Int. J. Nonlinear Anal. Appl. 14 (2023) 2, 359–367
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.22570.2385

On the common zero of a finite family of monotone operators in Hadamard spaces and its applications

Sajad Ranjbar

Department of Mathematics, Higher Education Center of Eghlid, Eghlid, Iran

(Communicated by Ali Jabbari)

Abstract

In this paper, a common zero of a finite family of monotone operators on Hadamard spaces is approximated via Mann-type proximal point algorithm. Some applications in convex minimization and fixed point theory are also presented.

Keywords: Hadamard space, Monotone operator, Mann-type proximal point algorithm, Δ-convergence; Convex minimization, Fixed point theory
2020 MSC: Primary 47H05; Secondary 47J05, 47H09

1 Basic definitions and preliminaries

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \longrightarrow X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

Definition 1.1. A geodesic space (X, d) is a CAT(0) space if satisfies the following inequality:
$CN - inequality$: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2)$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

A complete CAT(0) space is called a $Hadamard$ space.

It is known that a CAT(0) space is an uniquely geodesic space. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [5, 8, 11, 16, 20]. Some examples of CAT(0) spaces are pre-Hilbert spaces (see [8]), $\mathbb{R}$-trees (see [25]), Euclidean buildings (see [9]), the complex Hilbert ball with a hyperbolic metric (see [15]), Hadamard manifolds and many others.

For all x and y belongs to a CAT(0) space X, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z, x) = td(x, y) and d(z, y) = (1 - t)d(x, y)$. Set $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$, a subset C of X is called convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Email address: sranjbar@eghlid.ac.ir; sranjbar74@yahoo.com. (Sajad Ranjbar)

Received: February 2021 *Accepted:* July 2022

Lemma 1.2. [8] A geodesic space (X,d) is a CAT(0) space if and only if, for all $x,y,z,w \in X$ and all $t \in [0,1]$,

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y). \tag{1.1}$$

In this case:

(1) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$; and

(2) $d(tx \oplus (1-t)y, tz \oplus (1-t)w) \leq td(x,z) + (1-t)d(y,w)$.

A Hadamard space X is called a flat Hadamard space if and only if the inequality in (1.1) is an equality. Every closed convex subset of a Hilbert space is a flat Hadamard space.

A notation of convergence in complete CAT(0) spaces was introduced by Lim [27] that is called Δ-convergence as follows:

Let (x_n) be a bounded sequence in complete CAT(0) space (X,d) and $x \in X$. Set $r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of (x_n) is given by $r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}$ and the asymptotic center of (x_n) is the set $A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}$. It is known that in the complete CAT(0) spaces, $A((x_n))$ consists exactly one point (see [26]). A sequence (x_n) in the complete CAT(0) space (X,d) is said Δ-convergent to $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence (x_{n_k}) of (x_n). The concept of Δ-convergence has been studied by many authors (e.g. [12, 14]).

Berg and Nikolaev [6] have introduced the concept of *quasilinearization* for CAT(0) space X. They denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

It can be easily verified that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ are satisfied for all $a,b,c,d,e \in X$. Also, we can formally add compatible vectors, more precisely $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a,b,c \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \qquad (a,b,c,d \in X).$$

It is known [6, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality. Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space of a complete CAT(0) space X, based on a work of Berg and Nikolaev [6], as follows.
Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \qquad (t \in \mathbb{R},\ a,b,x \in X),$$

where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$ $(t \in \mathbb{R}, \quad a,b \in X)$, where $L(\varphi) = sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)), \qquad (t,s \in \mathbb{R},\ a,b,c,d \in X).$$

For a Hadamard space (X,d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X,\mathbb{R}), L)$. It is obtained [2, Lemma 2.1], that $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x,y \in X$. Thus, D can impose an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]) := D((t,a,b),(s,c,d))$, which is called the dual space of (X,d). It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a,b \in X$. Fix $o \in X$, we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. In [2], it is shown that the dual of a closed and convex subset of Hilbert space H with nonempty interior is H and $t(b-a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R},\ a,b \in H$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy}\rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle, \quad (x^* = [t\overrightarrow{ab}] \in X, x, y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy}\rangle := \alpha\langle x^*, \overrightarrow{xy}\rangle + \beta\langle y^*, \overrightarrow{xy}\rangle, \quad (\alpha, \beta \in \mathbb{R},\ x, y \in X,\ x^*, y^* \in X^*).$$

Ahmadi Kakavandi [1] proved that (x_n) Δ-converges to $x \in X$ if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0, \ \ \forall y \in X$.

Definition 1.3. [19] Let X be a Hadamard space with dual space X^*. The multi-valued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{x \in X : A(x) \neq \emptyset\}$, is monotone iff for all $x, y \in \mathbb{D}(A), x \neq y,\ x^* \in Ax, y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0.$$

In the following, we present some properties of resolvent operator of a monotone operator in CAT(0) spaces which verified in [24].

Definition 1.4. [24] Let $\lambda > 0$ and $A : X \to 2^{X^*}$ be a set-valued operator. The resolvent of A of order λ is the set-valued mapping $J_\lambda : X \to 2^X$ defined by $J_\lambda(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$.

Definition 1.5. [24] Let $T : C \subset X \to X$ be a mapping. We say that T is firmly nonexpansive if $d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle$, for any $x, y \in C$.

By the definition and Cauchy-Schwarz inequality, it is clear that any firmly nonexpansive mapping T is nonexpansive.

Theorem 1.6. [24] Let X be a CAT(0) space and J_λ is resolvent of the operator A of order λ. We have;
(i) For any $\lambda > 0$, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$, $F(J_\lambda) = A^{-1}(\mathbf{0})$.
(ii) If A is monotone then J_λ is a single-valued and firmly nonexpansive mapping.
(iii) If A is monotone and $\lambda \leq \mu$ then $d(x, J_\lambda x) \leq 2d(x, J_\mu x)$.

Remark 1.7. [24, Remark 3.10] It is well-known that if T is a nonexpansive mapping on subset C of CAT(0) space X then $F(T)$ is closed and convex. Thus, if A is a monotone operator on CAT(0) space X then, by parts (i) and (ii) of Theorem 1.6, $A^{-1}(\mathbf{0})$ is closed and convex.

The following lemma is generalization of Opial lemma in CAT(0) space.

Lemma 1.8. [31, Lemma 2.1] Let (X, d) be a CAT(0) space and (x_n) a sequence in X. If there exists a nonempty subset F of X verifying:

(i) For every $z \in F$, $\lim_n d(x_n, z)$ exists.

(ii) If a subsequence (x_{n_j}) of (x_n) is Δ-convergent to $x \in X$, then $x \in F$.

Then, there exists $p \in F$ such that (x_n) Δ-converges to p in X.

2 The direction of the research

Monotone operator theory is an area of research in nonlinear and convex analysis that has continued to attract the interest of many researchers due to the role it plays in mathematical problems such as optimization, equilibrium problems, variational inequality, evolution equations and semigroup theory. Approximation of a common zero of a finite family of monotone operators is one of the most important parts in monotone operator theory.

Martinet [28] and, systematically, Rockafellar [33] introduced the proximal point algorithm for approximation of a zero of a monotone operator which is one of the most popular methods in this field. This algorithm is defined by:

$$x_{n-1} - x_n \in \lambda_n A(x_n), \qquad x_0 \in H, \tag{2.1}$$

where (λ_n) is a sequence of positive real numbers and A is a monotone operator. In fact, Rockafellar [33] showed the weak convergence of the sequence generated (2.1) to a zero of the maximal monotone operator in Hilbert spaces provided $\lambda_n \geq \lambda > 0, \quad \forall n \geq 1$. Another algorithm for approximation a zero of the monotone operator A, is proposed by Kamimura and Takahashi [21] that is Mann-type proximal point algorithm

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n) J^A_{\lambda_n} x_n, \\ x_0 \in H, \end{cases} \tag{2.2}$$

where $(\alpha_n) \subset [0,1]$. They proved weak convergence of (2.2) in Hilbert space with the conditions $\alpha_n \leq \alpha < 1$ and $\lim \lambda_n = \infty$ on the control sequences (α_n) and (λ_n). For some generalization of (2.1) in Hilbert spaces the reader can consult [10, 13, 17, 22, 29]. In the last decade, the development and generalization of the results from the Hilbert spaces to the Hadamard spaces has attracted the attention of many researchers.

Bačák [4] proved the Δ-convergence of proximal point algorithm in CAT(0) spaces when the operator A is the subdifferential of a convex, proper, and lower semicontinuous function. Khatbzaedeh and author [24] considered some properties of a monotone operator and its resolvent operator in CAT(0) spaces and extended the proximal point algorithm in general case to Hadamrd spaces, as:

$$\begin{cases} [\frac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}] \in Ax_n, \\ x_0 \in X. \end{cases} \tag{2.3}$$

They proved Δ-convergence of (2.3) to a zero of monotone operator A in Hadamard spaces. Also, w-convergence (in the sense of Ahmadi Kakavandi and Amini [2]) of (2.3) to a zero of monotone operator A is considered in [30]. Recently, the author and Khatibzadeh [32] improved and generalized the result of Kamimura and Takahashi [21] to Hadamard spaces. In fact, they proved Δ-convergence of

$$\begin{cases} x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \\ x_0 \in X, \end{cases}$$

to a zero of the monotone operator A on Hadamard space X. Very recently, Heydari, Khadem and the author [18] considered the modified proximal point algorithm

$$\begin{cases} x_{n+1} = \prod_{i=1}^k J^{A_i}_{\lambda_{(n,i)}} x_n := J^{A_1}_{\lambda_{(n,1)}} o J^{A_2}_{\lambda_{(n,2)}} o ... o J^{A_k}_{\lambda_{(n,k)}} x_n, \\ x_0 \in X, \end{cases} \tag{2.4}$$

where $\{A_i\}_{i=1}^k$ is a finite family of monotone operators on the Hadamard space X with dual X^* and $(\lambda_{(n,i)})$ for $i = 1, 2, ..., k$ are some sequences of nonnegative real numbers. They established Δ-convergence of (2.4) to a common zero of finite family $\{A_i\}_{i=1}^k$ of monotone operators in Hadamard spaces. Motivated and inspired by the research going on in this direction, our purpose in this paper is to consider the modified Mann-type proximal point algorithm

$$\begin{cases} x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) \prod_{i=1}^k J^{A_i}_{\lambda_{(n,i)}} x_n, \\ x_0 \in X, \end{cases} \tag{2.5}$$

where $\{A_i\}_{i=1}^k$ is a finite family of monotone operators on the Hadamard space X with dual X^*, $(\alpha_n) \subseteq [0,1]$ and $(\lambda_{(n,i)})$ for $i = 1, 2, ..., k$ are some sequences of nonnegative real numbers. We prove Δ-convergence of the sequence generated by (2.5) to a common zero of finite family $\{A_i\}_{i=1}^k$ of monotone operators in Hadamard spaces under conditions $\limsup_n \alpha_n < 1$ and $\lambda_{(n,i)} \geq \lambda > 0$. Some applications in convex minimization and fixed point theory are also peresented to support our results. Our results improved and extended some results in [7, 12, 18, 21, 24, 32] and many others in the literatures.

3 Δ-convergence to a common zero of monotone operators

Let X be a Hadamard space with dual X^*. We say that the operator $A : X \to 2^{X^*}$ satisfies the *range condition* if for every $\lambda > 0$, $D(J^A_\lambda) = X$. It is known that if A is a maximal monotone operator on the Hilbert space H then $R(I + \lambda A) = H, \quad \forall \lambda > 0$, where I is the identity operator. Thus, every maximal monotone operator A on a Hilbert space satisfies the *range condition*. Also as it has shown in [3], if A is a maximal monotone operator on a Hadamard manifold, then A satisfies the range condition. For presenting some examples of monotone operators that satisfy the range condition in CAT(0) spaces, refer to [24, Sections 5 and 6].

Let $A_1, A_2, ..., A_k : X \to 2^{X^*}$ be a multi-valued monotone operators on the Hadamard space X with dual X^* that satisfy the *range condition* and $(\lambda_{(n,i)})$ for $i = 1, 2, ..., k$ be some sequences of nonnegative real numbers. In this section, for approximate a common zero of a finite family of monotone operators in Hadamard spaces, we propose Mann-type proximal point algorithm

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) z_n^1, \tag{3.1}$$

where $x_0 \in X$,

$$z_n^i = J_{\lambda_{(n,i)}}^{A_i} z_n^{i+1}, \quad \text{for } i \in \{1, 2, ..., k\},$$

and $z_n^{k+1} = x_n$, for all $n \in \mathbb{N}$. In following theorem, Δ-convergence of the sequence generated by (3.1) to a common zero of a finite family of monotone operators in Hadamard spaces is established.

Theorem 3.1. Let X be a Hadamard space with dual X^* and $A_1, A_2, ..., A_k : X \to 2^{X^*}$ be multi-valued monotone operators that satisfy the range condition and $\bigcap_{i=1}^k A_i^{-1}(0) \neq \emptyset$. If $(\lambda_{(n,i)})$ for $i = 1, 2, ..., k$ are some sequences of nonnegative real numbers such that for each (n, i), $\lambda_{(n,i)} \geq \lambda$ for some $\lambda > 0$, and (α_n) is a sequence in $[0, 1]$ such that $\limsup_n \alpha_n < 1$ then the sequence generated by the Mann-type proximal point algorithm (3.1) is Δ-converges to a point $p \in A^{-1}(\mathbf{0})$.

Proof . Let $p \in \bigcap_{i=1}^k A_i^{-1}(0)$. Then $0 \in A_i p$ for $i \in \{1, 2, ..., k\}$. By the definition of the resolvent operator, we get

$$[\frac{1}{\lambda_{(n,i)}} \overrightarrow{z_n^i z_n^{i+1}}] \in A_i(z_n^i), \text{ for } \ i \in \{1, 2, ..., k\},$$

hence by the monotonicity of A_i, for $i \in \{1, 2, ..., k\}$ one has

$$\langle [\frac{1}{\lambda_{(n,i)}} \overrightarrow{z_n^i z_n^{i+1}}] - 0, \overrightarrow{p z_n^i} \rangle \geq 0,$$

or equivalently,

$$d^2(z_n^{i+1}, p) - d^2(z_n^i, p) \geq d^2(z_n^{i+1}, z_n^i), \quad \text{for } i \in \{1, 2, ..., k\}. \tag{3.2}$$

By summing the inequality (3.2) from $i = 1$ to $i = k$, we get

$$d^2(p, x_n) - d^2(p, z_n^1) \geq \sum_{i=1}^k d^2(z_n^{i+1}, z_n^i) \geq 0. \tag{3.3}$$

This follows that

$$d(p, z_n^1) \leq d(p, x_n). \tag{3.4}$$

Thus,

$$d(x_{n+1}, p) \leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(z_n^1, p) \leq d(x_n, p).$$

Therefore, there exists $\lim_n d(x_n, p)$ and the sequence (x_n) is bounded. Hence,

$$\begin{aligned}
0 &= \lim_n (d(x_{n+1}, p) - d(x_n, p)) \\
&\leq \liminf_n (\alpha_n d(x_n, p) + (1 - \alpha_n) d(z_n^1, p) - d(x_n, p)) \\
&= \liminf_n (1 - \alpha_n)(d(z_n^1, p) - d(x_n, p)) \\
&\leq \limsup_n (1 - \alpha_n)(d(z_n^1, p) - d(x_n, p)) \\
&\leq \limsup_n (1 - \alpha_n)(d(x_n, p) - d(x_n, p)) = 0,
\end{aligned}$$

which implies $\lim_n (1 - \alpha_n)(d(z_n^1, p) - d(x_n, p)) = 0$. By the assumptions, we get

$$\lim_n (d(z_n^1, p) - d(x_n, p)) = 0. \tag{3.5}$$

which by (3.3) implies

$$\lim_{n \to \infty} d(z_n^i, z_n^{i+1}) = 0, \quad \text{for } i \in \{1, 2, ..., k\}. \tag{3.6}$$

This, together with the convexity of the metric d, follows

$$\lim_{n\to\infty} d(x_n, z_n^i) = 0, \quad \text{for } i \in \{1, 2, ..., k\}. \tag{3.7}$$

Moreover, by the assumptions and part (iii) of the Theorem 1.6, we have

$$d(J_\lambda^{A_i} z_n^{i+1}, z_n^{i+1}) \leq 2d(z_n^i, z_n^{i+1}), \qquad \text{for } i \in \{1, 2, ..., k\}, \tag{3.8}$$

which, by (3.6), implies

$$\lim_{n\to\infty} d(J_\lambda^{A_i} z_n^{i+1}, z_n^{i+1}) = 0, \quad \text{for } i \in \{1, 2, ..., k\}. \tag{3.9}$$

Now if (x_{n_j}) is a subsequence of (x_n) which is Δ-convergent to the point q then for $i \in \{1, 2, ..., k\}$, by (3.7), we get

$$\begin{aligned}
\limsup_j \langle \overrightarrow{z_{n_j}^i q}, \overrightarrow{xq} \rangle &= \limsup_j (\langle \overrightarrow{z_{n_j}^i x_{n_j}}, \overrightarrow{xq} \rangle + \langle \overrightarrow{x_{n_j} q}, \overrightarrow{xq} \rangle) \\
&\leq \limsup_j \langle \overrightarrow{z_{n_j}^i x_{n_j}}, \overrightarrow{xq} \rangle + \limsup_j \langle \overrightarrow{x_{n_j} q}, \overrightarrow{xq} \rangle \\
&\leq \limsup_j d(x_{n_j}, z_{n_j}^i) d(x, q) + \limsup_j \langle x_{n_j} q, xq \rangle \\
&\leq 0,
\end{aligned}$$

which implies the subsequence $(z_{n_j}^i)$ is Δ-convergent to q for $i \in \{1, 2, ..., k\}$. Hence by (3.9) and Δ-demicloseness of nonexpansive mapping $J_\lambda^{A_i}$, we obtain $q \in A_i^{-1}(0)$, for $i \in \{1, 2, ..., k\}$ which follows $q \in \bigcap_{i=1}^k A_i^{-1}(0)$. Therefore, we proved that

(1) for every $p \in \bigcap_{i=1}^k A_i^{-1}(0)$, $\lim_n d(x_n, p)$ exists,

(2) if subsequence (x_{n_j}) of (x_n) is Δ-convergent to $q \in X$, then $q \in \bigcap_{i=1}^k A_i^{-1}(0)$.

Hence, Lemma 1.8 completes the proof. □

Remark 3.2. If for every $n \in \mathbb{N}$, $\alpha_n = 0$ in theorem 3.1, then the result [18, Theorem 3.1] is obtaind. Therefore Theorem 3.1 improved and extended several results in [18, 21, 24, 32] and many others in the literatures.

4 Application in convex minimization

Approximation of a common minimizer of a finite family of the proper, convex and lower semicontinuous functions is one of the most important applications of Theorem 3.1. Let (X, d) be a Hadamard space. In [2], the subdifferential of a proper function on a Hadamard space X was defined, as follows.

Definition 4.1. [2] Let X be a Hadamard space with dual X^* and $f : X \to]-\infty, +\infty]$ be a proper function with efficient domain $D(f) := \{x : f(x) < +\infty\}$, then the subdifferential of f is the multi-valued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle \quad (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

Theorem 4.2. [2, Theorem 4.2] and [24, Proposition 5.2] Let $f : X \to]-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*, then

(i) f attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$.

(ii) $\partial f : X \to 2^{X^*}$ is a monotone operator.

(iii) for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha \overrightarrow{xy}] \in \partial f(x)$. (i.e. $D(J_\lambda^{\partial f}) = X$, for all $\lambda > 0$).

Note that part (iii) of Theorem 4.2 shows that subdifferential of a convex, proper and lower semicontinuous function satisfies the range condition. Let $f : X \to] - \infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*. By [24, Proposition 5.3], for all $\lambda \geq 0$ and $x \in X$, we have

$$J_\lambda^{\partial f} x = \arg\min_{z \in X} \{f(z) + \frac{1}{2\lambda} d^2(z, x)\},$$

In the following theorem, Δ-convergence of Mann type proximal point algorithm to a common minimizer of a finite family of the proper, convex and lower semicontinuous functions is established.

Theorem 4.3. Let (X, d) be a Hadamard space with dual X^* and $f_1, f_2, ..., f_k$ be proper, convex and lower semicontinuous functions from Hadamard space X to $] - \infty, +\infty]$ such that $\bigcap_{i=1}^k Argminf_i \neq \emptyset$. If $(\lambda_{(n,i)})$ for $i = 1, 2, ..., k$ are some sequences of nonnegative real numbers such that for each (n, i), $\lambda_{(n,i)} \geq \lambda$ for some $\lambda > 0$ and (α_n) is a sequence in $[0, 1]$ such that $\limsup_n \alpha_n < 1$ then the sequence generated by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) z_n^1, \tag{4.1}$$

where $x_0 \in X$,

$$z_n^i = J_{\lambda_{(n,i)}}^{\partial f_i} z_n^{i+1}, \quad \text{for } i \in \{1, 2, ..., k\},$$

and $z_n^{k+1} = x_n$, for all $n \in \mathbb{N}$ is Δ-convergent to the point $p \in \bigcap_{i=1}^k \arg\min f_i$ that is a common minimum of f_i.

Proof . Define $A_i := \partial f_i$, for $i = 1, 2, ..., k$, then each operator $A_i = \partial f_i$ is a monotone operator that satisfies the range condition. Therefore we can use Theorem 3.1 to get the desired result. □

Example 4.4. Let (X, d) be a Hadamard space. Given a finite number of points $a_1, a_2, ..., a_n$ and $(w_1, w_2, ...w_n) \in S$ where S is the convex hull of the canonical basis $e_1, e_2, ..., e_n \in \mathbb{R}^n$. Define $f : X \to \mathbb{R}$ with $f(x) = \sum_{k=1}^n w_k d^2(x, a_k)$. The function f is convex and continuous which has a unique minimizer (see [5, Proposition 2.2.17]) that is called the mean of the points $a_1, a_2, ..., a_n$. Employing the the Theorems 4.3 for this function, then the sequences generated by (4.1) is Δ-convergent to the mean of the points $a_1, a_2, ..., a_n$. For some applications of this function, we refer the reader to the book by Bač'ak [5].

5 Application in fixed point theory

Let I be the identity mapping and T be a nonexpansive selfmapping on Hilbert space H, then $I - T$ is maximal monotone and hence it satisfies the range condition. Suppose (X, d) is a Hadamard space with dual X^* and $T : X \to X$ is a nonexpansive mapping. Define $A : X \longrightarrow 2^{X^*}$ with $Az = [\overrightarrow{Tzz}]$, then $F(T) = A^{-1}(0)$ and Proposition 4.2 of [23] shows the operator $Az = [\overrightarrow{Tzz}]$ is a monotone operator. We refer the reader to [24, Section 6], to consider the *range condition* for the operator $Az = [\overrightarrow{Tzz}]$.

In the following theorem, we approximate a common fixed point of a finite family of nonexpansive mappings by a modified version of Mann iteration with condition $\limsup_n \alpha_n < 1$ on the control sequence (α_n). This condition is better than the condition $\sum \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$ and is different from the condition $\sum \alpha_n(1 - \alpha_n) = \infty$ which are used in literatures.

Theorem 5.1. Let (X, d) be a Hadamard space and $T_1, T_2, ..., T_k$ be nonexpansive selfmappings on X such that $\bigcap_{i=1}^k F(T_i) \neq \emptyset$ and the operators $A_i z = [\overrightarrow{T_i zz}]$, for $i = 1, 2, ..., k$, satisfy the range condition. If $(\lambda_{(n,i)})$ is a sequence of nonnegative real numbers such that for each (n, i), $\lambda_{(n,i)} \geq \lambda$ for some $\lambda > 0$ and (α_n) is a sequence in $[0, 1]$ such that $\limsup_n \alpha_n < 1$ then the sequence generated by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) z_n^1, \tag{5.1}$$

where $x_0 \in X$,

$$z_n^i = J_{\lambda_{(n,i)}}^{A_i} z_n^{i+1}, \quad \text{for } i \in \{1, 2, ..., k\},$$

and $z_n^{k+1} = x_n$, for all $n \in \mathbb{N}$ is Δ-convergent to the point $p \in \bigcap_{i=1}^k F(T_i)$.

Proof . The proof is deducted from Theorem 3.1, taking into account that the fixed points of the operators T_i are the zeroes of the operators A_i for all $i \in \{1, 2, ..., n\}$. □

Example 5.2. Let (X, d) be a flat Hadamard space and $y \in X$. For $i = 1, 2, 3, ..., k$, set $T_i x = \frac{1}{i+1} x \oplus \frac{i}{i+1} y$. Then T_i is a nonexpansive mapping for every $i \in \{1, 2, ..., k\}$. Therefore, by [24, Proposition 6.4.], the monotone operators $A_i x = [\overrightarrow{T_i x x}]$, for $i = 1, 2, ..., k$, satisfy the range condition. Set $\lambda_{n,i} = n + i$, for all $n, i \in \mathbb{N}$ and $\alpha_n = \frac{1}{n^2+2}$ for all $n \in \mathbb{N}$. Then the conditions of Theorem 5.1 are satisfied. Hence the sequence generated by (5.1) Δ-converges to the point $y \in \bigcap_{i=1}^{k} F(T_i)$.

Acknowledgements

The author is grateful to the referee for his/her valuable comments and suggestions.

References

[1] B. Ahmadi Kakavandi, *Weak topologies in complete CAT(0) metric spaces*, Proc. Amer. Math. Soc. **141** (2013), 1029–1039.

[2] B. Ahmadi Kakavandi and M. Amini, *Duality and subdifferential for convex functions on complete CAT(0) metric spaces*, Nonlinear Anal. **73** (2010), 3450–3455.

[3] QH. Ansari and F. Babu, *Existence and boundedness of solutions to inclusion problems for maximal monotone vector fields in Hadamard manifolds*, Optim. Lett. **14** (2020), 711–727.

[4] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701.

[5] M. Bačák, *Convex Analysis and Optimization in Hadamard Spaces*, Walter de Gruyter GmbH, Berlin, 2014.

[6] I.D. Berg and IG. Nikolaev, *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedicata **133** (2008), 195–218.

[7] B.A. Bin Dehaish and M.A. Khamsi, *Mann iteration process for monotone nonexpansive mappings*, Fixed Point Theory Appl. **177** (2015).

[8] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Fundamental Principles of Mathematical Sciences. Springer, Berlin, 1999.

[9] K.S. Brown, *Buildings*, Springer, New York, 1989.

[10] H. Brézis and P.L. Lions, *Produits infinis derésolvantes*, Israel J. Math. **29** (1978), 329–345.

[11] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Math., 33, Amer. Math. Soc., Providence, RI, 2001.

[12] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (2008), 2572–2579.

[13] B. Djafari Rouhani and H. Khatibzadeh, *On the proximal point algorithm*, J. Optim. Theory Appl. **137** (2008), 411–417.

[14] R. Espínola, *A. Fernández-León, CAT(κ)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410–427.

[15] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc, New York, 1984.

[16] M. Gromov and SM. Bates, *Metric Structures for Riemannian and Non-Riemannian Spaces*, with appendices by M. Katz, P. Pansu and S. Semmes, ed. by J. Lafontaine and P. Pansu, Progr. Math. 152, BirkhNauser, Boston, 1999.

[17] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[18] M.T. Heydari, A. Khadem and S. Ranjbar, *Approximating a common zero of finite family of monotone operators in Hadamard spaces*, Optim. **66** (2017), 2233–2244.

[19] M.T. Heydari and S. Ranjbar, *Halpern-type proximal point algorithm in complete CAT(0) metric spaces*, An. tiint. Univ. Ovidius Constanta Ser. Mat. **24** (2016), 141–159.

[20] J. Jöst, *Nonpositive Curvature: Geometric and Analytic Aspects*, Lectures Math. ETH ZNurich, BirkhNauser, Basel, 1997.

[21] S. Kamimura and W. Takahashi, *Approximating Solutions of Maximal Monotone Operators in Hilbert Spaces*, J. Approx. Theory **106** (2000), 226–240.

[22] H. Khatibzadeh, *Some remarks on the proximal point algorithm*, J. Optim. Theory Appl. **153** (2012), 769–778.

[23] H. Khatibzadeh and S. Ranjbar, *A variational inequality in complete CAT(0) spaces*, J. Fixed Point Theory Appl. **17** (2015), 557–574.

[24] H. Khatibzadeh and S. Ranjbar, *Monotone operators and the proximal point algorithm in complete CAT(0) metric spaces*, J. Aus. Math. Soc. **103** (2017), 70–90.

[25] W.A. Kirk, *Fixed point theorems in CAT(0) spaces and* $\mathbb{R}$*-trees*, Fixed Point Theory Appl. **4** (2004), 309–316.

[26] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.

[27] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. **60** (1976), 179–182.

[28] B. Martinet, *Régularisation d'Inéquations Variationnelles par Approximations Successives*, Revue Fran. Inf. Rech. Opér. **3** (1970), 154–158.

[29] G. Morosanu, *Nonlinear Evolution Equations and Applications*, Editura Academiei Romane (and D. Reidel publishing Company), Bucharest, 1988.

[30] S. Ranjbar, *W-convergence of the proximal point algorithm in complete CAT(0) metric spaces*, Bull. Iranian Math. Soc. **43** (2017), 817–834.

[31] S. Ranjbar and H. Khatibzadeh, Δ*-convergence and w-convergence of the modified Mann iteration for a family of asymptotically nonexpansive type mappings in complete CAT(0) spaces*, Fixed Point Theory **17** (2016), 151–158.

[32] S. Ranjbar and H. Khatibzadeh, *Strong and* Δ*-convergence to a zero of a monotone operator in CAT(0) spaces*, Mediterr. J. Math. **14** (2017), 56.

[33] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.