-t)^q$. In particular, taking $q = 2$ and $t = \frac{1}{2}$

$$\|\frac{u+v}{2}\|^2 \le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}g(\|u-v\|). \tag{2.19}$$

Lemma 2.17. [40] Let $\Upsilon : \zeta \to P_{px}(\zeta)$ and $P_\Upsilon = \{v \in \Upsilon u : d(u, \Upsilon u) = \|u - v\|\}$. Then the following are equivalent:

(1) $u \in F(\Upsilon)$
(2) $P_\Upsilon(u) = \{u\}$
(3) $u \in F(P_\Upsilon)$.
Moreover, $F(\Upsilon) = F(P_\Upsilon)$.

3 Main Result

In this section, we introduce the multivalued UD-iteration scheme and use it to prove some existence and convergence results for multivalued generalized α-nonexpansive mappings in a uniformly convex Banach space. We also analyze the convergence and stability of the scheme for multivalued contraction and multivalued generalized α-nonexpansive mappings.

We now define the multivalued UD-iteration scheme as follows:

Let $\Upsilon : \zeta \to P(\zeta)$, and ζ a non empty subset of a uniformly convex Banach space Ω. Define the four step multivalued UD-iteration process by:

$$\begin{cases} \xi_1 \in \zeta, \\ \xi_{n+1} = (1-\gamma_n)t_n + \gamma_n v_n \\ y_n = (1-\beta_n)s_n + \beta_n z_n, \qquad \forall n \in \mathbb{N}. \\ z_n \in P_\Upsilon(w_n), \\ w_n = (1-\alpha_n)\xi_n + \alpha_n s_n \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$, and $s_n \in P_\Upsilon(\xi_n),\ t_n \in P_\Upsilon(y_n),\ v_n \in P_\Upsilon(z_n)$.

3.1 Existence of Fixed Point for Multivalued Generalized α-nonexpansive Mappings

Lemma 3.1. [17] Let ζ be a nonempty, closed and convex subset of a uniformly convex Banach space Ω and $\Upsilon : \zeta \to P_{cb}(\zeta)$ a generalized α-nonexpansive mappings. Then the following results hold:

(1) For any $u \in \Omega$ and $z \in \Upsilon u, d(z, \Upsilon z) \le \|u - z\|$
(2) For any $u, v \in \Omega$ and $z \in \Upsilon u$ either $\frac{1}{2}d(u, \Upsilon u) \le \|u - v\|$ or $\frac{1}{2}d(z, \Upsilon z) \le \|z - v\|$.

Proof .

(1) Since $\frac{1}{2}d(u, \Upsilon u) \le \|u - z\|$ for any $z \in \Upsilon u$, we obtain

$$\begin{aligned} \mathcal{H}(\Upsilon u, \Upsilon z) &\le \alpha d(z, \Upsilon u) + \alpha d(u, \Upsilon z) + (1-2\alpha)\|u - z\| \\ &= \alpha d(u, \Upsilon z) + (1-2\alpha)\|u - z\|. \end{aligned}$$

Hence,

$$d(z, \Upsilon z) \le \alpha d(z, \Upsilon z) + \alpha\|u - z\| + (1-2\alpha)\|u - z\|.$$

This implies that,

$$d(z, \Upsilon z) \le \|u - z\| \quad \text{for any} \quad z \in \Upsilon u.$$

(2) Assume on the contrary that for any $u, v \in \Omega$ and $z \in \Upsilon u$, we have $\frac{1}{2}d(u, \Upsilon u) > \|u-v\|$ and $\frac{1}{2}d(z, \Upsilon z) > \|z-v\|$. It follows from (1) that

$$\begin{aligned}\|u-z\| &\leq \|u-v\| + \|v-z\| \\ &< \frac{1}{2}d(u, \Upsilon u) + \frac{1}{2}d(z, \Upsilon z) \\ &\leq \frac{1}{2}\|u-z\| + \frac{1}{2}\|u-z\| = \|u-z\|,\end{aligned}$$

a contradiction. Hence the result. □

Lemma 3.2. Let ζ be a nonempty, closed and convex subset of a uniformly convex Banach space Ω and $\Upsilon : \zeta \to P_{cb}(\zeta)$ a generalized α-nonexpansive mappings. Then for any $u, v \in \zeta$ and $z \in \Upsilon u$, we have

$$d(u, \Upsilon v) \leq 3d(u, \Upsilon u) + \|u-v\|. \tag{3.2}$$

Proof . Lets consider the following cases:

Case 1: If $\frac{1}{2}d(u, \Upsilon u) \leq \|u-z\|$ for any $z \in \Upsilon u$ and $d(u, v) = \|u-v\|$, then from Lemma (3.1) we have

$$\begin{aligned}d(u, \Upsilon v) &\leq d(u, \Upsilon u) + \mathcal{H}(\Upsilon u, \Upsilon v) \\ &\leq d(xu, \Upsilon u) + \alpha d(u, \Upsilon v) + \alpha d(v, \Upsilon u) + (1-2\alpha)d(u, v) \\ &\leq d(u, \Upsilon u) + \alpha d(u, \Upsilon v) + \alpha(d(u, v) + d(u, \Upsilon u)) + (1-2\alpha)d(u, v) \\ &\leq 2d(u, \Upsilon u) + \alpha d(u, \Upsilon v) + (1-\alpha)(d(u, v) + d(u, \Upsilon u)) \\ &\leq 2d(u, \Upsilon u) + (1-\alpha)d(u, \Upsilon u) + \alpha d(u, \Upsilon v) + (1-\alpha)d(u, v) \\ &\leq (3-\alpha)d(u, \Upsilon u) + \alpha d(u, \Upsilon v) + (1-\alpha)d(u, v)\end{aligned}$$

so that

$$(1-\alpha)d(u, \Upsilon v) \leq (3-\alpha)d(u, \Upsilon u) + (1-\alpha)d(u, v),$$

and we have that

$$d(u, \Upsilon v) \leq \frac{(3-\alpha)}{(1-\alpha)}d(u, \Upsilon u) + \|u-v\|. \tag{3.3}$$

Case 2: If $\frac{1}{2}d(z, \Upsilon z) \leq \|z-v\|$ for any $z \in \Upsilon u$ and $d(u, v) = \|u-v\|$, then by Lemma (3.1)(1) we have

$$\begin{aligned}d(u, \Upsilon v) &\leq d(u, z) + d(z, \Upsilon v) \\ &\leq d(u, z) + \mathcal{H}(\Upsilon u, \Upsilon v) \\ &\leq d(u, z) + \alpha d(u, \Upsilon v) + \alpha d(v, \Upsilon u) + (1-2\alpha)d(u, v) \\ &\leq d(u, z) + \alpha d(u, \Upsilon v) + \alpha(d(z, v) + d(z, \Upsilon u)) + (1-2\alpha)(d(u, z) + d(z, v)) \\ &= (2-2\alpha)d(u, z) + \alpha d(u, \Upsilon v) + \alpha d(z, v) + \alpha d(z, \Upsilon u) + (1-2\alpha)d(z, v) \\ &\leq (3-3\alpha)d(u, z) + \alpha d(u, \Upsilon v) + (1-\alpha)d(u, v)\end{aligned}$$

so that

$$(1-\alpha)d(u, \Upsilon v) \leq (3-3\alpha)d(u, z) + (1-\alpha)d(u, v),$$

and we have that

$$d(u, \Upsilon v) \leq \frac{3(1-\alpha)}{(1-\alpha)}d(u, z) + \frac{(1-\alpha)}{(1-\alpha)}d(u, v).$$

Since $z \in \Upsilon u$, we have

$$d(u, \Upsilon v) \leq 3d(u, \Upsilon u) + \|u-v\|. \tag{3.4}$$

□

Lemma 3.3. Let ζ be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $\varOmega$ and $\varUpsilon : \zeta \to P_{cb}(\zeta)$ satisfies (3.2). Suppose $\{\xi_n\}$ is a bounded approximate fixed point sequence (a.f.p.s.) for $\varUpsilon$ in ζ, then $A(\zeta, \{\xi_n\})$ is $\varUpsilon$-invariant.

Proof . Let $v \in A(\zeta, \xi_n)$. As $\varUpsilon$ satisfies (3.2), we obtain

$$d(\xi_n, \varUpsilon v) \leq 3d(\xi_n, \varUpsilon \xi_n) + \|\xi_n - v\|.$$

By the definition of asymptotic center, we have

$$\begin{aligned} r(\varUpsilon v, \{\xi_n\}) &= \limsup_{n\to\infty} d(\xi_n, \varUpsilon v) \\ &\leq 3 \limsup_{n\to\infty} d(\xi_n, \varUpsilon \xi_n) + \limsup_{n\to\infty} \|\xi_n - v\| \\ &= \limsup_{n\to\infty} \|\xi_n - v\| \\ &= r(v, \{\xi_n\}). \end{aligned}$$

Therefore, $\varUpsilon v \in A(\zeta, \{\xi_n\})$ □

Lemma 3.4. Let ζ be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $\varOmega$ and $\varUpsilon : \zeta \to P_{cp}(\zeta)$ satisfies (3.2). Suppose $\{\xi_n\}$ is an approximate fixed point sequence (a.f.p.s.) for $\varUpsilon$ in ζ, then

$$\limsup_{n\to\infty} d(\xi_n, \varUpsilon v) \leq \limsup_{n\to\infty} \|\xi_n - v\|,$$

for each $v \in \zeta$

Proof . Since $\varUpsilon$ satisfies (3.2), for any $v \in \zeta$ we have

$$d(\xi_n, \varUpsilon v) \leq 3d(\xi_n, \varUpsilon \xi_n) + \|\xi_n - v\|.$$

As $\{\xi_n\}$ is an approximate fixed point sequence for $\varUpsilon$, we have that

$$\begin{aligned} \limsup_{n\to\infty} d(\xi_n, \varUpsilon v) &\leq 3 \limsup_{n\to\infty} d(\xi_n, \varUpsilon \xi_n) + \limsup_{n\to\infty} \|\xi_n - v\| \\ &= \limsup_{n\to\infty} \|\xi_n - v\|. \end{aligned}$$

□

Lemma 3.5. Let ζ be a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $\varOmega$ and $\varUpsilon : \zeta \to P_{cp}(\zeta)$ satisfies (3.2). Suppose that the asymptotic center is nonempty and compact for each approximate fixed point sequence for $\varUpsilon$ in ζ. Then $\varUpsilon$ has a fixed point.

Proof . Since ζ is nonempty, closed, bounded, convex and $\varUpsilon$-invariant, there exists an approximate fixed point sequence (a.f.p.s.) for $\varUpsilon$, say $\{\xi_n\}$ in ζ. Since the asymptotic center is nonempty and compact, by Lemma (3.3), ζ is also compact, then there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_i}\}$ converges to some $z \in \zeta$. By Lemma (3.4)

$$\limsup_{n\to\infty} d(\{\xi_{n_i}\}, \varUpsilon z) \leq \limsup_{n\to\infty} \|\xi_{n_i} - z\| = 0,$$

and hence, $d(z, \varUpsilon z) = 0$, that is $z \in \varUpsilon z$. □

Lemma 3.6. Let $\varOmega$ be a uniformly convex Banach space and ζ a nonempty, closed and convex subset of $\varOmega$. Let $\varUpsilon : \zeta \to P_{px}(\zeta)$ be a multivalued mapping such that $F(\varUpsilon) \neq 0$ and $P_\varUpsilon$ is a generalized α-nonexpansive mapping. Let $\{\xi_n\}$ be the sequence defined by (3.1). Then

(1) $\lim_{n\to\infty} \|\xi_n - p\|$ exists for all $p \in F(\Upsilon)$ and
(2) $\lim_{n\to\infty} \|\xi_n - P_\Upsilon(\xi_n)\| = 0.$

Proof . Let $p \in F(\Upsilon)$. By Lemma (2.16), $P_\Upsilon(p) = \{p\}$ and $F(\Upsilon) = F(P_\Upsilon)$. Since P_Υ is generalized α-nonexpansive and $d(\xi_n, p) = \|\xi_n - p\|$, we have that

$$\begin{aligned} d(p, P_\Upsilon(p)) &\leq d(p, P_\Upsilon(\xi_n)) + \mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \\ &\leq 2\|\xi_n - p\| \end{aligned}$$

so that

$$\frac{1}{2} d(p, P_\Upsilon(p)) \leq \|\xi_n - p\| \tag{3.5}$$

Thus, $\frac{1}{2} d(p, P_\Upsilon(p)) = 0 = \|\xi_n - p\|$. We note that

$$\begin{aligned} \mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) &\leq \alpha d(\xi_n, P_\Upsilon(p)) + \alpha d(p, P_\Upsilon(\xi_n)) + (1-2\alpha) d(\xi_n, p) \\ &\leq \alpha\|\xi_n - p\| + \alpha(d(p, P_\Upsilon(p)) + \mathcal{H}(P_\Upsilon(p), P_\Upsilon(\xi_n))) + (1-2\alpha)\|\xi_n - p\| \\ &= \alpha\|\xi_n - p\| + \alpha\mathcal{H}(P_\Upsilon(p), P_\Upsilon(\xi_n)) + (1-2\alpha)\|\xi_n - p\|, \end{aligned}$$

so that

$$(1-\alpha)\mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \leq (1-\alpha)\|\xi_n - p\|$$

and

$$\mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \leq \|\xi_n - p\|. \tag{3.6}$$

(1) From (3.1) and (3.6), we have

$$\begin{aligned} \|w_n - p\| &= \|((1-\alpha_n)\xi_n + \alpha_n s_n) - p\| \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n d(s_n, P_\Upsilon(p)) \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n \mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n\|\xi_n - p\| \\ &= \|\xi_n - p\| \end{aligned} \tag{3.7}$$

Also, from (3.1) and (3.7), we have

$$\begin{aligned} \|z_n - p\| &\leq d(z_n, P_\Upsilon(p)) \\ &\leq \mathcal{H}(P_\Upsilon(w_n), P_\Upsilon(p)) \\ &\leq \|w_n - p\| \\ &\leq \|\xi_n - p\| \end{aligned} \tag{3.8}$$

Again, from (3.1) , (3.7) and (3.8) we have

$$\begin{aligned} \|y_n - p\| &= \|((1-\beta_n)s_n + \beta_n z_n) - p\| \\ &\leq (1-\beta_n) d(s_n - P_\Upsilon(p)) + \beta_n d(z_n - P_\Upsilon(p)) \\ &\leq (1-\beta_n)\mathcal{H}(P_\Upsilon(\xi_n) - P_\Upsilon(p)) + \beta_n \mathcal{H}(P_\Upsilon(w_n) - P_\Upsilon(p)) \\ &\leq (1-\beta_n)\|\xi_n - p\| + \beta_n\|\xi_n - p\| \\ &= \|\xi_n - p\|. \end{aligned} \tag{3.9}$$

Further, from (3.1) and (3.9) we have that

$$\begin{aligned} \|\xi_{n+1} - p\| &= \|((1-\gamma_n)t_n + \gamma_n v_n) - p\| \\ &\leq (1-\gamma_n) d(t_n - P_\Upsilon(p)) + \gamma_n d(v_n - P_\Upsilon(p)) \\ &\leq (1-\gamma_n)\mathcal{H}(P_\Upsilon(y_n) - P_\Upsilon(p)) + \gamma_n \mathcal{H}(P_\Upsilon(z_n) - P_\Upsilon(p)) \\ &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n\|z_n - p\| \\ &\leq (1-\gamma_n)\|\xi_n - p\| + \gamma_n\|\xi_n - p\| \\ &= \|\xi_n - p\|. \end{aligned} \tag{3.10}$$

Thus, the sequence $\{\|\xi_n - p\|\}$ is bounded and nonincreasing, hence $\lim_{n\to\infty} \|\xi_n - p\|$ exists for all $p \in F(\Upsilon)$.

(2) Now, we show that $\lim_{n\to\infty} \|\xi_n - s_n\| = 0$.

Lets suppose that

$$\lim_{n\to\infty} \|\xi_n - p\| = c. \tag{3.11}$$

So from (3.7), (3.8) and (3.9), we have that

$$\limsup_{n\to\infty} \|w_n - p\| \leq c, \tag{3.12}$$

$$\limsup_{n\to\infty} \|z_n - p\| \leq c, \tag{3.13}$$

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \tag{3.14}$$

Also, we obtain the following inequalities as follows:

$$\begin{aligned} \|s_n - p\| &\leq \mathcal{H}(P_\Upsilon(\xi_n) - P_\Upsilon(p)) \leq \|\xi_n - p\| \\ \|v_n - p\| &\leq \mathcal{H}(P_\Upsilon(z_n) - P_\Upsilon(p)) \leq \|z_n - p\| \\ \|t_n - p\| &\leq \mathcal{H}(P_\Upsilon(y_n) - P_\Upsilon(p)) \leq \|y_n - p\| \end{aligned} \tag{3.15}$$

Taking the $\limsup_{n\to\infty}$ on both sides of the inequalities (3.15) above, we have that

$$\limsup_{n\to\infty} \|s_n - p\| \leq c, \tag{3.16}$$

$$\limsup_{n\to\infty} \|v_n - p\| \leq c, \tag{3.17}$$

$$\limsup_{n\to\infty} \|t_n - p\| \leq c. \tag{3.18}$$

Now from (3.1), (3.10) and (3.11), we have that

$$\begin{aligned} c = \lim_{n\to\infty} \|\xi_{n+1} - p\| &= \lim_{n\to\infty} \|((1-\gamma_n)t_n + \gamma_n v_n) - p\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)(t_n - p) + \gamma_n(v_n - p)\|. \end{aligned} \tag{3.19}$$

By Lemma (2.8), since $\limsup_{n\to\infty} \|t_n - p\| \leq c$ and $\limsup_{n\to\infty} \|v_n - p\| \leq c$, then we have that

$$\lim_{n\to\infty} \|t_n - v_n\| = 0. \tag{3.20}$$

Again, from (3.1), we have that

$$\begin{aligned} \|\xi_{n+1} - p\| &\leq \|((1-\gamma_n)t_n + \gamma_n v_n) - p\| \\ &\leq \|t_n - p\| + \gamma_n \|t_n - v_n\|, \end{aligned} \tag{3.21}$$

which gives

$$c \leq \liminf_{n\to\infty} \|t_n - p\|. \tag{3.22}$$

From (3.18) and (3.22), we have that

$$\limsup_{n\to\infty} \|t_n - p\| \leq c \leq \liminf_{n\to\infty} \|t_n - p\|, \tag{3.23}$$

so that

$$\lim_{n\to\infty} \|t_n - p\| = c. \tag{3.24}$$

From (3.1), (3.8), (3.15) and (3.24),

$$\begin{aligned} c &= \lim_{n\to\infty} \|t_n - p\| \\ &\leq \lim_{n\to\infty} \mathcal{H}(P_\Upsilon(y_n) - P_\Upsilon(p)) \\ &\leq \lim_{n\to\infty} \|y_n - p\| \\ &\leq \lim_{n\to\infty} (1-\beta_n)\|s_n - p\| + \lim_{n\to\infty} \beta_n \|z_n - p\| && (3.25) \\ &\leq \lim_{n\to\infty} (1-\beta_n)\|s_n - p\| + \lim_{n\to\infty} \beta_n \|\xi_n - p\| && (3.26) \\ &\leq \lim_{n\to\infty} (1-\beta_n)\|\xi_n - p\| + \lim_{n\to\infty} \beta_n \|\xi_n - p\| && (3.27) \\ &= c. && (3.28) \end{aligned}$$

Consequently, from (3.26)

$$\lim_{n\to\infty} \|(1-\beta_n)(s_n-p)+\beta_n(\xi_n-p)\| = c. \tag{3.29}$$

Thus from (3.11), (3.15), (3.16) and Lemma (2.8), we have that

$$\lim_{n\to\infty} \|\xi_n - s_n\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|\xi_n - P_{\Upsilon}(\xi_n)\| = 0,$$

and this completes the proof.

□

3.2 Convergence Results

Theorem 3.7. Let Ω be a uniformly convex Banach space and ζ a nonempty compact convex subset of Ω. Let $\Upsilon : \zeta \to P_{px}(\zeta)$ be such that $F(\Upsilon) \neq 0$ and P_{Υ} is a generalized α-nonexpansive mapping. Let $\{\xi_n\}$ be the sequence defined by (3.1). Then $\{\xi_n\}$ converges strongly to a fixed point of Υ.

Proof . From Lemma (3.5), we have that $\{\xi_n\}$ is bounded and $\lim_{n\to\infty} d(\xi_n, P_{\Upsilon}(\xi_n)) = 0$. Since ζ is compact, then there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_i}\}$ converges to some $y \in \zeta$. Since P_{Υ} is generalized α-nonexpansive mapping, it satisfies (3.2) and hence

$$d(\xi_{n_i}, P_{\Upsilon}(y)) \leq 3d(\xi_{n_i}, P_{\Upsilon}(\xi_{n_i})) + \|\xi_{n_i} - y\|.$$

As $F(\Upsilon) = F(P_{\Upsilon})$, so on taking limit as $i \to \infty$, we have that $y \in \Upsilon y$. Therefore, $\{\xi_n\}$ converges strongly to $y \in F(\Upsilon)$. □

Theorem 3.8. Let Ω be a uniformly convex Banach space satisfying Opial's condition and ζ a nonempty, closed and convex subset of Ω. Suppose that P_{Υ} is generalized α-nonexpansive mapping, where Let $\Upsilon : \zeta \to P_{px}(\zeta)$. Let $\{\xi_n\}$ be the sequence defined by (3.1). If $F(\Upsilon) \neq \emptyset$ be such that $\liminf_{n\to\infty} d(\xi_n, F(\Upsilon)) = 0$. Then $\{\xi_n\}$ converges strongly to a fixed point of Υ.

Proof . By Lemma (3.6), we know that $\lim_{n\to\infty} \|\xi_n - p\|$ exists, for all $p \in F(P_{\Upsilon}) = F(\Upsilon)$. Thus, $\lim_{n\to\infty} d(\xi_n, F(\Upsilon))$ exists. Now, $\liminf_{n\to\infty} d(\xi_n, F(\Upsilon)) = 0$ implies that $\lim_{n\to\infty} d(\xi_n, F(\Upsilon)) = 0$. Therefore, there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ and $y_i \in F(\Upsilon)$ such that $\|\xi_{n_i} - y_i\| \leq \frac{1}{2^i}$ for all $i \in \mathbb{N}$. Also since $\{\xi_n\}$ is nonincreasing, we have that

$$\|\xi_{n_{i+1}} - y_i\| \leq \|\xi_{n_i} - y_i\| \leq \frac{1}{2^i}.$$

Consequently,

$$\begin{aligned}\|y_{i+1} - y_i\| &\leq \|y_{i+1} - \xi_{i+1}\| + \|\xi_{i+1} - y_i\| \\ &\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} \\ &= \frac{1}{2^{i-1}} \to 0 \quad \text{as } i \to \infty.\end{aligned}$$

Hence, $\{\xi_n\}$ is a cauchy sequence in ζ and converges to a point say, $p \in \zeta$. As P_{Υ} satisfies (3.2), we have

$$\begin{aligned}d(p, P_{\Upsilon}(p)) &\leq \|\xi_n - p\| + d(\xi_n, P_{\Upsilon}(p)) \\ &\leq \|\xi_n - p\| + d(\xi_n, P_{\Upsilon}(p)) + \|\xi_n - p\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

This implies that $p \in P_{\Upsilon}(p)$ and $p \in F(P_{\Upsilon})$ By Lemma (2.17), we have $p \in F(\Upsilon)$ and hence $\{\xi_n\}$ converges strongly to a fixed point of Υ. □

Theorem 3.9. Let Ω be a uniformly convex Banach space satisfying Opial's condition and ζ a nonempty, closed and convex subset of Ω. Let $\Upsilon : \zeta \to P_{px}(\zeta)$ be a multivalued mapping satisfying Condition (I) such that $F(\Upsilon) \neq \emptyset$. Let $\{\xi_n\}$ be the sequence defined by (3.1). Suppose P_Υ is a generalized α-nonexpansive mapping. Then $\{\xi_n\}$ converges strongly to a fixed point of Υ.

Proof . By Lemma (3.6)(1), $\{\xi_n\}$ is nonincreasing and $\lim_{n\to\infty} \|\xi_n - p\|$ exists, for all $p \in F(\Upsilon)$. Let $c = \lim_{n\to\infty} \|\xi_n - p\|$, for some $c \geq 0$. If $c = 0$, then the result follows. Now suppose that $c > 0$. Then

$$\|\xi_{n+1} - p\| \leq \|\xi_n - p\|,$$

implies that

$$\liminf_{n\to\infty} \|\xi_{n+1} - p\| \leq \liminf_{n\to\infty} \|\xi_n - p\|.$$

Therefore, $d(\xi_{n+1}, F(\Upsilon)) \leq d(\xi_n, F(\Upsilon))$. Consequently, $\lim_{n\to\infty} d(x_n, F(\Upsilon))$ exists. As $F(P_\Upsilon) = F(\Upsilon)$, and by Lemma (3.6) and Condition (I), we have

$$\lim_{n\to\infty} f(d(\xi_n, F(\Upsilon))) \leq \lim_{n\to\infty} d(\xi_n, P_\Upsilon(\xi_n)) = 0.$$

As f is nondecreasing and $f(0) = 0$, we have that

$$\lim_{n\to\infty} d(\xi_n, F(\Upsilon)) = 0.$$

Thus, the result follows from Theorem (3.8). □

Theorem 3.10. Let Ω be a uniformly convex Banach space satisfying Opial's condition and ζ a nonempty, closed and convex subset of Ω. Let $\Upsilon : \zeta \to P_{px}(\zeta)$ be a multivalued mapping such that $F(\Upsilon) \neq \emptyset$. Suppose P_Υ is a generalized α-nonexpansive mapping and $I - P_\Upsilon$ is demiclosed with respect to zero. If $\{\xi_n\}$ is a sequence defined by (3.1). Then $\{\xi_n\}$ converges weakly to a fixed point of Υ.

Proof . Let $p \in F(P_\Upsilon) = F(\Upsilon)$. From Lemma (3.6), it is shown that $\{\xi_n\}$ is bounded and $\lim_{n\to\infty} \|\xi_n - p\|$ exists, for all $p \in F(\Upsilon)$. Since Ω is uniformly convex, then Ω is reflexive. Thus there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_i}\}$ converges weakly to some $y_1 \in \zeta$. Since $(I - P_\Upsilon)$ is demiclosed with respect to zero, then $y_1 \in F(P_\Upsilon) = F(\Upsilon)$. If $\xi_n \nrightarrow y_1$, then there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_i}\} \to y_2$, where $y_2 \neq y_1$. Clearly, $y_2 \in F(P_\Upsilon) = F(\Upsilon)$. By the Opial's property, we have that

$$\begin{aligned}
\lim_{n\to\infty} \|\xi_n - y_1\| &\leq \lim_{n\to\infty} \|\xi_{n_i} - y_1\| \\
&< \lim_{n\to\infty} \|\xi_{n_i} - y_2\| \\
&= \lim_{n\to\infty} \|\xi_n - y_2\| \\
&< \lim_{n\to\infty} \|\xi_{n_i} - y_2\| \\
&< \lim_{n\to\infty} \|\xi_{n_i} - y_1\| \\
&= \lim_{n\to\infty} \|\xi_n - y_1\|,
\end{aligned}$$

a contradiction. Hence, $y_1 = y_2$. Thus, $\{\xi_n\}$ converges weakly to a fixed point of Υ. □

3.3 Stability Result

In this section, we analyze the stability of the iteration process (3.1) with respect to multivalued contraction mapping.

Theorem 3.11. Let ζ a nonempty, closed and convex subset of a uniformly convex Banach space Ω. Let $\Upsilon : \zeta \to P_{px}(\zeta)$ be a multivalued mapping and P_Υ a multivalued contraction with $\delta \in [0, 1)$. If $\{\xi_n\}$ is a sequence defined in (3.1) with the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in (0, 1)$ satisfying $\sum_{s=0}^{\infty} \alpha_n\beta_n(1 - \gamma_n) = \infty$. Then $\{\xi_n\}$ converges strongly to a fixed point of Υ.

Proof . From (3.1), we have

$$\begin{aligned}\|w_n - p\| &= \|((1-\alpha_n)\xi_n + \alpha_n s_n) - p\| \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n d(s_n, P_\Upsilon(p)) \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n \mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \\ &\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n \delta \|\xi_n - p\| \\ &= (1-(1-\delta)\alpha_n)\|\xi_n - p\|. \end{aligned} \tag{3.30}$$

Also, from (3.1) and (3.30), we have

$$\begin{aligned}\|z_n - p\| &\leq d(z_n, P_\Upsilon(p)) \\ &\leq \mathcal{H}(P_\Upsilon(w_n), P_\Upsilon(p)) \\ &\leq \delta\|w_n - p\| \\ &\leq \delta(1-(1-\delta)\alpha_n)\|\xi_n - p\|. \end{aligned} \tag{3.31}$$

Again, from (3.1) , (3.30) and (3.31) we have

$$\begin{aligned}\|y_n - p\| &= \|((1-\beta_n)s_n + \beta_n z_n) - p\| \\ &\leq (1-\beta_n)d(s_n, P_\Upsilon(p)) + \beta_n d(z_n, P_\Upsilon(p)) \\ &\leq (1-\beta_n)\mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) + \beta_n \mathcal{H}(P_\Upsilon(w_n), P_\Upsilon(p)) \\ &\leq (1-\beta_n)\delta\|\xi_n - p\| + \beta_n \delta \|w_n - p\| \\ &\leq \delta(1-\alpha_n\beta_n(1-\delta))\|\xi_n - p\|. \end{aligned} \tag{3.32}$$

Further, from (3.1), (3.31) and (3.32) we have that

$$\begin{aligned}\|\xi_{n+1} - p\| &= \|((1-\gamma_n)t_n + \gamma_n v_n) - p\| \\ &\leq (1-\gamma_n)d(t_n, P_\Upsilon(p)) + \gamma_n d(v_n, P_\Upsilon(p)) \\ &\leq (1-\gamma_n)\mathcal{H}(P_\Upsilon(y_n), -P_\Upsilon(p)) + \gamma_n \mathcal{H}(P_\Upsilon(z_n), P_\Upsilon(p)) \\ &\leq (1-\gamma_n)\delta\|y_n - p\| + \gamma_n \delta \|z_n - p\| \\ &\leq \delta^2((1-\gamma_n)(1-\alpha_n\beta_n(1-\delta))\|\xi_n - p\|) + \gamma_n\delta^2((1-(1-\delta)\alpha_n)\|\xi_n - p\|) \\ &= \delta^2(1-\alpha_n\beta_n(1-\gamma_n)(1-\delta))\|\xi_n - p\|. \end{aligned} \tag{3.33}$$

From (3.33) we deduce that

$$\begin{aligned}\|\xi_{n+1} - p\| &\leq \delta^2(1-\alpha_n\beta_n(1-\gamma_n)(1-\delta))\|\xi_n - p\| \\ \|\xi_n - p\| &\leq \delta^2(1-\alpha_{n-1}\beta_{n-1}(1-\gamma_{n-1})(1-\delta))\|\xi_{n-1} - p\| \\ \|\xi_{n-1} - p\| &\leq \delta^2(1-\alpha_{n-2}\beta_{n-2}(1-\gamma_{n-2})(1-\delta))\|\xi_{n-2} - p\| \\ &\vdots \\ \|\xi_1 - p\| &\leq \delta^2(1-\alpha_0\beta_0(1-\gamma_0)(1-\delta))\|\xi_0 - p\|. \end{aligned} \tag{3.34}$$

Thus, from (3.34) we have

$$\|\xi_1 - p\| \leq \delta^{2(n+1)} \prod_{k=0}^{n} (1-(1-\delta)\alpha_k\beta_k(1-\gamma_k))\|\xi_0 - p\|. \tag{3.35}$$

Since for all $n \in \mathbb{N}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ and $\delta \in [0,1)$, it follows that $(1-(1-\delta)\alpha_k\beta_k(1-\gamma_k)) < 1$ and from classical analysis, we know that $1-\xi \leq \exp^{-\xi}$ for all $\xi \in [0,1]$. Then from (3.34), we have that

$$\|\xi_1 - p\| \leq \frac{\delta^{2(n+1)}\|\xi_0 - p\|}{\exp^{(1-\delta)\sum_{k=0}^{n}\alpha_k\beta_k(1-\gamma_k)}} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.36}$$

Consequently, $\{\xi_n\}$ converges to a fixed point of P_Υ. Since $p \in F(P_\Upsilon)$, by Lemma (2.17), we have that $p \in F(\Upsilon)$ and hence $\{\xi_n\} \to p \in F(\Upsilon)$. □

Theorem 3.12. Let ζ a nonempty, closed and convex subset of a uniformly convex Banach space Ω. Let $\Upsilon : \zeta \to P_{px}(\zeta)$ be a multivalued mapping and P_Υ a multivalued contraction with $\delta \in [0,1)$. If $\{\xi_n\}$ is a sequence defined in (3.1) with the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in (0,1)$ and $\sum_{s=0}^{\infty} \alpha_n\beta_n(1-\gamma_n) = \infty$. Then the iteration process (3.1) is Υ-stable.

Proof . Let $\{g_n\}$ be an arbitrary sequence in ζ and suppose that the UD iterative sequence generated by (3.1) is $g_{n+1} = f(\Upsilon, g_n)$. Then by Theorem (3.11), $\{g_n\}$ converges to a unique fixed point $q\ of\ \Upsilon$. Lets define $\epsilon_n = \|g_{n+1} - f(\Upsilon, g_n)\|$ for $n \geq 1$. To prove that $\{\xi_n\}$ is Υ-stable, we have to show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} g_n = q$. If $\lim_{n\to\infty} \varepsilon_n = 0$. Then from (3.33) we have,

$$\begin{aligned}
\|g_{n+1} - q\| &= \|g_{n+1} - f(\Upsilon, g_n) + f(\Upsilon, g_n) - q\| \\
&\leq \|g_{n+1} - f(\Upsilon, g_n)\| + \|f(\Upsilon, g_n) - q\| \\
&\leq \epsilon_n + (1-\gamma_n)\|t_n - q\| + \gamma_n\|v_n - q\| \\
&\leq \epsilon_n + \delta^2(1 - \alpha_n\beta_n(1-\gamma_n)(1-\delta))\|g_n - q\|. \qquad (3.37)
\end{aligned}$$

Now, let

$$\begin{aligned}
a_n &= \|g_n - q\| \qquad (3.38)\\
b_n &= \alpha_n\beta_n(1-\gamma_n)(1-\delta). \qquad (3.39)
\end{aligned}$$

Then from (3.38), (3.39), and (3.37) becomes

$$a_{n+1} \leq (1-b_n)a_n + \epsilon_n \qquad (3.40)$$

Since $\sum_{n=0}^{\infty} b_n = \infty$ and $\lim_{n\to\infty} \epsilon_n = 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n\beta_n(1-\gamma_n)(1-\delta)} \to 0\ as\ n \to \infty.$$

From Lemma (2.13), since all the conditions are satisfied by inequality (3.40), then we obtain that $\lim_{n\to\infty} a_n = 0$. Consequently, $\lim_{n\to\infty} g_n = q$

Conversely, suppose that $\lim_{k\to\infty} g_n = q$, then from (3.33) we have that

$$\begin{aligned}
\epsilon_n &= \|g_{n+1} - f(\Upsilon, g_n)\| \\
&= \|(g_{n+1} - q) + (q - f(\Upsilon, g_n)\| \\
&\leq \|g_{n+1} - q\| + \|f(\Upsilon, g_n) - q\| \\
&\leq \|g_{n+1} - q\| + \delta^2(1 - \alpha_n\beta_n(1-\gamma_n)(1-\delta))\|g_n - q\|. \qquad (3.41)
\end{aligned}$$

Now, since $\lim_{n\to\infty} g_n = q$, and taking the limit as $n \to \infty$ in (3.40), we have that $\lim_{n\to\infty} \epsilon_n = 0$. Hence the iteration process defined in (3.1) is stable with respect to Υ. □

3.4 Data Dependence Analysis

Theorem 3.13. Let $\tilde{\Upsilon}$ be an approximate operator of a mapping Υ satisfying the contraction mapping (2.1). Let $\{\xi_n\}$ be the multivalued UD iterative sequence generated by (3.1) for Υ and define an iterative sequence $\{\tilde{\xi_n}\}$ as follows:

$$\begin{cases} \tilde{\xi}_1 \in \zeta, \\ \tilde{\xi}_{n+1} = (1-\gamma_n)\tilde{t_n} + \gamma\tilde{v}_n, \\ \tilde{y}_n = (1-\beta_n)\tilde{s}_n + \beta_n\tilde{z}_n, \qquad \forall n \geq 1. \\ \tilde{z}_n = \tilde{P}_{\Upsilon}\tilde{w}_n), \\ \tilde{w}_n = (1-\alpha_n)\tilde{\xi}_n + \alpha_n\tilde{s}_n \end{cases} \tag{3.42}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$, $n \in \mathbb{N}$ satisfy the following conditions:

(i) $\frac{1}{2} \leq \alpha_n\beta_n(1-\gamma_n)$, $\forall\, n \in \mathbb{N}$,

(ii) $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n(1-\gamma_n) = \infty$,

and $\tilde{s}_n \in \tilde{P}_{\Upsilon}(\tilde{\xi}_n)$, $\tilde{v}_n \in \tilde{P}_{\Upsilon}(\tilde{z}_n)$, $\tilde{t}_n \in \tilde{P}_{\Upsilon}(\tilde{y}_n)$. If $\Upsilon q = q$ and $\tilde{\Upsilon}\tilde{q} = \tilde{q}$ such that $\lim\limits_{n\to\infty} \tilde{\xi}_n = \tilde{q}$, then we have

$$\|q - \tilde{q}\| \leq \frac{7\epsilon}{1-\delta}, \tag{3.43}$$

where $\epsilon > 0$ is a fixed number.

Proof . From (3.1), (3.42) and (2.1) we have

$$\begin{aligned} \|w_n - \tilde{w}_n\| &= \|((1-\alpha_n)\xi_n + \alpha_n s_n) - ((1-\alpha_n)\tilde{\xi}_n + \alpha_n\tilde{s}_n)\| \\ &\leq (1-\alpha_n)\|\xi_n - \tilde{\xi}_n\| + \alpha_n\|s_n - \tilde{s}_n\| \\ &\leq (1-\alpha_n)\|\xi_n - (\tilde{\xi}_n)\| + \alpha_n\|P_{\Upsilon}(\xi_n) - \tilde{P}_{\Upsilon}(\tilde{\xi}_n)\| \\ &\leq (1-\alpha_n)\|\xi_n - \tilde{\xi}_n\| + \alpha_n\|P_{\Upsilon}(\xi_n) - P_{\Upsilon}(\tilde{\xi}_n)\| + \alpha_n\|P_{\Upsilon}(\tilde{\xi}_n) - \tilde{P}_{\Upsilon}(\tilde{\xi}_n)\| \\ &\leq (1-\alpha_n)\|\xi_n - \tilde{\xi}_n\| + \alpha_n\delta\|\xi_n - \tilde{\xi}_n\| + \alpha_n\epsilon \\ &= (1-\alpha_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\epsilon \end{aligned} \tag{3.44}$$

Also, from (3.1), (3.42) and (2.1) we have

$$\begin{aligned} \|z_n - \tilde{z}_n\| &\leq \|P_{\Upsilon}(w_n) - \tilde{P}_{\Upsilon}(\tilde{w}_n)\| \\ &\leq \|P_{\Upsilon}(w_n) - P_{\Upsilon}(\tilde{w}_n)\| + \|P_{\Upsilon}(\tilde{w}_n) - \tilde{P}_{\Upsilon}(\tilde{w}_n)\| \\ &\leq \delta\|w_n - \tilde{w}_n\| + \epsilon. \end{aligned} \tag{3.45}$$

Substituting (3.44) into (3.45), we have

$$\|z_n - \tilde{z}_n\| \leq \delta(1-\alpha_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\delta\epsilon + \epsilon. \tag{3.46}$$

Again, from (3.1), (3.42) and (2.1) we have

$$\begin{aligned} \|y_n - \tilde{y}_n\| &= \|((1-\beta_n)s_n + \beta_n z_n) - ((1-\beta_n)\tilde{s}_n + \beta_n\tilde{z}_n)\| \\ &\leq (1-\beta_n)\|s_n - \tilde{s}_n\| + \beta_n\|z_n - \tilde{z}_n\| \\ &\leq (1-\beta_n)\|P_{\Upsilon}(\xi_n) - \tilde{P}_{\Upsilon}(\tilde{\xi}_n)\| + \beta_n\|z_n - \tilde{z}_n\|. \end{aligned} \tag{3.47}$$

Substituting (3.46) into (3.47), we have

$$\begin{aligned} \|y_n - \tilde{y}_n\| &\leq (1-\beta_n)\|P_{\Upsilon}(\xi_n) - \tilde{P}_{\Upsilon}(\tilde{\xi}_n)\| + \beta_n(\delta(1-\alpha_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\delta\epsilon + \epsilon) \\ &\leq (1-\beta_n)\|P_{\Upsilon}(\xi_n) - P_{\Upsilon}(\tilde{\xi}_n)\| + (1-\beta_n)\|P_{\Upsilon}(\tilde{\xi}_n) - \tilde{P}_{\Upsilon}(\tilde{\xi}_n)\| \\ &\quad + \beta_n\delta(1-\alpha_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\beta_n\delta\epsilon + \beta_n\epsilon \\ &\leq (1-\beta_n)\delta\|\xi_n - \tilde{\xi}_n\| + (1-\beta_n)\epsilon + \beta_n\delta(1-\alpha_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\beta_n\delta\epsilon + \beta_n\epsilon \\ &= \delta(1-\alpha_n\beta_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\beta_n\delta\epsilon + \epsilon \qquad (3.48) \\ &\leq (1-\alpha_n\beta_n(1-\delta))\|\xi_n - \tilde{\xi}_n\| + \alpha_n\beta_n\delta\epsilon + \epsilon. \qquad (3.49) \end{aligned}$$

Furthermore, from (3.1), (3.42) and (2.1), we have

$$\begin{aligned}\|\xi_{n+1}-\tilde{\xi}_{n+1}\| &= \|((1-\gamma_n)t_n+\gamma_n v_n)-((1-\gamma_n)\tilde{t}_n+\gamma_n\tilde{v}_n)\|\\ &\leq (1-\gamma_n)\|t_n-\tilde{t}_n\|+\gamma_n\|v_n-\tilde{v}_n\|\\ &\leq (1-\gamma_n)\|P_\Upsilon(y_n)-\tilde{P}_\Upsilon(\tilde{y}_n)\|+\gamma_n\|P_\Upsilon(z_n)-\tilde{P}_\Upsilon(\tilde{z}_n)\|\\ &\leq (1-\gamma_n)\delta\|y_n-\tilde{y}_n\|+\gamma_n\delta\|z_n-\tilde{z}_n\|+\epsilon. \qquad (3.50)\end{aligned}$$

Substituting (3.46) and (3.49) into (3.50), we have

$$\begin{aligned}\|\xi_{n+1}-\tilde{\xi}_{n+1}\| &\leq (1-\gamma_n)\delta(\delta(1-\alpha_n\beta_n(1-\delta))\|\xi_n-\tilde{\xi}_n\|+\alpha_n\beta_n\delta\epsilon+\epsilon)\\ &\quad +\gamma_n\delta(\delta(1-\alpha_n(1-\delta))\|\xi_n-\tilde{\xi}_n\|+\alpha_n\delta\epsilon+\epsilon)+\epsilon\\ &= (\delta^2(1-\alpha_n\beta_n(1-\delta))-\gamma_n\delta^2(1-\alpha_n\beta_n(1-\delta))+\gamma_n\delta^2\\ &\quad -\alpha_n\gamma_n\delta^2(1-\delta))\|\xi_n-\tilde{\xi}_n\|+(1-\gamma_n)\alpha_n\beta_n\delta^2\epsilon+(1-\gamma_n)\delta\epsilon+\alpha_n\gamma_n\delta^2\epsilon+\gamma_n\delta\epsilon+\epsilon\\ &= \delta^2(1-\alpha_n\beta_n(1-\delta)+\alpha_n\beta_n\gamma_n(1-\delta)-\alpha_n\gamma_n(1-\delta))\|\xi_n-\tilde{\xi}_n\|\\ &\quad +(1-\gamma_n)\alpha_n\beta_n\delta^2\epsilon+\alpha_n\gamma_n\delta^2\epsilon+\delta\epsilon+\epsilon\\ &\leq \delta^2(1-(1-\delta)\alpha_n\beta_n(1-\gamma_n))\|\xi_n-\tilde{\xi}_n\|+(1-\gamma_n)\alpha_n\beta_n\delta^2\epsilon+\alpha_n\gamma_n\delta^2\epsilon+\delta\epsilon+\epsilon. \qquad (3.51)\end{aligned}$$

Since $\delta,\delta^2\in(0,1)$, $\alpha_n,\beta_n,\gamma_n\subset(0,1)$ and by condition (i), $1\leq 2\alpha_n\beta_n(1-\gamma_n)$ for all $n\in\mathbb{N}$, then (3.51) becomes

$$\begin{aligned}\|\xi_{n+1}-\tilde{\xi}_{n+1}\| &\leq (1-(1-\delta)\alpha_n\beta_n(1-\gamma_n))\|\xi_n-\tilde{\xi}_n\|+\alpha_n\beta_n(1-\gamma_n)\epsilon+\alpha_n\gamma_n\epsilon+2\epsilon\\ &\leq (1-(1-\delta)\alpha_n\beta_n(1-\gamma_n))\|\xi_n-\tilde{\xi}_n\|+7\alpha_n\beta_n(1-\gamma_n)\epsilon \qquad (3.52)\\ &= (1-(1-\delta)\alpha_n\beta_n(1-\gamma_n))\|\xi_n-\tilde{\xi}_n\|+\frac{(1-\delta)\alpha_n\beta_n(1-\gamma_n)7\epsilon}{(1-\delta)}. \qquad (3.53)\end{aligned}$$

Now, we set $a_n=\|\xi_n-\tilde{\xi}_n\|$; $\sigma_n=(1-\delta)\alpha_n\beta_n(1-\gamma_n)$; $b_n=\frac{7\epsilon}{(1-\delta)}$ From Theorem (3.11), we have that $\lim_{n\to\infty}\xi_n=q$. Thus by Lemma (2.12) $\sum_{n=0}^{\infty}\alpha_n\beta_n(1-\gamma_n)=\infty$ and $\frac{7\epsilon}{(1-\delta)}>0$, so we have that

$$0\leq\limsup_{n\to\infty}\|\xi_n-\tilde{\xi}_n\|\leq\limsup_{n\to\infty}\frac{7\epsilon}{(1-\delta)}. \qquad (3.54)$$

Since by Theorem(3.11), $\lim_{n\to\infty}\xi_n=q$, and by our assumption that $\lim_{n\to\infty}\tilde{\xi}_n=\tilde{q}$, then from (3.54)

$$\|q-\tilde{q}\|\leq\frac{7\epsilon}{(1-\delta)}. \qquad (3.55)$$

Thus, the multivalued UD iterative scheme (3.1) is data dependent. This completes the proof. □

4 Rate of convergence

4.1 Rate of Convergence of Multivalued UD Iteration Scheme for Multivalued Contraction Mapping

In this section, we show that the multivalued UD iteration scheme (3.1) converges faster than the multivalued modified Piri *et al* iteration scheme (1.3) for multivalued contraction mapping.

Theorem 4.1. Let P_Υ be a contraction mapping defined on a nonempty, closed and convex subset ζ of a uniformly convex Banach space Ω with a contraction factor $\delta\in(0,1)$ and $F(P_\Upsilon)\neq\phi$. If $\{\xi_n\}$ is a sequence defined by (3.1), then $\{\xi_n\}$ converges faster than the iteration process (1.3).

Proof .

$$\begin{aligned}
\|w_n - p\| &= \|((1-\alpha_n)\xi_n + \alpha_n s_n) - p\| \\
&\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n d(s_n, P_\Upsilon(p)) \\
&\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n \mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) \\
&\leq (1-\alpha_n)\|\xi_n - p\| + \alpha_n \delta\|\xi_n - p\| \\
&= (1-(1-\delta)\alpha_n)\|\xi_n - p\| \\
&\leq \|\xi_n - p\|, \quad \text{since} \quad (1-(1-\delta)\alpha_n) < 1. \qquad (4.1)
\end{aligned}$$

Also, from (3.1) and (4.1), we have

$$\begin{aligned}
\|z_n - p\| &\leq d(z_n, P_\Upsilon(p)) \\
&\leq \mathcal{H}(P_\Upsilon(w_n), P_\Upsilon(p)) \\
&\leq \delta\|w_n - p\| \\
&\leq \delta\|\xi_n - p\|. \qquad (4.2)
\end{aligned}$$

Again, from (3.1) , (4.1) and (4.2) we have

$$\begin{aligned}
\|y_n - p\| &= \|((1-\beta_n)s_n + \beta_n z_n) - p\| \\
&\leq (1-\beta_n)d(s_n, P_\Upsilon(p)) + \beta_n d(z_n, P_\Upsilon(p)) \\
&\leq (1-\beta_n)\mathcal{H}(P_\Upsilon(\xi_n), P_\Upsilon(p)) + \beta_n \mathcal{H}(P_\Upsilon(w_n), P_\Upsilon(p)) \\
&\leq \delta((1-\beta_n)\|\xi_n - p\| + \beta_n\|w_n - p\|) \\
&= \delta\|\xi_n - p\|. \qquad (4.3)
\end{aligned}$$

Further, from (3.1), (4.1), (4.2) and (4.3) we have that

$$\begin{aligned}
\|\xi_{n+1} - p\| &= \|((1-\gamma_n)t_n + \gamma_n v_n) - p\| \\
&\leq (1-\gamma_n)\mathcal{H}(P_\Upsilon(y_n), -P_\Upsilon(p)) + \gamma_n \mathcal{H}(P_\Upsilon(z_n), P_\Upsilon(p)) \\
&\leq \delta((1-\gamma_n)\|y_n - p\| + \gamma_n\|z_n - p\|) \\
&\leq \delta^2((1-\gamma_n)\|\xi_n - p\| + \gamma_n\|\xi_n - p\|) \\
&= \delta^2\|\xi_n - p\| \\
&\vdots \\
&\leq \delta^{2n}\|\xi_1 - p\|. \qquad (4.4)
\end{aligned}$$

Let

$$p_n = \delta^{2n}\|\xi_1 - p\|. \qquad (4.5)$$

Also from (1.3), we have

$$\begin{aligned}
\|z_n - p\| &= \|((1-\beta_n)\xi_n + \beta_n s_n) - p\| \\
&\leq (1-\beta_n)\|\xi_n - p\| + \beta_n d(s_n, P_\Upsilon(p)) \\
&\leq (1-(1-\delta)\beta_n)\|\xi_n - p\| \\
&\leq \|\xi_n - p\|, \quad \text{since} \quad (1-(1-\delta)\beta_n) < 1. \qquad (4.6)
\end{aligned}$$

Using (1.3) and (4.6)

$$\begin{aligned}
\|y_n - p\| &\leq d(P_\Upsilon(z_n), p) \\
&\leq H(P_\Upsilon(z_n), P_\Upsilon(p)) \\
&\leq \delta\|z_n - p\| \\
&\leq \delta\|\xi_n - p\|. \qquad (4.7)
\end{aligned}$$

Using (1.3), (4.6) and (4.7), we have

$$
\begin{aligned}
\|\xi_{n+1} - p\| &= \|((1-\alpha_n)v_n + \alpha_n t_n) - p\| \\
&\leq (1-\alpha_n)d(v_n, P_\Upsilon(p)) + \alpha_n d(t_n, P_\Upsilon(p)) \\
&\leq (1-\alpha_n)H(P_\Upsilon(z_n), P_\Upsilon(p)) + \alpha_n \mathcal{H}(P_\Upsilon(y_n), P_\Upsilon(p)) \\
&\leq (1-\alpha_n)\delta\|z_n - p\| + \alpha_n\delta\|y_n - p\| \\
&\leq \delta(1-(1-\delta)\alpha_n)\|\xi_n - p\| \\
&\leq \delta\|\xi_n - p\|, \quad \text{since} \quad (1-(1-\delta)\alpha_n) < 1. \\
&\vdots \\
&\leq \delta^n\|\xi_1 - p\|. \qquad (4.8)
\end{aligned}
$$

Let

$$r_n = \delta^n\|\xi_1 - q\| \qquad (4.9)$$

So from (4.5) and (4.9), we have that

$$\frac{p_n}{r_n} = \frac{\delta^{2n}\|\xi_1 - q\|}{\delta^n\|\xi_1 - q\|} = \delta^n \longrightarrow 0, \qquad as \;\; n \to \infty.$$

Hence (3.1) converges faster than (1.3). □

4.2 Numerical Example

We now show the comparison between the rate of convergence of the multivalued UD iteration process (3.1) and other well known iteration algorithms in literature.

Example 4.2. Let $(\Re, \|.\|)$ be a normed space with the usual norm and $\zeta = [0,1]$. Let $\Upsilon : \zeta \longrightarrow P(\zeta)$ be a multivalued contraction defined by

$$\Upsilon u = \left[\; 0, \frac{3u}{4} \;\right] \qquad (4.10)$$

For Table $A1$/Table $B1$ (see (Appendix A) and (Appendix B) respectively), we use the following parameters: $\alpha_n = \frac{n}{n+1}$, $\beta = \frac{n}{n+7}$, $\gamma = \frac{2n}{7n+5}$, and the initial value $t_1 = 0.5$. Obviously, the fixed point of Υ is $p = 0$, with a contraction constant $\delta = \frac{3}{4}$. Table $A1$/Table $B1$ shows the behavior of the multivalued UD iteration process (3.1) in comparison with the multivalued iteration processes (see Table (1)) of Mann , Ishikawa , Noor , Agarwal *et al.* (S-iteration), Abbas and Nazir , Thakur et al. , modified Piri et al. to the fixed point of Υ in 80-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

From the Table $A1$ and Table $B1$, we observed that since Mann converges after 173 iterations, Ishikawa converges after 157 iterations, Noor converges after 155 iterations, they have been excluded from the table, but S-iteration converges at the 76th iteration, Thakur converges at the 67th iteration, Abbas and Nazir converges at the 52nd iteration, modified Piri *et al.* converges at the 51st iteration, and NEW UD converges at the 38th iteration.

For Table $C1$ (see (Appendix C)), we use the following parameters: $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, and the initial value $t_1 = 0.5$. Number of iterations: 55-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

Again, from Table $C1$, we observe that since Mann converges after 111 iterations and Ishikawa converges after 67 iterations, they have been excluded from the table, but Noor converges at the 54th iteration, S-iteration converges at the 53rd iteration, Thakur converges at the 36th iteration, Abbas-Nazir converges at the 35th iteration, modified Piri et al. converges at the 33rd iteration, and NEW UD converges at the 30th iteration.

Clearly, the tabulations in Table $A1$, Table $B1$ (See Figure 1) and Table $C1$ (See Figure 2) in (Appendix A), (Appendix B) and (Appendix C) respectively shows that the multivalued UD iteration process (3.1) has a faster rate of convergence than the modified Piri *et al.* iterative scheme (1.3) and some well known iteration schemes in literature for multivalued contraction mapping.

Example 4.3. Let $(\Re, \|.\|)$ be a normed space with the usual norm and $\zeta = [1,4]$. Let $\Upsilon : \zeta \longrightarrow P(\zeta)$ be defined by

$$\Upsilon u = \begin{cases} [0, \frac{u}{2}], & u \in [0,2] \\ 0, & u \in (2,4] \end{cases} \tag{4.11}$$

Then

(1) Υ is multivalued generalized α-nonexpansive mapping.
(2) Υ does not satisfy Condition (C).

Proof .

(1) We now consider the following cases:

Case (i): If $u \in [0,2]$, then

$$\mathcal{H}(\Upsilon u, \Upsilon v) = |\Upsilon u - \Upsilon v| = |\frac{u}{2} - \frac{v}{2}| = \frac{1}{2}|u-v|$$

and,

$$\begin{aligned} \alpha d(\Upsilon u - v) + \alpha d(\Upsilon v - u) + (1-2\alpha)\|u-v\| &= \frac{1}{3}|\frac{u}{2} - v| + \frac{1}{3}|\frac{v}{2} - u| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}|\frac{3u}{2} - \frac{3v}{2}| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{2}|u-v| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{2}|u-v| = \mathcal{H}(\Upsilon u, \Upsilon v). \end{aligned}$$

Case (ii): If $u \in (2,4]$, then

$$\mathcal{H}(\Upsilon u, \Upsilon v) = |\Upsilon u - \Upsilon v| = |1-1| = 0$$

and,

$$\begin{aligned} \alpha d(\Upsilon u - v) + \alpha d(\Upsilon v - u) + (1-2\alpha)\|u-v\| &= \frac{1}{3}|0-v| + \frac{1}{3}|0-u| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}|u-v| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}|0| \\ &= 0 = \mathcal{H}(\Upsilon u, \Upsilon v). \end{aligned}$$

Case (iii): If $u \in [0,2]$ and $v \in (2,4]$, then

$$\mathcal{H}(\Upsilon u, \Upsilon v) = |\Upsilon u - \Upsilon v| = |\frac{u}{2} - 0| = \frac{u}{2}$$

and,

$$\begin{aligned} \alpha d(\Upsilon u - v) + \alpha d(\Upsilon v - u) + (1-2\alpha)\|u-v\| &= \frac{1}{3}|\frac{u}{2} - v| + \frac{1}{3}|0-u| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}|v - \frac{u}{2} + u| + \frac{1}{3}|u-v| \\ &= \frac{1}{3}|v + \frac{u}{2}| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}|v + \frac{u}{2} + u - v| \\ &= \frac{1}{3}|\frac{3u}{2}| \\ &= \frac{u}{2} = \mathcal{H}(\Upsilon u, \Upsilon v). \end{aligned}$$

Hence, Υ is multivalued generalized $\frac{1}{3}$-nonexpansive mapping.

(2) We now show that Υ does not satisfy Condition (C). Let $u = \frac{9}{5}$ and $v = \frac{14}{5}$, then we have

$$\begin{aligned}
\mathcal{H}(\Upsilon u, \Upsilon v) &= \alpha d(\Upsilon u - v) + \alpha d(\Upsilon v - u) + (1-2\alpha)\|u - v\| \\
&= \frac{1}{3}\left|\frac{9}{10} - \frac{14}{5}\right| + \frac{1}{3}\left|\frac{7}{5} - \frac{9}{5}\right| + \frac{1}{3}\left|\frac{9}{5} - \frac{14}{5}\right| \\
&= \frac{1}{3}\left(\frac{19}{10}\right) + \frac{1}{3}\left(\frac{2}{5}\right) + \frac{1}{3}(1) \\
&= \frac{19}{30} + \frac{12}{15} + \frac{1}{3} \\
&= \frac{11}{10}.
\end{aligned}$$

Also, $|u - v| = \left|\frac{9}{5} - \frac{14}{5}\right| = 1$, and,

$$\begin{aligned}
\frac{1}{2}d(u, \Upsilon u) &= \frac{1}{2}d(\frac{9}{5}, \Upsilon\frac{9}{5}) = \frac{1}{2}d(\frac{9}{5}, \frac{9}{10}) \\
&= \frac{1}{2}\left|\frac{9}{5} - \frac{9}{10}\right| \\
&= \frac{1}{2}\left|\frac{9}{10}\right| \\
&= \frac{9}{20} \\
&< 1 = |u - v|.
\end{aligned}$$

But on the other hand, $\mathcal{H}(\Upsilon u, \Upsilon v) = \frac{11}{10} > 1 = |x - y|$.

Thus, Υ does not satisfy Condition (C). □

Hence, we show that P_Υ is multivalued generalized α-nonexpansive mapping. We consider the following cases:

Case 1: If $u \in [0, 2]$, then we have

$$\begin{aligned}
P_\Upsilon &= \left\{v \in \Upsilon u : |v - u| = d\left(u, \left[1, \frac{u}{2}\right]\right)\right\} \\
&= \left\{v \in \Upsilon u : |v - u| = \left|u - \frac{u}{2}\right|\right\} \\
&= \left\{v \in \Upsilon u : |v - u| = \left|\frac{x}{2}\right|\right\} \\
&= \left\{v \in \Upsilon u : u - v = \frac{u}{2}\right\} \\
&= \left\{v \in \Upsilon u : v = \frac{u}{2}\right\}.
\end{aligned}$$

Case 2: If $u \in (2, 4]$, then we have

$$\begin{aligned}
P_\Upsilon &= \{v \in \Upsilon u : |v - u| = d(u, \{0\})\} \\
&= \{v \in \Upsilon u : |v - u| = |u - 0|\} \\
&= \{v \in \Upsilon u : u - v = u\} \\
&= \{v \in \Upsilon u : v = 0\}.
\end{aligned}$$

By following the same arguments as in Example (4.3), we can clearly show that P_Υ is a multivalued generalized α-nonexpansive mapping.

Finally, using the above Example (4.3), we show that the multivalued UD iteration process (3.1) converges faster than iterative processes of Mann, Ishikawa, Noor, Agarwal *et al.* (S-iteration) , Abbas and Nazir, Thakur *et al.*, and modified Piri *et al.* to the fixed point of multivalued generalized α-nonexpansive mapping.

Example 4.4. Let $(\Re, \|.\|)$ be a normed space with the usual norm and $\zeta = [0,4]$. Let $P_\Upsilon : \zeta \longrightarrow P(\zeta)$ be a multivalued generalized α-nonexpansive mapping defined by

$$P_\Upsilon u = \begin{cases} [0, \frac{u}{2}], & u \in [0,2] \\ 0, & u \in (2,4] \end{cases}$$

For Table $D1$ (see (Appendix D)), we use the following parameters:

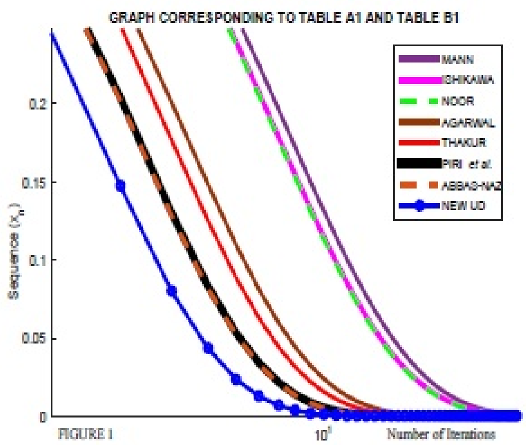
Choose $\alpha_n = \frac{3n}{8n+4}$, $\beta_n = \frac{1}{n+4}$, $\gamma_n = \frac{2n}{5n+2}$, and the initial value $t_1 = 0.5$. Obviously, the fixed point of P_Υ is $p = 0 \in P_\Upsilon u$. Table $D1$ shows the behavior of the multivalued UD iteration process (3.1) in comparison with the multivalued iteration processes (see Table (1)) of Mann, Ishikawa, Noor, Agarwal et al. (S-iteration) , Abbas and Nazir, Thakur et al., and modified Piri et al. to the fixed point of Υ in 40-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

From Table $D1$, we observed that since Mann converges after 173 iterations, Ishikawa converges after 156 iterations, Noor converges after 154 iterations, they have been excluded from the table, but S-iteration converges at the 33rd iteration, Thakur converges at the 27th iteration, modified Piri *et al.* converges at the 25th iteration, Abbas-Nazir converges at the 24th iteration and NEW UD converges at the 20th iteration.

For Table $E1$/Table $F1$ (see (Appendix E) and (Appendix F)), we use the following parameters:
Choose $\alpha_n = \beta_n = \gamma_n = \frac{2}{3}$, and the initial value $t_1 = 0.5$, Number of iterations:.'/ 60-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

Again, from Table $E1$/Table $F1$, we observe that Mann converges at the 57th iteration, Ishikawa converges at the 40th iteration, Noor converges at the 36th iteration, S-iteration converges at the 25th iteration, Thakur converges at the 18th iteration, Abbas-Nazir converges at the 17th iteration, modified Piri *et al.* converges at the 16th iteration, and NEW UD converges at the 14th iteration.

Clearly, the tabulations in Table Table $D1$ (See Figure 3), Table $E1$ and Table $F1$ (See Figure 4) in (Appendix D), (Appendix E) and (Appendix F) respectively shows that the multivalued UD iteration process (3.1) has a faster rate of convergence than some well known iteration schemes in literature for multivalued generalized α-nonexpansive mapping.



Appendix A Table A1

Convergence Analysis of Multivalued UD-Iteration Process in Comparison with Mann, Ishikawa, Noor, Agarwal, Thakur, Abbas-Nazir and Piri et al. Iteration Schemes for Multivalued Contraction Mapping in 80-iterations

n	AGARWAL	THAKUR	ABBAS-NAZIR	PIRI ET AL	NEW UD
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.3691406250	0.3537597656	0.3195800781	0.3178710938	0.2717285156
2	0.2725296021	0.2502919436	0.2042628527	0.2020840645	0.1476727724
3	0.2012034953	0.1770864386	0.1305566768	0.1284733652	0.0802538065
4	0.1485447680	0.1252921140	0.0834466260	0.0816759382	0.0436144954
5	0.1096678170	0.0886466178	0.0533357585	0.0519248396	0.0237026042
6	0.0809656930	0.0627192135	0.0340900917	0.0330108111	0.0128813469
7	0.0597754530	0.0443750685	0.0217890284	0.0209863653	0.0070004586
8	0.0441310962	0.0313962277	0.0139266788	0.0133419178	0.0038044484
9	0.0325811609	0.0222134443	0.0089013782	0.0084820200	0.0020675542
10	0.0240540602	0.0157164457	0.0056894063	0.0053923779	0.0011236269
11	0.0177586616	0.0111196923	0.0036364418	0.0034281621	0.0006106429
12	0.0131108869	0.0078673995	0.0023242687	0.0021794273	0.0003318582
13	0.0096795220	0.0055663388	0.0014855800	0.0013855539	0.0001803507
14	0.0071462096	0.0039382934	0.0009495235	0.0008808551	0.0000980128
15	0.0052759125	0.0027864195	0.0006068976	0.0005599967	0.0000532658
16	0.0038951073	0.0019714462	0.0003879048	0.0003560135	0.0000289477
17	0.0028756847	0.0013948367	0.0002479333	0.0002263328	0.0000157318
18	0.0021230641	0.0009868742	0.0001584691	0.0001438893	0.0000085496
19	0.0015674184	0.0006982328	0.0001012871	0.0000914765	0.0000046463
20	0.0011571956	0.0004940133	0.0000647387	0.0000581555	0.0000025251
21	0.0008543358	0.0003495241	0.0000413784	0.0000369719	0.0000013723
22	0.0006307401	0.0002472951	0.0000264474	0.0000235046	0.0000007458
23	0.0004656636	0.0001749661	0.0000169041	0.0000149429	0.0000004053
24	0.0003437907	0.0001237919	0.0000108045	0.0000094998	0.0000002203
25	0.0002538142	0.0000875852	0.0000069058	0.0000060394	0.0000001197
26	0.0001873863	0.0000619683	0.0000044139	0.0000038395	0.0000000651
27	0.0001383438	0.0000438438	0.0000028212	0.0000024409	0.0000000354
28	0.0001021366	0.0000310203	0.0000018032	0.0000015518	0.0000000192
29	0.0000754056	0.0000219475	0.0000011525	0.0000009866	0.0000000104
30	0.0000556705	0.0000155283	0.0000007366	0.0000006272	0.0000000057
31	0.0000411005	0.0000109866	0.0000004708	0.0000003987	0.0000000031
32	0.0000303437	0.0000077732	0.0000003009	0.0000002535	0.0000000017
33	0.0000224022	0.0000054997	0.0000001923	0.0000001612	0.0000000009
34	0.0000165391	0.0000038911	0.0000001229	0.0000001025	0.0000000005
35	0.0000122105	0.0000027531	0.0000000786	0.0000000651	0.0000000003
36	0.0000090148	0.0000019478	0.0000000502	0.0000000414	0.0000000001
37	0.0000066555	0.0000013781	0.0000000321	0.0000000263	0.0000000001
38	0.0000049136	0.0000009751	0.0000000205	0.0000000167	0.0000000000
39	0.0000036276	0.0000006899	0.0000000131	0.0000000106	0.0000000000
40	0.0000026782	0.0000004881	0.0000000084	0.0000000068	0.0000000000

Appendix B Table B1: *Continuation of Table* ***A1***

n	AGARWAL	THAKUR	ABBAS-NAZIR	PIRI ET AL	NEW UD
41	0.0000019773	0.0000003453	0.0000000054	0.0000000043	0.0000000000
42	0.0000014598	0.0000002443	0.0000000034	0.0000000027	0.0000000000
43	0.0000010777	0.0000001729	0.0000000022	0.0000000017	0.0000000000
44	0.0000007957	0.0000001223	0.0000000014	0.0000000011	0.0000000000
45	0.0000005874	0.0000000865	0.0000000009	0.0000000007	0.0000000000
46	0.0000004337	0.0000000612	0.0000000006	0.0000000004	0.0000000000
47	0.0000003202	0.0000000433	0.0000000004	0.0000000003	0.0000000000
48	0.0000002364	0.0000000306	0.0000000002	0.0000000002	0.0000000000
49	0.0000001745	0.0000000217	0.0000000001	0.0000000001	0.0000000000
50	0.0000001288	0.0000000153	0.0000000001	0.0000000001	0.0000000000
51	0.0000000951	0.0000000109	0.0000000001	0.0000000000	0.0000000000
52	0.0000000702	0.0000000077	0.0000000000	0.0000000000	0.0000000000
53	0.0000000518	0.0000000054	0.0000000000	0.0000000000	0.0000000000
54	0.0000000383	0.0000000038	0.0000000000	0.0000000000	0.0000000000
55	0.0000000283	0.0000000027	0.0000000000	0.0000000000	0.0000000000
56	0.0000000209	0.0000000019	0.0000000000	0.0000000000	0.0000000000
57	0.0000000154	0.0000000014	0.0000000000	0.0000000000	0.0000000000
58	0.0000000114	0.0000000010	0.0000000000	0.0000000000	0.0000000000
59	0.0000000084	0.0000000007	0.0000000000	0.0000000000	0.0000000000
60	0.0000000062	0.0000000005	0.0000000000	0.0000000000	0.0000000000
61	0.0000000046	0.0000000003	0.0000000000	0.0000000000	0.0000000000
62	0.0000000034	0.0000000002	0.0000000000	0.0000000000	0.0000000000
63	0.0000000025	0.0000000002	0.0000000000	0.0000000000	0.0000000000
64	0.0000000018	0.0000000001	0.0000000000	0.0000000000	0.0000000000
65	0.0000000014	0.0000000001	0.0000000000	0.0000000000	0.0000000000
66	0.0000000010	0.0000000001	0.0000000000	0.0000000000	0.0000000000
67	0.0000000007	0.0000000000	0.0000000000	0.0000000000	0.0000000000
68	0.0000000005	0.0000000000	0.0000000000	0.0000000000	0.0000000000
69	0.0000000004	0.0000000000	0.0000000000	0.0000000000	0.0000000000
70	0.0000000003	0.0000000000	0.0000000000	0.0000000000	0.0000000000
71	0.0000000002	0.0000000000	0.0000000000	0.0000000000	0.0000000000
72	0.0000000002	0.0000000000	0.0000000000	0.0000000000	0.0000000000
73	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
74	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
75	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
76	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
77	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
78	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
79	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
80	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Appendix C Table C1

Convergence Analysis of Multivalued UD-Iteration Process in Comparison with Thakur, Abbas-Nazir and Piri *et al.* Iteration Schemes for Multivalued Generalized α-Nonexpansive Mapping in 55-iterations

n	NOOR	AGARWAL	THAKUR	ABBAS-NAZIR	PIRI ET AL	NEW UD
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.3238525391	0.3222656250	0.2618408203	0.2574462891	0.2475585938	0.2318115234
2	0.2097609341	0.2077102661	0.1371212304	0.1325571835	0.1225705147	0.1074731648
3	0.1358632222	0.1338757575	0.0718078709	0.0682527100	0.0606867685	0.0498270361
4	0.0879992990	0.0862871093	0.0376044636	0.0351428138	0.0300470621	0.0231009623
5	0.0569975928	0.0556147384	0.0196927672	0.0180947740	0.0148768169	0.0107101385
6	0.0369176303	0.0358454369	0.0103127406	0.0093168648	0.0073657677	0.0049654671
7	0.0239117366	0.0231035042	0.0054005929	0.0047971846	0.0036469182	0.0023021050
8	0.0154877532	0.0148909305	0.0028281914	0.0024700347	0.0018056519	0.0010673089
9	0.0100314964	0.0095976700	0.0014810719	0.0012718025	0.0008940093	0.0004948290
10	0.0064974512	0.0061859983	0.0007756102	0.0006548417	0.0004426394	0.0002294141
11	0.0042084321	0.0039870692	0.0004061728	0.0003371731	0.0002191584	0.0001063617
12	0.0027258229	0.0025697907	0.0002127052	0.0001736079	0.0001085091	0.0000493117
13	0.0017655293	0.0016563104	0.0001113898	0.0000893894	0.0000537247	0.0000228621
14	0.0011435423	0.0010675438	0.0000583328	0.0000460260	0.0000266000	0.0000105994
15	0.0007406782	0.0006880653	0.0000305478	0.0000236984	0.0000131701	0.0000049141
16	0.0004797410	0.0004434796	0.0000159973	0.0000122021	0.0000065208	0.0000022783
17	0.0003107307	0.0002858365	0.0000083775	0.0000062828	0.0000032285	0.0000010563
18	0.0002012618	0.0001842305	0.0000043871	0.0000032350	0.0000015985	0.0000004897
19	0.0001303583	0.0001187423	0.0000022975	0.0000016657	0.0000007914	0.0000002270
20	0.0000844337	0.0000765331	0.0000012031	0.0000008576	0.0000003919	0.0000001053
21	0.0000546882	0.0000493280	0.0000006301	0.0000004416	0.0000001940	0.0000000488
22	0.0000354218	0.0000317934	0.0000003300	0.0000002274	0.0000000961	0.0000000226
23	0.0000229429	0.0000204919	0.0000001728	0.0000001171	0.0000000476	0.0000000105
24	0.0000148602	0.0000132076	0.0000000905	0.0000000603	0.0000000235	0.0000000049
25	0.0000096250	0.0000085127	0.0000000474	0.0000000310	0.0000000117	0.0000000023
26	0.0000062342	0.0000054867	0.0000000248	0.0000000160	0.0000000058	0.0000000010
27	0.0000040379	0.0000035364	0.0000000130	0.0000000082	0.0000000029	0.0000000005
28	0.0000026154	0.0000022793	0.0000000068	0.0000000042	0.0000000014	0.0000000002
29	0.0000016940	0.0000014691	0.0000000036	0.0000000022	0.0000000007	0.0000000001
30	0.0000010972	0.0000009469	0.0000000019	0.0000000011	0.0000000003	0.0000000000
31	0.0000007107	0.0000006103	0.0000000010	0.0000000006	0.0000000002	0.0000000000
32	0.0000004603	0.0000003933	0.0000000005	0.0000000003	0.0000000001	0.0000000000
33	0.0000002981	0.0000002535	0.0000000003	0.0000000002	0.0000000000	0.0000000000
34	0.0000001931	0.0000001634	0.0000000001	0.0000000001	0.0000000000	0.0000000000
35	0.0000001251	0.0000001053	0.0000000001	0.0000000000	0.0000000000	0.0000000000
36	0.0000000810	0.0000000679	0.0000000000	0.0000000000	0.0000000000	0.0000000000
..	....	....	....	....	....	....
50	0.0000000002	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
51	0.0000000001	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
52	0.0000000001	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
53	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
54	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
55	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

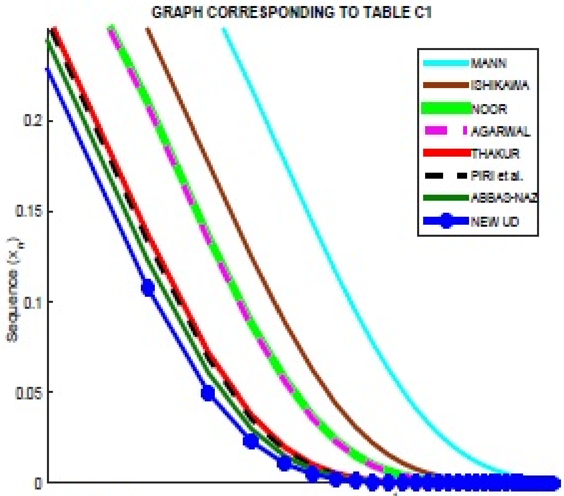


FIGURE 2:

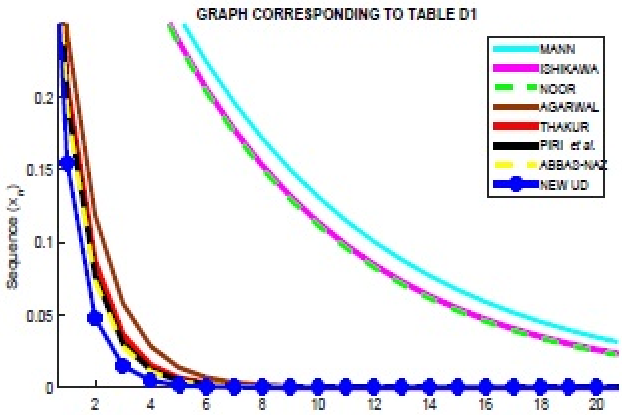


FIGURE 3

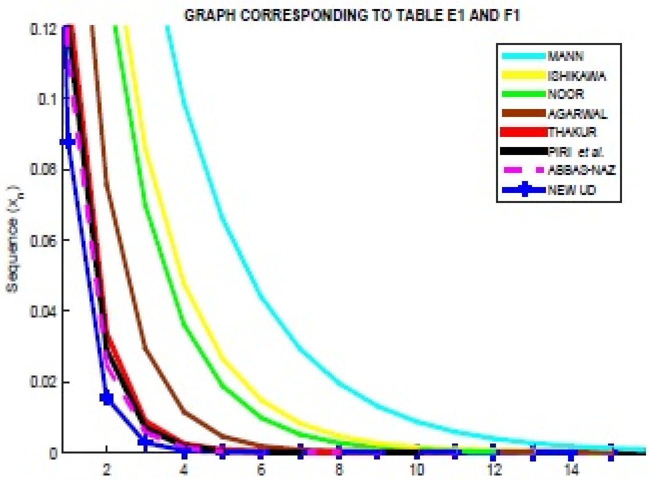


FIGURE 4

Appendix D Table D1

Convergence Analysis of Multivalued UD-Iteration Process in Comparison with Agarwal, Thakur, Piri *et al.* and Abbas-Nazir Iteration Schemes for Multivalued Generalized α-Nonexpansive Mapping in 40-iterations

n	AGARWAL	THAKUR	PIRI *et al.*	ABBAS-NAZIR	NEW UD
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.2437500000	0.2089285714	0.1968750000	0.1910714286	0.1547619048
2	0.1188281250	0.0873022959	0.0775195313	0.0730165816	0.0479024943
3	0.0579287109	0.0364798879	0.0305233154	0.0279027651	0.0148269625
4	0.0282402466	0.0152433817	0.0120185555	0.0106628424	0.0045892979
5	0.0137671202	0.0063695559	0.0047323062	0.0040747291	0.0014204970
6	0.0067114711	0.0026615644	0.0018633456	0.0015571286	0.0004396776
7	0.0032718422	0.0011121537	0.0007336923	0.0005950456	0.0001360907
8	0.0015950231	0.0004647214	0.0002888914	0.0002273924	0.0000421233
9	0.0007775737	0.0001941871	0.0001137510	0.0000868964	0.0000130382
10	0.0003790672	0.0000811425	0.0000447894	0.0000332068	0.0000040356
11	0.0001847953	0.0000339060	0.0000176358	0.0000126898	0.0000012491
12	0.0000900877	0.0000141679	0.0000069441	0.0000048493	0.0000003866
13	0.0000439177	0.0000059201	0.0000027342	0.0000018531	0.0000001197
14	0.0000214099	0.0000024738	0.0000010766	0.0000007082	0.0000000370
15	0.0000104373	0.0000010337	0.0000004239	0.0000002706	0.0000000115
16	0.0000050882	0.0000004319	0.0000001669	0.0000001034	0.0000000035
17	0.0000024805	0.0000001805	0.0000000657	0.0000000395	0.0000000011
18	0.0000012092	0.0000000754	0.0000000259	0.0000000151	0.0000000003
19	0.0000005895	0.0000000315	0.0000000102	0.0000000058	0.0000000001
20	0.0000002874	0.0000000132	0.0000000040	0.0000000022	0.0000000000
21	0.0000001401	0.0000000055	0.0000000016	0.0000000008	0.0000000000
22	0.0000000683	0.0000000023	0.0000000006	0.0000000003	0.0000000000
23	0.0000000333	0.0000000010	0.0000000002	0.0000000001	0.0000000000
24	0.0000000162	0.0000000004	0.0000000001	0.0000000000	0.0000000000
25	0.0000000079	0.0000000002	0.0000000000	0.0000000000	0.0000000000
26	0.0000000039	0.0000000001	0.0000000000	0.0000000000	0.0000000000
27	0.0000000019	0.0000000000	0.0000000000	0.0000000000	0.0000000000
28	0.0000000009	0.0000000000	0.0000000000	0.0000000000	0.0000000000
29	0.0000000004	0.0000000000	0.0000000000	0.0000000000	0.0000000000
30	0.0000000002	0.0000000000	0.0000000000	0.0000000000	0.0000000000
31	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
32	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
33	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
34	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
35	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
36	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
37	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
38	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
39	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
40	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Appendix E Table E1

Convergence Analysis of Multivalued UD-Iteration Process in Comparison with Mann, Ishikawa, Noor and Agarwal Iteration Schemes for Multivalued Generalized α-Nonexpansive Mapping in 60-iterations

n	MANN	ISHIKAWA	NOOR	AGARWAL	NEW UD
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.3333333333	0.2777777778	0.2592592593	0.1944444444	0.0879629630
2	0.2222222222	0.1543209877	0.1344307270	0.0756172840	0.0154749657
3	0.1481481481	0.0857338820	0.0697048214	0.0294067215	0.0027224477
4	0.0987654321	0.0476299345	0.0361432407	0.0114359473	0.0004789491
5	0.0658436214	0.0264610747	0.0187409396	0.0044473128	0.0000842596
6	0.0438957476	0.0147005971	0.0097175243	0.0017295105	0.0000148234
7	0.0292638317	0.0081669984	0.0050387163	0.0006725874	0.0000026078
8	0.0195092212	0.0045372213	0.0026126677	0.0002615618	0.0000004588
9	0.0130061474	0.0025206785	0.0013547166	0.0001017185	0.0000000807
10	0.0086707650	0.0014003769	0.0007024456	0.0000395572	0.0000000142
11	0.0057805100	0.0007779872	0.0003642311	0.0000153833	0.0000000025
12	0.0038536733	0.0004322151	0.0001888606	0.0000059824	0.0000000004
13	0.0025691155	0.0002401195	0.0000979277	0.0000023265	0.0000000001
14	0.0017127437	0.0001333997	0.0000507773	0.0000009047	0.0000000000
15	0.0011418291	0.0000741110	0.0000263290	0.0000003518	0.0000000000
16	0.0007612194	0.0000411728	0.0000136521	0.0000001368	0.0000000000
17	0.0005074796	0.0000228738	0.0000070788	0.0000000532	0.0000000000
18	0.0003383197	0.0000127076	0.0000036705	0.0000000207	0.0000000000
19	0.0002255465	0.0000070598	0.0000019032	0.0000000080	0.0000000000
20	0.0001503643	0.0000039221	0.0000009869	0.0000000031	0.0000000000
21	0.0001002429	0.0000021790	0.0000005117	0.0000000012	0.0000000000
22	0.0000668286	0.0000012105	0.0000002653	0.0000000005	0.0000000000
23	0.0000445524	0.0000006725	0.0000001376	0.0000000002	0.0000000000
24	0.0000297016	0.0000003736	0.0000000713	0.0000000001	0.0000000000
25	0.0000198011	0.0000002076	0.0000000370	0.0000000000	0.0000000000
26	0.0000132007	0.0000001153	0.0000000192	0.0000000000	0.0000000000
27	0.0000088005	0.0000000641	0.0000000099	0.0000000000	0.0000000000
28	0.0000058670	0.0000000356	0.0000000052	0.0000000000	0.0000000000
29	0.0000039113	0.0000000198	0.0000000027	0.0000000000	0.0000000000
30	0.0000026075	0.0000000110	0.0000000014	0.0000000000	0.0000000000
31	0.0000017384	0.0000000061	0.0000000007	0.0000000000	0.0000000000
32	0.0000011589	0.0000000034	0.0000000004	0.0000000000	0.0000000000
33	0.0000007726	0.0000000019	0.0000000002	0.0000000000	0.0000000000
34	0.0000005151	0.0000000010	0.0000000001	0.0000000000	0.0000000000
35	0.0000003434	0.0000000006	0.0000000001	0.0000000000	0.0000000000
36	0.0000002289	0.0000000003	0.0000000000	0.0000000000	0.0000000000
37	0.0000001526	0.0000000002	0.0000000000	0.0000000000	0.0000000000
38	0.0000001017	0.0000000001	0.0000000000	0.0000000000	0.0000000000
39	0.0000000678	0.0000000001	0.0000000000	0.0000000000	0.0000000000
40	0.0000000452	0.0000000000	0.0000000000	0.0000000000	0.0000000000
..	....	....	....	....	....
57	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
..	....	....	....	....	....
60	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Appendix F Table F1

Convergence Analysis of Multivalued UD-Iteration Process in Comparison with Thakur, Abbas-Nazir and Piri *et al.* Iteration Schemes for Multivalued Generalized α-Nonexpansive Mapping in 60-iterations

n	THAKUR	ABBAS-NAZIR	PIRI *et al.*	NEW UD
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.1296296296	0.1203703704	0.1111111111	0.0879629630
2	0.0336076818	0.0289780521	0.0246913580	0.0154749657
3	0.0087131027	0.0069761977	0.0054869684	0.0027224477
4	0.0022589525	0.0016794550	0.0012193263	0.0004789491
5	0.0005856544	0.0004043132	0.0002709614	0.0000842596
6	0.0001518363	0.0000973347	0.0000602136	0.0000148234
7	0.0000393650	0.0000234324	0.0000133808	0.0000026078
8	0.0000102057	0.0000056411	0.0000029735	0.0000004588
9	0.0000026459	0.0000013581	0.0000006608	0.0000000807
10	0.0000006860	0.0000003269	0.0000001468	0.0000000142
11	0.0000001778	0.0000000787	0.0000000326	0.0000000025
12	0.0000000461	0.0000000189	0.0000000073	0.0000000004
13	0.0000000120	0.0000000046	0.0000000016	0.0000000001
14	0.0000000031	0.0000000011	0.0000000004	0.0000000000
15	0.0000000008	0.0000000003	0.0000000001	0.0000000000
16	0.0000000002	0.0000000001	0.0000000000	0.0000000000
17	0.0000000001	0.0000000000	0.0000000000	0.0000000000
18	0.0000000000	0.0000000000	0.0000000000	0.0000000000
19	0.0000000000	0.0000000000	0.0000000000	0.0000000000
20	0.0000000000	0.0000000000	0.0000000000	0.0000000000
21	0.0000000000	0.0000000000	0.0000000000	0.0000000000
22	0.0000000000	0.0000000000	0.0000000000	0.0000000000
23	0.0000000000	0.0000000000	0.0000000000	0.0000000000
24	0.0000000000	0.0000000000	0.0000000000	0.0000000000
25	0.0000000000	0.0000000000	0.0000000000	0.0000000000
26	0.0000000000	0.0000000000	0.0000000000	0.0000000000
27	0.0000000000	0.0000000000	0.0000000000	0.0000000000
28	0.0000000000	0.0000000000	0.0000000000	0.0000000000
29	0.0000000000	0.0000000000	0.0000000000	0.0000000000
30	0.0000000000	0.0000000000	0.0000000000	0.0000000000
31	0.0000000000	0.0000000000	0.0000000000	0.0000000000
32	0.0000000000	0.0000000000	0.0000000000	0.0000000000
33	0.0000000000	0.0000000000	0.0000000000	0.0000000000
34	0.0000000000	0.0000000000	0.0000000000	0.0000000000
35	0.0000000000	0.0000000000	0.0000000000	0.0000000000
36	0.0000000000	0.0000000000	0.0000000000	0.0000000000
37	0.0000000000	0.0000000000	0.0000000000	0.0000000000
38	0.0000000000	0.0000000000	0.0000000000	0.0000000000
39	0.0000000000	0.0000000000	0.0000000000	0.0000000000
40	0.0000000000	0.0000000000	0.0000000000	0.0000000000
..	....	....	....	....
60	0.0000000000	0.0000000000	0.0000000000	0.0000000000

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