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COMMENTS ON RELAXED (γ, r)-COCOERCIVE MAPPINGS

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Abstract. We show that the variational inequality $VI(C, A)$ has a unique solution for a relaxed (γ, r)-cocoercive, μ-Lipschitzian mapping $A : C \to H$ with $r > \gamma\mu^2$, where C is a nonempty closed convex subset of a Hilbert space H. From this result, it can be derived that, for example, the recent algorithms given in the references of this paper, despite their becoming more complicated, are not general as they should be.

Let H be a Hilbert space, whose inner product is denoted by $\langle .\ ,.\rangle$. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which is denoted by $VI(C, A)$ is used to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0$$

for all $v \in C$. For a given $x \in H$, $u \in C$ satisfies the inequality

$$\langle x - u, u - y\rangle \geq 0, \ \ \forall\, y \in C,$$

if and only if $u = P_C x$. It is known that projection operator P_C is nonexpansive.

It is easy to see that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad (1)$$

where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. Recall that:

(i) A is called r-strongly monotone, if for each $x, y \in C$ we have

$$\langle Ax - Ay, x - y\rangle \geq r\|x - y\|^2$$

for a constant $r > 0$.

(ii) A is called relaxed (γ, r)-cocoercive if there exists two constants $\gamma > 0$ and $r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq -\gamma\|Ax - Ay\|^2 + r\|x - y\|^2.$$

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This class of maps is more general than the class of strongly monotone maps.

In [10], Verma considered a relaxed (γ, r)-cocoercive μ-Lipschitz mapping to present approximate solvability of a system of variational inequality problems. In [11], the same author proved the following theorem:

Theorem 1 (Verma [11]). *Let C be a nonempty closed convex subset of H and let $A : C \to H$ be r-strongly monotone and μ-Lipschitz. Suppose that $x^*, y^* \in C$ be chosen such that*

$$\begin{cases} x^* = P_C(y^* - \rho A y^*) & \text{for } \rho > 0, \\ y^* = P_C(x^* - \eta A x^*) & \text{for } \eta > 0. \end{cases} \tag{2}$$

For arbitrary chosen initial points $x_0, y_0 \in C$, define $\{x_n\}$ and $\{y_n\}$ as

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n P_C(y_n - \rho A y_n) \\ y_n = (1 - b_n)x_n + b_n P_C(x_n - \eta A x_n) \end{cases} \tag{3}$$

for all $n \geq 0$, where, $0 \leq a_n,\, b_n \leq 1$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$. Then sequences $\{x_n\}$ and $\{y_n\}$, respectively, converge to x^* and y^* for

$$0 < \rho < \frac{2r}{\mu^2} \text{ and } 0 < \eta < \frac{2r}{\mu^2}.$$

Very recently, Noor [1], Noor and Huang [2, 3, 4, 6] and Qin et al. [9] have considered some iterative methods for finding a common element of the set of the fixed points of nonexpansive mapping and the set of the solution of a variational inequality $VI(C, A)$, where A is a relaxed (γ, r)-cocoercive μ-Lipschitzian mapping of C into H such that $r > \gamma\mu^2$.

Moreover, Gao and Guo [5] and Qin et al. [7, 8] proposed iterative algorithms to find a common element of the set of solutions of variational inequalities for a relaxed (γ, r)-cocoercive μ-Lipschitzian mapping of C into H such that $r > \gamma\mu^2$, the set of solutions of an equilibrium problem and also the set of the fixed points of nonexpansive mapping.

By proving the following proposition we give some important comments on the above mentioned works.

Proposition 2. *Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a relaxed (γ, r)-cocoercive, $0 < \mu$-Lipschitzian mapping such that $r > \gamma\mu^2$. Then $VI(C, A)$ is singleton.*

Proof. Let s be a real number such that

$$0 < s < \frac{2(r - \gamma\mu^2)}{\mu^2}.$$

Then, for every $x, y \in C$ we have

$$\begin{aligned} &\|P_C(I - sA)x - P_C(I - sA)y\|^2 \\ &\leq \|(I - sA)x - (I - sA)y\|^2 = \|(x - y) - s(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2s\langle x - y, Ax - Ay\rangle + s^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2s(-\gamma\|Ax - Ay\|^2 + r\|x - y\|^2) + s^2\|Ax - Ay\|^2 \end{aligned}$$

$$\leq \|x-y\|^2 + 2s\mu^2\gamma\|x-y\|^2 - 2sr\|x-y\|^2 + \mu^2 s^2\|x-y\|^2$$

$$= (1 + 2s\mu^2\gamma - 2sr + \mu^2 s^2)\|x-y\|^2$$

$$= (1 - s\mu^2[\frac{2(r-\gamma\mu^2)}{\mu^2} - s])\|x-y\|^2.$$

Now, since $1-s\mu^2[\frac{2(r-\gamma\mu^2)}{\mu^2}-s] < 1$, the mapping $P_C(I-sA) : C \to C$ is a contraction and Banach' s Contraction Mapping Principle guarantees that it has a unique fixed point u; i.e., $P_C(I-sA)u = u$, which is the unique solution of $VI(C, A)$ by (1). □

Since r-strongly monotone mappings are relaxed (γ, r)-cocoercive, we get the following.

Proposition 3. *Let C be a nonempty closed convex subset of H and let $A : C \to H$ be an $0 < r$-strongly monotone and $0 < \mu$-Lipschitzian mapping. Then $VI(C, A)$ is singleton.*

The followings are our comments:

Comment 1. In Verma' s theorem, mentioned above, $A : C \to H$ is an $0 < r$-strongly monotone, $0 < \mu$-Lipschitzian mapping. Therefore, according to Proposition 3, $VI(C, A)$ is singleton, e.g., $VI(C, A) = \{u\}$. Takeing $x^* = y^* = u$, and considering (1), the equations in (2) hold and indeed the sequences $\{x_n\}$ and $\{y_n\}$ in (3) converge to the unique solution u of $VI(C, A)$. Therefore, x^*, y^* in Verma' s theorem are indeed equal the unique solution of $VI(C, A)$; thus the system of variational inequality problems that is considered by Verma in [11] comprises only one problem.

Comment 2. The iterative methods considered by Noor [1], Noor and Huang [2, 3, 4, 6] and Qin et al. [9] for finding the common element of the set of the solution of the variational inequality $VI(C, A)$, where A is a relaxed (γ, r)-cocoercive μ-Lipschitzian mapping of C into H such that $r > \gamma\mu^2$, and the set of the fixed points of nonexpansive mapping, can not be really considered as algorithms for finding fixed points of nonexpansive mappings; because, according to Proposition 2, $VI(C, A)$ is singleton. Therefore, they must be considered as algorithms converging only to the unique solution of $VI(C, A)$.

Comment 3. The iterative algorithms proposed by Gao and Guo [5] and Qin et al. [7, 8] for finding a common element of the set of solutions of variational inequalities for a relaxed (γ, r)-cocoercive μ-Lipschitzian mapping of C into H such that $r > \gamma\mu^2$, the set of solutions of an equilibrium problem and set of the fixed points of nonexpansive mapping, are not general as they should be and despite their becoming more complicated, they converge only to the unique solution of $VI(C, A)$.

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