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Growth of solutions to complex linear differential equations in which the coefficients are analytic functions except at a finite singular point

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Abstract

In this paper, we study the growth of solutions to complex higher order linear differential equations, where the coefficients are analytic in the closed complex plane except at a finite singular point. We obtain some results on the $[p,q]$-order and on the lower $[p,q]$-order which improve and extend those of Long and Zeng.

Keywords: Linear differential equations, Growth of solutions, Finite singular point, Nevanlinna theory.
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1 Introduction and main results

For $k \geq 2$, we consider the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_j(z)$ $(j = 0, \ldots, k-1)$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. By using the concepts of $[p,q]$-order and $[p,q]$-type which were firstly introduced by Juneja and his co-authors for entire functions (see [10, 11]), many authors have investigated the complex linear differential equation (1.1) for the cases when the coefficients are entire functions, meromorphic functions, analytic in the unit disk and recently when they are analytic except at a finite singular point (see e.g. [1, 2, 3, 9, 13, 14, 15, 17, 18, 19, 20]), including Long and Zeng who they made a minor modification to these concepts for functions which are analytic except at a finite singular point and they also obtained some results about the growth of solutions of (1.1) (see [17]). In this paper using these concepts of the $[p,q]$-order and the $[p,q]$-type giving in [17], also using the lower $[p,q]$-order and lower $[p,q]$-type which we define them similarly, we study the growth of solutions of (1.1).

Nevanlinna's theory is the main tool in this work, so we assume the reader is familiar with its fundamental results and its standard notations (see [6, 8, 12, 21].
Before stating our results, we need to introduce some notations and definitions which can be found in [4, 5, 17], we also mention some previous results. Firstly, we define for all $r \in (0, \infty)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, we also

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define for all sufficiently large r, $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We denote $\exp_0 r := r$, $\log_0 r := r$, $\exp_{-1} r := \log_1 r$, $\log_{-1} r := \exp_1 r$, we also denote the logarithmic measure of a set $E \subset (0,1)$ by $m_l(E) = \int_E \frac{dt}{t}$.

Definition 1.1. [4, 5] Let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \mathbb{C}$. The counting function of f near z_0 is defined by

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t} dt - n(\infty,f)\log r,$$

where $n(t,f)$ counts the number of poles of f in $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$, each pole according to its multiplicity. The proximity function of f near z_0 is defined by

$$m_{z_0}(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi.$$

The characteristic function of f near z_0 is defined by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Definition 1.2. [17] Let f be an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, p and q be two integers with $p \geq q \geq 1$. The $[p,q]$-order of f near z_0 is defined by

$$\rho_{[p,q]}(f,z_0) = \limsup_{r \longrightarrow 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \limsup_{r \longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}} = \limsup_{r \longrightarrow 0} \frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$ and $V_{z_0}(r,f)$ is the central index of f near z_0. If $\rho_{[p,q]}(f,z_0) = \rho \in (0,\infty)$, then the $[p,q]$-type of f near z_0 is defined by

$$\tau_{[p,q],M}(f,z_0) = \limsup_{r \longrightarrow 0} \frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1} \frac{1}{r})^{\rho}}.$$

Similarly, we define the lower $[p,q]$-order and the lower $[p,q]$-type by using "lim inf" instead of "lim sup".

Definition 1.3. Let f be an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, p and q be two integers with $p \geq q \geq 1$. The lower $[p,q]$-order of f near z_0 is defined by

$$\mu_{[p,q]}(f,z_0) = \liminf_{r \longrightarrow 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \liminf_{r \longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}} = \liminf_{r \longrightarrow 0} \frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

If $\mu_{[p,q]}(f,z_0) = \mu \in (0,\infty)$, then the lower $[p,q]$-type of f is defined by

$$\underline{\tau}_{[p,q],M}(f,z_0) = \liminf_{r \longrightarrow 0} \frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1} \frac{1}{r})^{\mu}}.$$

Long and Zeng investigated the growth of solutions of (1.1) and they obtained the following results on the $[p,q]$-order. Firstly, when there is a dominating coefficient $A_0(z)$ with $[p,q]$-order.

Theorem 1.4. [17] *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j = 1,\ldots,k-1\} < \rho_{[p,q]}(A_0,z_0) < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.1) satisfies $\rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0)$.

Secondly, when there are other coefficients having the same $[p,q]$-order as $A_0(z)$.

Theorem 1.5. [17] *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j = 1,\ldots,k-1\} \leq \rho_{[p,q]}(A_0,z_0) < +\infty$$

and

$$\max\{\tau_{[p,q],M}(A_j,z_0) : \rho_{[p,q]}(A_j,z_0) = \rho_{[p,q]}(A_0,z_0) > 0\} < \tau_{[p,q],M}(A_0,z_0) < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.1) satisfies $\rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0)$.

The aim of this paper is to investigate the growth of solutions of (1.1) under different hypotheses. First, when $A_0(z)$ is a dominating coefficient with lower $[p,q]$-order instead of $[p,q]$-order. Next, when there are other coefficients having $[p,q]$-order equals the lower $[p,q]$-order of $A_0(z)$, we obtain the following theorems.

Theorem 1.6. *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0): j=1,\ldots,k-1\} < \mu_{[p,q]}(A_0,z_0) \le \rho_{[p,q]}(A_0,z_0) < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (1.1) satisfies

$$\mu_{[p,q]}(A_0,z_0) = \mu_{[p+1,q]}(f,z_0) \le \rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0).$$

Theorem 1.7. *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0): j=1,\ldots,k-1\} \le \mu_{[p,q]}(A_0,z_0) \le \rho_{[p,q]}(A_0,z_0) = \rho < +\infty$$

and

$$\tau_1 = \max\{\tau_{[p,q],M}(A_j,z_0): \rho_{[p,q]}(A_j,z_0) = \mu_{[p,q]}(A_0,z_0) > 0\} < \underline{\tau}_{[p,q],M}(A_0,z_0) = \underline{\tau} < +\infty.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (1.1) satisfies $\mu_{[p,q]}(A_0,z_0) = \mu_{[p+1,q]}(f,z_0) \le \rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0)$.

Theorem 1.8. *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0): j=1,\ldots,k-1\} \le \mu_{[p,q]}(A_0,z_0) \le \rho_{[p,q]}(A_0,z_0) < +\infty$$

and

$$\limsup_{r\longrightarrow 0} \frac{\sum_{j=1}^{k-1} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)} < 1.$$

Then every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (1.1) satisfies

$$\mu_{[p,q]}(A_0,z_0) = \mu_{[p+1,q]}(f,z_0) \le \rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0).$$

Remark 1.9. The following example shows that there exists a solution f of (1.1) such that f is not analytic in $\overline{\mathbb{C}}-\{z_0\}$ provided that all coefficients $A_j(z)$ $(j=0,\ldots,k-1)$ of (1.1) are analytic in $\overline{\mathbb{C}}-\{z_0\}$. For instance, we consider the equation

$$f'' + \left(\exp_2\left\{\frac{1}{z_0-z}\right\} + \frac{1}{z_0-z}\right) f' + \frac{2}{z_0-z}\exp_2\left\{\frac{1}{z_0-z}\right\} f = 0. \tag{1.2}$$

The function $f(z) = (z_0-z)^2$ solves (1.2), and f is not analytic in $\overline{\mathbb{C}}-\{z_0\}$. So, in our results, we suppose always that f is analytic in $\overline{\mathbb{C}}-\{z_0\}$.

2 Some useful lemmas

The following lemmas are important to prove our results.

Lemma 2.1. *Let f be a nonconstant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\mu_{[p,q]}(f,z_0)=\mu<\infty$. Then there exists a set $E\subset(0,1)$ having infinite logarithmic measure such that for all $|z-z_0|=r\in E$, we have*

$$\mu = \lim_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0}\frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0}\frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

and for any given $\varepsilon>0$ and all $|z-z_0|=r\in E$

$$M_{z_0}(r,f) \le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu+\varepsilon}\right\}.$$

Proof . We use a similar proof as in ([16], Lemma 2.5). By the definition of the lower $[p,q]$-order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and

$$\lim_{n\longrightarrow\infty} \frac{\log_{p+1}^{+} M_{z_0}(r_n, f)}{\log_q \frac{1}{r_n}} = \mu.$$

Therefore, there exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$ and for any $r \in [\frac{n}{n+1} r_n, r_n]$, we get

$$\frac{\log_{p+1}^{+} M_{z_0}(r_n, f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} \leq \frac{\log_{p+1}^{+} M_{z_0}(r, f)}{\log_q \frac{1}{r}} \leq \frac{\log_{p+1}^{+} M_{z_0}(\frac{n}{n+1} r_n, f)}{\log_q \frac{1}{r_n}}.$$

Since

$$\lim_{n\longrightarrow+\infty} \frac{\log_{p+1}^{+} M_{z_0}(r_n, f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} = \lim_{n\longrightarrow+\infty} \frac{\log_{p+1}^{+} M_{z_0}(\frac{n}{n+1} r_n, f)}{\log_q \frac{1}{r_n}} = \mu,$$

for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$\lim_{r\longrightarrow 0} \frac{\log_{p+1}^{+} M_{z_0}(r, f)}{\log_q \frac{1}{r}} = \mu.$$

Set $E = \bigcup_{n=n_0}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then for any given $\varepsilon > 0$ and $|z - z_0| = r \in E$

$$M_{z_0}(r, f) \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r}\right)^{\mu+\varepsilon} \right\},$$

where

$$m_l(E) = \sum_{n=n_0}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_0}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty.$$

Similarly, we can prove the other results. □

Lemma 2.2. [17] *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\rho_{[p,q]}(f, z_0) = \rho < \infty$. Then there exists a set $E_1 \subset (0,1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E_1$, we have*

$$\rho = \lim_{r\longrightarrow 0} \frac{\log_p^{+} T_{z_0}(r, f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log_{p+1}^{+} M_{z_0}(r, f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0} \frac{\log_p^{+} V_{z_0}(r, f)}{\log_q \frac{1}{r}},$$

and for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1$

$$T_{z_0}(r, f) \geq \exp_p \left\{ (\rho - \varepsilon) \log_q \frac{1}{r} \right\}.$$

Lemma 2.3. *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $0 < \mu_{[p,q]}(f, z_0) = \mu < \infty$ and $0 < \underline{\tau}_{[p,q],M}(f, z_0) = \underline{\tau} < \infty$. Then there exists a set $E_2 \subset (0,1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E_2$, we have*

$$M_{z_0}(r, f) < \exp_p \left\{ (\underline{\tau} + \varepsilon) \left(\log_{q-1} \frac{1}{r}\right)^{\mu} \right\}.$$

Proof . By the definition of lower $[p,q]$-order and lower $[p,q]$-type, there exists a sequence $\{r_m\}_{m=1}^{\infty}$ tending to 0 satisfying $r_{m+1} < \frac{m}{m+1} r_m$ and

$$\lim_{m\rightarrow+\infty} \frac{\log_p^{+} M_{z_0}(r_m, f)}{\left(\log_{q-1} \frac{1}{r_m}\right)^{\mu}} = \underline{\tau}.$$

For any $r \in \left[\frac{m}{m+1}r_m, r_m\right]$, we have

$$\frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^\mu} \leq \frac{\log_p^+ M_{z_0}(\frac{m}{m+1}r_m, f)}{\left(\log_{q-1}\frac{1}{r_m}\right)^\mu} = \frac{\log_p^+ M_{z_0}(\frac{m}{m+1}r_m, f)}{\left(\log_{q-1}\frac{1}{\frac{m}{m+1}r_m}\right)^\mu} \cdot \frac{\left(\log_{q-1}\frac{1}{\frac{m}{m+1}r_m}\right)^\mu}{\left(\log_{q-1}\frac{1}{r_m}\right)^\mu} \underset{m\to+\infty}{\longrightarrow} \underline{\tau}.$$

Then, for any given $\varepsilon > 0$, there exists a positive integer m_0 such that for all $m \geq m_0$ and for all $r \in \left[\frac{m}{m+1}r_m, r_m\right]$, we have

$$M_{z_0}(r,f) < \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^\mu\right\}.$$

Set $E_2 = \bigcup_{m=m_0}^{+\infty}\left[\frac{m}{m+1}r_m, r_m\right]$. Then for any given $\varepsilon > 0$ and all $|z-z_0| = r \in E_2$

$$M_{z_0}(r,f) < \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^\mu\right\},$$

where

$$m_l(E_2) = \sum_{m=m_0}^{+\infty}\int_{\frac{m}{m+1}r_m}^{r_m}\frac{dt}{t} = \sum_{m=m_0}^{+\infty}\log\left(1+\frac{1}{m}\right) = +\infty.$$

□

Lemma 2.4. [5] *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, let $\alpha > 1$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$. Then there exist a set $E_3 \subset (0, r_0]$, $(r_0 \in (0,1))$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on α and j such that for all $|z-z_0| = r \in (0,r_0] \setminus E_3$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda\left[\frac{1}{r^2}T_{z_0}(\frac{1}{\alpha}r, f)\log T_{z_0}(\frac{1}{\alpha}r, f)\right]^j.$$

Lemma 2.5. [7] *Let f be a nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then, there exists a set $E_4 \subset (0,1)$ that has finite logarithmic measure, such that for all $j = 0, 1, \ldots, k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1+o(1))\left(\frac{V_{z_0}(r,f)}{z_0 - z_r}\right)^j,$$

as $r \longrightarrow 0$, $r \notin E_4$, where z_r is a point in the circle $|z-z_0| = r$ that satisfies $|f(z_r)| = \max\{|f(z)| : |z-z_0| = r\}$.

Lemma 2.6. [17] *Let $g : (0,1) \to \mathbb{R}$, $h : (0,1) \to \mathbb{R}$ be monotone decreasing functions such that $g(r) \geq h(r)$ possibly outside an exceptional set $E_5 \subset (0,1)$ that has finite logarithmic measure. Then for any given $\delta > 1$, there exists a constant $0 < r_1 < 1$, such that for all $r \in (0, r_1)$, we have $g(r^\delta) \geq h(r)$.*

Lemma 2.7. [17] *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. Then the following statements hold:*

(i) $T_{z_0}\left(r, \frac{1}{f}\right) = T_{z_0}(r,f) + O(1)$;

(ii) $T_{z_0}(r, f') < O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right)$, $r \in (0, r_2] \setminus E_6$, *where $E_6 \subset (0, r_2]$ with $m_l(E_6) < +\infty$, $r_2 \in (0,1)$* is a constant.

3 Proof of theorems

Proof of Theorem 1.6.

Proof . We only need to prove that every solution $f \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.1) satisfies $\mu_{[p+1,q]}(f, z_0) = \mu_{[p,q]}(A_0, z_0)$, because we already have from Theorem 1.4, $\rho_{[p+1,q]}(f, z_0) = \rho_{[p,q]}(A_0, z_0)$. We rewrite (1.1) as

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.1}$$

Set $\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\dots,k-1\} = \rho_1 < \mu_{[p,q]}(A_0,z_0)$. Then for any given ε $(0<2\varepsilon<\mu_{[p,q]}(A_0,z_0)-\rho_1)$, there exists $r_3\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_3)$, we have

$$M_{z_0}(r,A_0)\geq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\} \tag{3.2}$$

and

$$M_{z_0}(r,A_j)\leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\},\ (j=1,2,\dots,k-1). \tag{3.3}$$

By Lemma 2.4, there exist a set $E_3\subset(0,r_0]$, $(r_0\in(0,1))$ that has a finite logarithmic measure and a constant $\lambda>0$ that depends on $\alpha>1$ and $j=1,2,\dots,k$ such that for all $r=|z-z_0|$ satisfying $r\in(0,r_0]\setminus E_3$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\lambda\left[\frac{1}{r^2}T_{z_0}(\frac{1}{\alpha}r,f)\log T_{z_0}(\frac{1}{\alpha}r,f)\right]^j,\ (j=1,2,\dots,k). \tag{3.4}$$

Substituting (3.2)-(3.4) into (3.1), for the above ε and $r\in(0,r_0]\cap(0,r_3)\setminus E_3$, we have

$$\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\leq\lambda k\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\}\left[\frac{1}{r^2}T_{z_0}(\frac{1}{\alpha}r,f)\log T_{z_0}(\frac{1}{\alpha}r,f)\right]^k. \tag{3.5}$$

By (3.5), we get

$$\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\leq\lambda k\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\}\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k}, \tag{3.6}$$

for all $|z-z_0|=r\in(0,r_0]\cap(0,r_3)\setminus E_3$ and $|A_0(z)|=M_{z_0}(r,A_0)$. By (3.6) and Lemma 2.6, we obtain $\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0)-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get

$$\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0). \tag{3.7}$$

By (1.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|. \tag{3.8}$$

By Lemma 2.5, there exists a set $E_4\subset(0,1)$ that has a finite logarithmic measure, such that for all $j=0,1,\dots,k$ and $r\notin E_4$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|=|1+o(1)|\left(\frac{V_{z_0}(r,f)}{r}\right)^j,\ r\longrightarrow0, \tag{3.9}$$

where z is a point in the circle $|z-z_0|=r$ that satisfies $|f(z)|=M_{z_0}(r,f)$. By Lemma 2.1, there exists a set $E\subset(0,1)$ having infinite logarithmic measure, such that for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E$, we have

$$|A_0(z)|\leq M_{z_0}(r,A_0)\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}. \tag{3.10}$$

By substituting (3.3), (3.9) and (3.10) into (3.8), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E\cap(0,r_3)\setminus E_4$ and $|f(z)|=M_{z_0}(r,f)$, we get

$$|1+o(1)|\left(V_{z_0}(r,f)\right)^k\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|\left(V_{z_0}(r,f)\right)^{k-1}, \tag{3.11}$$

then we obtain

$$V_{z_0}(r,f)\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|,\ r\in E\cap(0,r_3)\setminus E_4. \tag{3.12}$$

By Lemma 2.1, Lemma 2.6 and (3.12), we get $\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain

$$\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0), \tag{3.13}$$

from (3.7) and (3.13), we obtain $\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0)$. The proof is complete. □

Proof of Theorem 1.7

Proof . By Theorem 1.5, we have $\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$. We only need to prove that $\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0)$. We set $\rho_2=\max\{\rho_{[p,q]}(A_j,z_0),\rho_{[p,q]}(A_j,z_0)<\mu_{[p,q]}(A_0,z_0):j=1,\dots,k-1\}$. If $\rho_{[p,q]}(A_j,z_0)<\mu_{[p,q]}(A_0,z_0)$, then for any given ε $\left(0<2\varepsilon<\mu_{[p,q]}(A_0,z_0)-\rho_2\right)$, there exists $r_4\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_4)$, we have

$$M_{z_0}(r,A_j)\le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}\le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\},\ (j=1,2,\dots,k-1). \tag{3.14}$$

If $\rho_{[p,q]}(A_j,z_0)=\mu_{[p,q]}(A_0,z_0)$, $\tau_{[p,q],M}(A_j,z_0)\le\tau_1<\underline{\tau}=\underline{\tau}_{[p,q],M}(A_0,z_0)$, then for any given ε $(0<2\varepsilon<\underline{\tau}-\tau_1)$, there exists $r_5\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_5)$, we have

$$M_{z_0}(r,A_j)\le \exp_p\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\} \tag{3.15}$$

and

$$M_{z_0}(r,A_0)\ge \exp_p\left\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}. \tag{3.16}$$

By substituting (3.4) and (3.14)-(3.16) into (3.1), then for any given ε $(0<2\varepsilon<\min\{\mu_{[p,q]}(A_0,z_0)-\rho_2,\underline{\tau}-\tau_1\})$, we obtain

$$\exp_p\left\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\le \lambda k\exp_p\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k}, \tag{3.17}$$

for all $|z-z_0|=r\in(0,r_0]\cap(0,r_4)\cap(0,r_5)\setminus E_3$, $r\longrightarrow0$ and $|A_0(z)|=M_{z_0}(r,A_0)$, where $\lambda>0$ is a constant. By Lemma 2.6 and (3.17), we have

$$\mu_{[p+1,q]}(f,z_0)\ge\mu_{[p,q]}(A_0,z_0). \tag{3.18}$$

By Lemma 2.3, there exists a set $E_2\subset(0,1)$ having infinite logarithmic measure, such that for all $|z-z_0|=r\in E_2$, we have

$$|A_0(z)|\le M_{z_0}(r,A_0)\le \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}. \tag{3.19}$$

By combining (3.8), (3.9), (3.14), (3.15) and (3.19), for all $|z-z_0|=r\in E_2\cap(0,r_4)\cap(0,r_5)\setminus E_4$, $r\longrightarrow0$ and $|f(z)|=M_{z_0}(r,f)$, we have

$$|1+o(1)|\,(V_{z_0}(r,f))^k\le kr\exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}|1+o(1)|\,(V_{z_0}(r,f))^{k-1},$$

so

$$V_{z_0}(r,f)\le kr\exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}|1+o(1)|. \tag{3.20}$$

By Lemma 2.1, Lemma 2.6 and (3.20), we obtain

$$\mu_{[p+1,q]}(f,z_0)\le\mu_{[p,q]}(A_0,z_0). \tag{3.21}$$

Thus, from (3.18) and (3.21) we have

$$\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0),$$

which completes the proof. □

Proof of Theorem 1.8

Proof . By (1.1), we have

$$m_{z_0}(r,A_0)\leq \sum_{j=1}^{k-1} m_{z_0}(r,A_j)+\sum_{j=1}^{k} m_{z_0}\left(r,\frac{f^{(j)}(z)}{f(z)}\right)+O(1). \tag{3.22}$$

By Lemma 2.7, for a constant $r_2\in(0,1)$, there is a set $E_6\subset(0,r_2]$ with $m_l(E_6)<+\infty$ such that for all $|z-z_0|=r\in(0,r_2]\setminus E_6$, we have

$$\sum_{j=1}^{k} m_{z_0}\left(r,\frac{f^{(j)}(z)}{f(z)}\right)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{3.23}$$

Setting $\limsup_{r\longrightarrow 0}\frac{\sum_{j=1}^{k-1} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)}<\beta<1$. Then for $r\to 0$, we have

$$\sum_{j=1}^{k-1} m_{z_0}(r,A_j)<\beta m_{z_0}(r,A_0). \tag{3.24}$$

By substituting (3.23) and (3.24) into (3.22), we obtain for all $|z-z_0|=r\in(0,r_2]\setminus E_6$, $r\to 0$

$$(1-\beta)m_{z_0}(r,A_0)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{3.25}$$

By the definition of lower $[p,q]$ −order, for any given $\varepsilon>0$, there exists $r_6\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_6)$, we have

$$m_{z_0}(r,A_0)=T_{z_0}(r,A_0)\geq \exp_p\left\{\left(\mu_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\}. \tag{3.26}$$

By (3.25) and (3.26), for any given $\varepsilon>0$ and $|z-z_0|=r\in(0,r_2]\cap(0,r_6)\setminus E_6$, $r\to 0$, we obtain

$$(1-\beta)\exp_p\left\{\left(\mu_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{3.27}$$

By Definition 1.3, Lemma 2.6 and (3.27), we have $\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0)-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain

$$\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0). \tag{3.28}$$

Set $\max\{\rho_{[p,q]}(A_j,z_0):j=1,\ldots,k-1\}=\rho_3\leq\mu_{[p,q]}(A_0,z_0)\leq\rho_{[p,q]}(A_0,z_0)$. Then for any given $\varepsilon>0$, there exists $r_7\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_7)$, we have

$$M_{z_0}(r,A_j)\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_3+\varepsilon}\right\}\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\},\ (j=1,2,\ldots,k-1). \tag{3.29}$$

By substituting (3.9), (3.10) and (3.29) into (3.8), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E\cap(0,r_7)\setminus E_4$ and $|f(z)|=M_{z_0}(r,f)$, we get

$$|1+o(1)|\left(V_{z_0}(r,f)\right)^k\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|\left(V_{z_0}(r,f)\right)^{k-1}. \tag{3.30}$$

By (3.30), for above ε, we get

$$V_{z_0}(r,f)\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|, \tag{3.31}$$

where $|z-z_0|=r\in E\cap(0,r_7)\setminus E_4$ and $|f(z)|=M_{z_0}(r,f)$. By (3.31), Lemma 2.6 and Lemma 2.1, we obtain $\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain

$$\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0). \tag{3.32}$$

Thus, from (3.28) and (3.32) we have

$$\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0).$$

By using similar method, from (3.25) we have for $|z - z_0| = r \in (0, r_2] \setminus E_6$, $r \to 0$

$$(1-\beta)T_{z_0}(r, A_0) = (1-\beta)m_{z_0}(r, A_0) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.33}$$

By Lemma 2.2, there exists a set $E_1 \subset (0,1)$ having infinite logarithmic measure such that for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1$

$$T_{z_0}(r, A_0) \geq \exp_p\left\{\left(\rho_{[p,q]}(A_0, z_0) - \varepsilon\right)\log_q\frac{1}{r}\right\}. \tag{3.34}$$

By substituting (3.34) into (3.33), we obtain for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1 \cap (0, r_2] \setminus E_6$, $r \to 0$

$$(1-\beta)\exp_p\left\{\left(\rho_{[p,q]}(A_0, z_0) - \varepsilon\right)\log_q\frac{1}{r}\right\} \leq (1-\beta)T_{z_0}(r, A_0) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.35}$$

Making use of Lemma 2.6 and Lemma 2.1, from (3.35), we get

$$\rho_{[p+1,q]}(f, z_0) \geq \rho_{[p,q]}(A_0, z_0). \tag{3.36}$$

By the definition of the $[p,q]$-order of $\rho_{[p,q]}(A_0, z_0)$ for any given $\varepsilon > 0$, there exists $r_8 \in (0,1)$ such that for all $|z - z_0| = r \in (0, r_8)$, we have

$$M_{z_0}(r, A_0) \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_{[p,q]}(A_0,z_0)+\varepsilon}\right\}. \tag{3.37}$$

Also by substituting (3.9), (3.29), (3.37) into (3.8), for any given $\varepsilon > 0$ and for all $|z - z_0| = r \in (0, r_7) \cap (0, r_8) \setminus E_4$ and $|f(z)| = M_{z_0}(r, f)$, we can find that

$$V_{z_0}(r, f) \leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1 + o(1)|. \tag{3.38}$$

By using Definition 1.2, Lemma 2.6 and (3.38), we get

$$\rho_{[p+1,q]}(f, z_0) \leq \rho_{[p,q]}(A_0, z_0). \tag{3.39}$$

Thus, from (3.36) and (3.39), we conclude that

$$\rho_{[p+1,q]}(f, z_0) = \rho_{[p,q]}(A_0, z_0).$$

The proof is complete. □

4 Examples

Example 4.1. $f(z) = \exp_3\left\{\frac{1}{(z_0-z)^{2n+1}}\right\}$ solves the following equation

$$f'' + A_1(z)f' + A_0(z)f = 0, \tag{4.1}$$

where

$$A_0(z) = -\frac{(2n+1)^2}{(z_0-z)^{4n+4}}\exp\left\{2\exp\left(\frac{1}{(z_0-z)^{2n+1}} + \frac{2}{(z_0-z)^{2n+1}}\right)\right\}$$

and

$$A_1(z) = \frac{2n+1}{(z_0-z)^{2n+2}}\exp\left\{\frac{1}{(z-z_0)^{2n+1}}\right\} + \frac{2n+1}{(z_0-z)^{2n+2}} + \frac{2n+2}{z_0-z}.$$

We have

$$\rho_{[2,1]}(A_1, z_0) = 0 < \mu_{[2,1]}(A_0, z_0) = \rho_{[2,1]}(A_0, z_0) = 2n+1$$

Obviously, the conditions of Theorem 1.6 are satisfied and we see that

$$\mu_{[2,1]}(A_0, z_0) = \rho_{[2,1]}(A_0, z_0) = \mu_{[3,1]}(f, z_0) = \rho_{[3,1]}(f, z_0) = 2n+1.$$

Example 4.2. $f(z) = \frac{1}{(z_0-z)^n} \exp_2\left\{\frac{1}{(z_0-z)^{n+1}}\right\}$ solves the following equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0, \tag{4.2}$$

where

$$A_0(z) = \frac{n(n+1)^2}{(z_0-z)^{3n+6}} \exp\left\{\frac{3}{(z_0-z)^{n+1}}\right\}$$
$$+\left(\frac{(n+1)^2(3n+2)}{(z_0-z)^{3n+6}} + \frac{(n+1)\left(5n^2+7n+3\right)}{(z_0-z)^{2n+5}}\right)\exp\left\{\frac{2}{(z_0-z)^{n+1}}\right\}$$
$$+\left(\frac{(n+1)^3}{(z_0-z)^{3n+6}} + \frac{6(n+1)^3}{(z_0-z)^{2n+5}} + \frac{6(n+1)^3}{(z_0-z)^{n+4}}\right)\exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\} + \frac{n(n+1)(n+2)}{(z_0-z)^3},$$
$$A_1(z) = -\frac{(n+1)^2}{(z_0-z)^{n+3}} \exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\}$$

and

$$A_2(z) = \frac{1}{(z_0-z)^{n+2}} \exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\}.$$

We have

$$\max\left\{\rho_{[1,1]}(A_2,z_0), \rho_{[1,1]}(A_1,z_0)\right\} = \max\{n+1, n+1\} = n+1$$
$$= \mu_{[1,1]}(A_0,z_0) = \rho_{[1,1]}(A_0,z_0)$$

and

$$\max\left\{\tau_{[1,1],M}(A_2,z_0), \tau_{[1,1],M}(A_1,z_0)\right\} = 1 < \underline{\tau}_{[1,1],M}(A_0,z_0) = 3.$$

It is clear that the conditions of Theorem 1.7 are satisfied and we see that

$$\mu_{[1,1]}(A_0,z_0) = \rho_{[1,1]}(A_0,z_0) = \mu_{[2,1]}(f,z_0) = \rho_{[2,1]}(f,z_0) = n+1.$$

Example 4.3. $f(z) = \exp_2\left\{\frac{1}{2(z_0-z)}\right\}$ is a solution to equation (4.2) for the following coefficients

$$A_0(z) = \frac{1}{8(z_0-z)^6} \exp\left\{\frac{3}{2(z_0-z)}\right\},$$

$$A_1(z) = \left(-\frac{3}{(z_0-z)^3} - \frac{1}{2(z_0-z)^4}\right)\exp\left\{\frac{1}{2(z_0-z)}\right\} - \frac{2}{(z_0-z)^3} - \frac{6}{(z_0-z)^2}$$

and

$$A_2(z) = \frac{1}{2(z_0-z)^2}.$$

We have

$$\max\left\{\rho_{[1,1]}(A_2,z_0), \rho_{[1,1]}(A_1,z_0)\right\} = \max\{0,1\} = 1$$
$$= \mu_{[1,1]}(A_0,z_0) = \rho_{[1,1]}(A_0,z_0),$$
$$\limsup_{r\longrightarrow 0} \frac{m_{z_0}(r,A_2) + m_{z_0}(r,A_1)}{m_{z_0}(r,A_0)} = \frac{1}{3} < 1.$$

Obviously the conditions of Theorem 1.8 are verified and we see that

$$\mu_{[1,1]}(A_0,z_0) = \rho_{[1,1]}(A_0,z_0) = \mu_{[2,1]}(f,z_0) = \rho_{[2,1]}(f,z_0) = 1.$$

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