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Existence and multiplicity of global classical solutions for approximate long water wave (ALWW) equations

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Abstract

In this paper, we investigate the Cauchy problem for long water wave equations for the existence and nonuniqueness of global classical solutions. We give sufficient conditions on the initial data of the considered equations that guarantee the existence and multiplicity of nonnegative global classical solutions. For this goal, a new topological approach to the fixed point theory of the sum of two operators in Banach spaces is used.

Keywords: Approximate long water wave equations, classical solution, fixed point, initial value problem
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1 Introduction

In this paper, we are concerned with a system of nonlinear evolution equations. Namely, we investigate the Cauchy problem for the approximate long water wave equations [1], [9], [16] in the form:

$$\begin{array}{lcl} u_t - uu_x - v_x + \alpha u_{xx} & = & 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ v_t - (uv)_x - \alpha v_{xx} & = & 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) & = & u_0(x), \quad x \in \mathbb{R}, \\ v(0,x) & = & v_0(x), \quad x \in \mathbb{R}, \end{array} \tag{1.1}$$

where the unknowns $u = u(t,x)$ and $v = v(t,x)$ denote respectively, the horizontal velocity and the height that deviates from equilibrium position of the liquid. Here, the constant α which represents a diffusion power, belongs to the set of all nonzero real numbers $\mathbb{R}^*$, and the initial conditions u_0, v_0 are given functions.

The equations of IVP (1.1) play a vital role in describing the properties of shallow water waves and they are using to study many physical phenomena, such as propagation of waves in dissipative and nonlinear media in hydrodynamics. They are also widely used in ocean and coastal engineering, see [5], [8] and the references therein. In [4], the authors have developed some approximate models for water waves. In [15] and [7], the travelling wave solutions of the

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approximate long water wave equations are found out by using the $(\frac{G'}{G})$-expansion method. Solitary wave solutions to the approximate long water wave equations are considered in [11] by using the first integral method. In [14], the q-homotopy analysis transform method is employed to study the approximate long wave equations of the Caputo fractional time derivative, see also [17] and [10] for fractional approximate long wave equations.

In this paper, we are especially interested in question of what conditions the initial data u_0, v_0 must verify in order that Problem (1.1) has classical solutions. Here, by a classical solution to the approximate long water wave equations we mean a solution which is at least two times continuously differentiable in x and once in t for any t ≥ 0. In other words, (u, v) belongs to the space $\mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R}))$ of continuously differentiable functions on $[0,\infty)$ with values in the Banach space $\mathcal{C}^2(\mathbb{R})$. The main assumption on the functions u_0, v_0 is

(H1) $u_0, v_0 \in \mathcal{C}^1(\mathbb{R})$, $0 \leq u_0, v_0 \leq B$ on $\mathbb{R}$ for some positive constant B.

To prove our main results we use the fixed-point index theory in cone spaces. During the last decades, the fixed-point index theory in cone spaces has intensively used for the proof of existence of solutions of different classes initial and boundary value problems for ordinary and partial differential equations (see [2], [13] and references therein). In many cases, via the fixed-point index theory in cone spaces are obtained new and complimentary results to the classical results.

The paper is organized as follows. In the next section, we give some auxiliary results concerning a new topological approach which uses fixed point theory of the sum of two operators. In Section 3, we give an integral representation and some estimates for solutions of IVP (1.1). In Section 4, we prove existence and multiplicity of solutions for the system (1.1). Finally, in section 5 we give an example to illustrate our main results.

2 Auxiliary Results

In this section, some definitions and results related to fixed points for the sum of two operators will be given.

Theorem 2.1. Let E be a Banach space and

$$E_1 = \{x \in E : \|x\| \leq R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = -\epsilon x, \ x \in E_1,$$

with $\epsilon > 0$ and $S : E_1 \to E$ be continuous and such that

(i) $(I - S)(E_1)$ resides in a compact subset of E and

(ii) $\{x \in E : x = \lambda(I - S)x, \quad \|x\| = R\} = \emptyset, \quad \text{for any} \quad \lambda \in \left(0, \frac{1}{\epsilon}\right)$.

Then there exists $x^* \in E_1$ such that

$$Tx^* + Sx^* = x^*.$$

Theorem 2.1 will be used to prove Theorem 4.1 and its proof can be found in [6].

Let E be a real Banach space.

Definition 2.2. A closed, convex set $\mathcal{P}$ in E is said to be cone if

1. $\beta x \in \mathcal{P}$ for any $\beta \geq 0$ and for any $x \in \mathcal{P}$,
2. $x, -x \in \mathcal{P}$ implies $x = 0$.

Definition 2.3. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.4. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

The following result (see details of its proof in [3]) will be used to prove Theorem 4.4.

Theorem 2.5. Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $w_0 \in \mathcal{P}\backslash\{0\}$ such that the following conditions hold:

(i) $Sx \neq (I - T)(x - \lambda w_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda w_0)$,

(ii) there exists $\varepsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,

(iii) $Sx \neq (I - T)(x - \lambda w_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda w_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

The following result will be used to prove the existence of three nonnegative solutions (at least two non zeros) of our proposal problem. More precisely, it will be used to prove Theorem 4.5. For the proof, we use the same arguments used in [3].

Theorem 2.6. Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to E$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous one and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exist $w_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(i) $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_1$ and $\lambda x \in \Omega$,

(ii) $Sx \neq (I - T)(x - \lambda w_0)$, for all $\lambda \geq 0$ and $x \in \partial U_2 \cap (\Omega + \lambda w_0)$,

(iii) $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_3$ and $\lambda x \in \Omega$.

Then $T + S$ has at least three non trivial fixed points $x_1, x_2, x_3 \in \mathcal{P}$ such that

$$x_1 \in \overline{U}_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3) \setminus \overline{U}_2) \cap \Omega.$$

3 Integral representation and some estimates for solutions of Problem (1.1)

Let $X = X^1 \times X^1$, where $X^1 = \mathcal{C}^1([0, \infty), \mathcal{C}^2(\mathbb{R}))$. For $(u, v) \in X$, define the operators S_1^1, S_1^2 and S_1 as follows.

$$\begin{aligned} S_1^1(u,v)(t,x) &= u(t,x) - u_0(x) + \int_0^t (-u(s,x)u_x(s,x) - v_x(s,x) + \alpha u_{xx}(s,x))ds, \\ S_1^2(u,v)(t,x) &= v(t,x) - v(0,x) + \int_0^t \left(-u_x(s,x)v(s,x) - u(s,x)v_x(s,x) - \alpha v_{xx}(s,x)\right) ds, \\ S_1(u,v)(t,x) &= \left(S_1^1(u,v)(t,x), S_1^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

Lemma 3.1. Suppose that $(H1)$ is satisfied. If $(u, v) \in X$ satisfies the equation

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{3.1}$$

then (u, v) is a solution of the IVP (1.1).

Proof . Let $(u,v) \in X$ is a solution to the equation (3.1). Then

$$S_1^1(u,v)(t,x) = 0, \quad S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.2}$$

We differentiate both equations of (3.2) with respect to t and we find

$$\begin{aligned} u_t(t,x) - u(t,x)u_x(t,x) - v_x(t,x) + \alpha u_{xx}(t,x) &= 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ v_t(t,x) - u_x(t,x)v(t,x) - u(t,x)v_x(t,x) - \alpha v_{xx}(t,x) &= 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

We put $t = 0$ in both equations of (3.2) and we arrive at

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}.$$

This completes the proof. □

Lemma 3.2. Suppose that $(H1)$ is satisfied. Let $h \in \mathcal{C}([0,\infty) \times \mathbb{R})$ be a positive function almost everywhere on $[0,\infty) \times \mathbb{R}$. If $(u,v) \in X$ satisfies the following integral equations:

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^2 h(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^2 h(t_1,x_1) S_1^2(u,v)(t_1,x_1) dx_1 dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then (u,v) is a solution to the IVP (1.1).

Proof . We differentiate two times with respect to t and three times with respect to x the integral equations of Lemma 3.2 and we find

$$h(t,x) S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}.$$

Hence and Lemma 3.1, we conclude that (u,v) is a solution to the IVP (1.1). □

Now, let us prove some estimates related to solutions of IVP (1.1). In the sequel, $X = X^1 \times X^1$ where $X^1 = \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R}))$ will be endowed with the norm

$$\|(u,v)\| = \max\{\|u\|_{X^1}, \quad \|v\|_{X^1}\}, \quad (u,v) \in X,$$

with

$$\begin{aligned} \|u\|_{X^1} = \max\Big\{ & \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u(t,x)|, \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u_t(t,x)|, \\ & \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u_x(t,x)|, \sup_{(t,x) \in [0,\infty) \times \mathbb{R}} |u_{xx}(t,x)| \Big\}, \end{aligned}$$

provided it exists. Let

$$B_1 = \max\{2B, B^2 + B + |\alpha|B, 2B^2 + |\alpha|B\}.$$

Lemma 3.3. Under hypothesis $(H1)$ and for $(u,v) \in X$ with $\|(u,v)\| \leq B$, the following estimates hold:

$$|S_1^1(u,v)(t,x)| \leq B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$|S_1^2(u,v)(t,x)| \leq B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof . Suppose that $(H1)$ is satisfied and let $(u,v) \in X$ with $\|(u,v)\| \leq B$.

(i) Estimation of $|S_1^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} |S_1^1(u,v)(t,x)| &= \left| u(t,x) - u_0(x) + \int_0^t (-u(s,x)u_x(s,x) - v_x(s,x) + \alpha u_{xx}(s,x))ds \right| \\ &\leq |u(t,x)| + |u_0(x)| + \int_0^t (|u(s,x)||u_x(s,x)| + |v_x(s,x)| + |\alpha||u_{xx}(s,x)|)ds \\ &\leq 2B + (B^2 + B + |\alpha|B)t \leq B_1(1+t). \end{aligned}$$

(ii) Estimation of $|S_1^2(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} \left|S_1^2(u,v)(t,x)\right| &= \Big| v(t,x) - v(0,x) \\ &\quad + \int_0^t \left(-u_x(s,x)v(s,x) - u(s,x)v_x(s,x) - \alpha v_{xx}(s,x)\right) ds \Big| \\ &\leq |v(t,x)| + |v(0,x)| \\ &\quad + \int_0^t \left(|u_x(s,x)||v(s,x)| + |u(s,x)||v_x(s,x)| + |\alpha||v_{xx}(s,x)|\right) ds \\ &\leq 2B + (2B^2 + |\alpha|B)t \\ &= B_1(1+t). \end{aligned}$$

This completes the proof. □

Suppose

(H2) $g \in \mathcal{C}([0,\infty) \times \mathbb{R})$ is a positive function almost everywhere on $[0,\infty) \times \mathbb{R}$ such that

$$8(1+t)^2 \left(1 + |x| + x^2\right) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \leq A, \ (t,x) \in [0,\infty) \times \mathbb{R},$$

for some constant $A > 0$.

In the last section, we will give an example for a function g that satisfies $(H2)$. For $(u,v) \in X$, define the operators S_2^1, S_2^2 and S_2 as follows.

$$\begin{aligned} S_2^1(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ S_2^2(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1) S_1^2(u,v)(t_1,x_1)dx_1dt_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

and

$$S_2(u,v)(t,x) = \left(S_2^1(u,v)(t,x), S_2^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.3}$$

Lemma 3.4. Under hypotheses $(H1)$ and $(H2)$ and for $(u,v) \in X$, with $\|(u,v)\| \leq B$, the following estimate holds:

$$\|S_2(u,v)\| \leq AB_1.$$

Proof . Suppose that $(H1)$ and $(H2)$ are satisfied. Let $(u,v) \in X$, with $\|(u,v)\| \leq B$.

(i) Estimation of $|S_2^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
|S_2^1(u,v)(t,x)| &= \left| \int_0^t \int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1 \right| \\
&\leq \int_0^t \left| \int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1 \right| dt_1 \\
&\leq B_1(1+t) \int_0^t \left| \int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 8B_1(1+t)^2x^2 \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 8B_1(1+t)^2(1+|x|+x^2) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

(ii) Estimation of $|\frac{\partial}{\partial t} S_2^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
\left| \frac{\partial}{\partial t} S_2^1(u,v)(t,x) \right| &= \left| \int_0^t \int_0^x (x-x_1)^2 g(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1 \right| \\
&\leq \int_0^t \left| \int_0^x (x-x_1)^2 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1 \right| dt_1 \\
&\leq B_1(1+t) \int_0^t \left| \int_0^x (x-x_1)^2 g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 8B_1(1+t)^2|x|^2 \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 8B_1(1+t)^2(1+|x|+x^2) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

(iii) Estimation of $|\frac{\partial}{\partial x} S_2^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
\left| \frac{\partial}{\partial x} S_2^1(u,v)(t,x) \right| &= 2\left| \int_0^t \int_0^x (t-t_1)(x-x_1) g(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1 \right| \\
&\leq 2\int_0^t \left| \int_0^x (t-t_1)|x-x_1| g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1 \right| dt_1 \\
&\leq 2B_1(1+t) \int_0^t \left| \int_0^x (t-t_1)|x-x_1| g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 4B_1(1+t)^2|x| \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq 8B_1(1+t)^2(1+|x|+x^2) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

(iv) Estimation of $|\frac{\partial^2}{\partial x^2} S_2^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x^2} S_2^1(u,v)(t,x)\right| &= 2\left|\int_0^t \int_0^x (t-t_1) g(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1\right| \\
&\leq 2\int_0^t \left|\int_0^x (t-t_1) g(t_1,x_1) |S_1^1(u,v)(t_1,x_1)| dx_1\right| dt_1 \\
&\leq 2B_1(1+t)\int_0^t \left|\int_0^x (t-t_1) g(t_1,x_1) dx_1\right| dt_1 \\
&\leq 2B_1(1+t)^2\int_0^t \left|\int_0^x g(t_1,x_1) dx_1\right| dt_1 \\
&\leq 8B_1(1+t)^2(1+|x|+x^2)\int_0^t \left|\int_0^x g(t_1,x_1) dx_1\right| dt_1 \\
&\leq AB_1.
\end{aligned}$$

Similarly, the same estimates $(i)-(iv)$ can be proved for the operator S_2^2. Finally,

$$\|S_2(u,v)\| \leq AB_1.$$

This completes the proof. □

4 Applications of the sum of two operators method

4.1 Existence of at least one nonnegative solution

In the sequel, suppose that the constants B and A which appear in the conditions $(H1)$ and $(H2)$, respectively, satisfy the following inequality:

(H3) $AB_1 < B$, where $B_1 = \max\{2B, B^2 + B + |\alpha|B, 2B^2 + |\alpha|B\}$.

Our first main result for existence of classical solutions of the IVP (1.1) is as follows.

Theorem 4.1. Under hypotheses $(H1)$, $(H2)$ and $(H3)$, the IVP (1.1) has at least one nonnegative solution $(u,v) \in \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$.

Remark 4.2. When we say that $(u,v) \in \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$ is a nonnegative solution to the IVP (1.1) we have in mind that it is a solution to the IVP (1.1) and $u(t,x) \geq 0$, $v(t,x) \geq 0$ for any $(t,x) \in [0,\infty) \times \mathbb{R}$.

Remark 4.3. Below, we will use the notation $(u,v)(t,x)$ to denote $(u(t,x), v(t,x))$.

Proof . Choose $\epsilon \in (0,1)$, such that $\epsilon B_1(1+A) < B$. For $(u,v) \in X = \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$, we will write

$$(u,v) \geq 0 \text{ if } u(t,x) \geq 0 \text{ and } v(t,x) \geq 0 \text{ for any } (t,x) \in [0,\infty) \times \mathbb{R}.$$

Let $\widetilde{\widetilde{\widetilde{Y}}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{\widetilde{Y}} = \overline{\widetilde{\widetilde{\widetilde{Y}}}}$ be the closure of $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{Y} = \widetilde{\widetilde{Y}} \cup \{(u_0,v_0)\}$,

$$Y = \{(u,v) \in \widetilde{Y} : (u,v) \geq 0, \quad \|(u,v)\| \leq B\}.$$

Note that Y is a compact set in X. For $(u,v) \in X$, define the operators T and S as follows.

$$\begin{aligned}
T(u,v)(t,x) &= -\epsilon(u,v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\
S(u,v)(t,x) &= (u,v)(t,x) + \epsilon(u,v)(t,x) + \epsilon S_2(u,v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},
\end{aligned}$$

where S_2 is the operator given by formula (3.3). For $(u, v) \in Y$ and by using Lemma 3.4, it follows that

$$\begin{aligned}\|(I-S)(u,v)\| &= \|\epsilon(u,v) - \epsilon S_2(u,v)\| \\ &\leq \epsilon\|(u,v)\| + \epsilon\|S_2(u,v)\| \\ &\leq \epsilon B_1 + \epsilon A B_1 \\ &= \epsilon B_1(1+A) \\ &< B.\end{aligned}$$

Thus, $S : Y \to X$ is continuous and $(I-S)(Y)$ resides in a compact subset of X. Now, suppose that there is $(u, v) \in X$ such that $\|(u, v)\| = B$ and

$$(u,v) = \lambda(I-S)(u,v)$$

or

$$\frac{1}{\lambda}(u,v) = (I-S)(u,v) = -\epsilon(u,v) - \epsilon S_2(u,v),$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(u,v) = -\epsilon S_2(u,v)$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(u, v)\| \leq AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|(u,v)\| = \epsilon\|S_2(u,v)\| < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T + S$ has a fixed point $(u^*, v^*) \in Y$. Therefore, for $(t, x) \in [0, \infty) \times \mathbb{R}$,

$$\begin{aligned}(u^*,v^*)(t,x) &= T(u^*,v^*)(t,x) + S(u^*,v^*)(t,x) \\ &= -\epsilon(u^*,v^*)(t,x) + (u^*,v^*)(t,x) + \epsilon(u^*,v^*)(t,x) + \epsilon S_2(u^*,v^*)(t,x),\end{aligned}$$

whereupon

$$0 = S_2(u^*,v^*)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

From here and from Lemma 3.2, it follows that (u^*, v^*) is a solution to the IVP (1.1). This completes the proof. □

4.2 Existence of at least two nonnegative solutions

In the sequel, suppose that the constants B and A which appear in the conditions $(H1)$ and $(H2)$, respectively, satisfy the following inequality:

(H4) $AB_1 < \frac{L}{5}$, where $B_1 = \max\{2B, B^2 + B + |\alpha|B, 2B^2 + |\alpha|B\}$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

with r and R_1 are positive constants and $m > 0$ is large enough.

Our second main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

Theorem 4.4. Under Hypotheses $(H1)$, $(H2)$ and $(H4)$, the IVP (1.1) has at least two nonnegative solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R}))$.

Proof . Set $X = \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^2(\mathbb{R}))$ and let

$$\widetilde{P} = \{(u,v) \in X : (u,v) \geq 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u,v) \in X$, define the operators T_1 and S_3 as follows. For $(t,x) \in [0,\infty) \times \mathbb{R}$,

$$\begin{aligned} T_1(u,v)(t,x) &= (1+m\epsilon)(u,v)(t,x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \\ S_3(u,v)(t,x) &= -\epsilon S_2(u,v)(t,x) - m\epsilon(u,v)(t,x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \end{aligned}$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (3.3). Note that any fixed point $(u,v) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Now, let us define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{(u,v) \in \mathcal{P} : \|(u,v)\| < r\}, \\ U_2 &= \mathcal{P}_L = \{(u,v) \in \mathcal{P} : \|(u,v)\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{(u,v) \in \mathcal{P} : \|(u,v)\| < R_1\}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{(u,v) \in \mathcal{P} : \|(u,v)\| \leq R_2\}, \text{ with } R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}. \end{aligned}$$

1. For $(u_1,v_1), (u_2,v_2) \in \Omega$, we have

$$\|T_1(u_1,v_1) - T_1(u_2,v_2)\| = (1+m\epsilon)\|(u_1,v_1) - (u_2,v_2)\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

2. For $(u,v) \in \overline{\mathcal{P}_{R_1}}$ and by using Lemma 3.4 we get

$$\begin{aligned} \|S_3(u,v)\| &\leq \epsilon\|S_2(u,v)\| + m\epsilon\|(u,v)\| + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

Therefore, $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently, $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

3. Let $(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$. Set

$$(u_2,v_2) = (u_1,v_1) + \frac{1}{m}S_2(u_1,v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

By Lemma 3.4, we have

$$\begin{aligned} |S_2^1(u_1,v_1)(t,x)| &\leq AB_1, \\ |S_2^2(u_1,v_1)(t,x)| &\leq AB_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

By $(H4)$, we have $AB_1 < \frac{L}{5}$. Hence,

$$\begin{aligned} |S_2^1(u_1,v_1)(t,x)| &< \frac{L}{5}, \\ |S_2^2(u_1,v_1)(t,x)| &< \frac{L}{5}, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

Thus, $u_2, v_2 \geq 0$ on $[0, \infty) \times \mathbb{R}$ and

$$\begin{aligned} \|(u_2, v_2)\| &\leq \|(u_1, v_1)\| + \frac{1}{m}\|S_2(u_1, v_1)\| + \frac{L}{5m} \\ &\leq R_1 + \frac{A}{m}B_1 + \frac{L}{5m} \\ &= R_2. \end{aligned}$$

Therefore, $(u_2, v_2) \in \Omega$ and

$$-\epsilon m(u_2, v_2) = -\epsilon m(u_1, v_1) - \epsilon S_2(u_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$\begin{aligned} (I - T_1)(u_2, v_2) &= -\epsilon m(u_2, v_2) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) \\ &= S_3(u_1, v_1). \end{aligned}$$

Consequently, $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $(w_0, z_0) \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$ there exist $\lambda > 0$ and $(u, v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(w_0, z_0))$ or $(u, v) \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda(w_0, z_0))$ such that

$$S_3(u, v) = (I - T_1)((u, v) - \lambda(w_0, z_0)).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u, v) - \lambda(w_0, z_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(u, v) = \lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2(u, v)\| = \left\|\lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $(u_1, v_1) \in \partial\mathcal{P}_L$ and $\lambda_1 > 1 + \varepsilon_1$ such that $\lambda_1(u_1, v_1) \in \overline{\mathcal{P}_{R_1}} \subset \Omega$ and

$$S_3(u_1, v_1) = (I - T_1)(\lambda_1(u_1, v_1)). \tag{4.1}$$

Since $(u_1, v_1) \in \partial\mathcal{P}_L$ and $\lambda_1(u_1, v_1) \in \overline{\mathcal{P}_{R_1}}$, it follows that

$$\left(\frac{2}{5m} + 1\right)L < \lambda_1 L = \lambda_1\|(u_1, v_1)\| \leq R_1.$$

Moreover,

$$-\epsilon S_2(u_1, v_1) - m\epsilon(u_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1, v_1) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right),$$

or

$$S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1)m(u_1, v_1).$$

From here,

$$2\frac{L}{5} \geq \left\|S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| = (\lambda_1 - 1)m\|(u_1, v_1)\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore, all conditions of Theorem 2.5 hold. Hence, the IVP (1.1) has at least two solutions (u_1, v_1) and (u_2, v_2) so that

$$\|(u_1, v_1)\| = L < \|(u_2, v_2)\| \le R_1$$

or

$$r \le \|(u_1, v_1)\| < L < \|(u_2, v_2)\| \le R_1.$$

□

4.3 Existence of at least three nonnegative solutions

In this section, we will use the notations of the proof of Theorem 4.4. Suppose

(H5) $\epsilon m r > \frac{2L}{5}$.

Our third main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

Theorem 4.5. Under Hypotheses $(H1)$, $(H2)$, $(H4)$ and $(H5)$, the IVP (1.1) has at least two nonnegative solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in \mathcal{C}^1([0, \infty), \mathcal{C}^2(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^2(\mathbb{R}))$.

Proof .

1. Assume that there are $\lambda \ge 1 + \epsilon$, $(u, v) \in \partial U_1$ and $\lambda(u, v) \in \Omega$ so that

$$S_3(u, v) = (I - T_1)(\lambda u, \lambda v).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right) = -m\epsilon\lambda(u, v) + \left(\epsilon \frac{L}{10}, \epsilon \frac{L}{10}\right)$$

or

$$S_2(u, v) = (\lambda - 1)m(u, v) - \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\begin{aligned} \|S_2(u, v)\| &= \left\| m(\lambda - 1)(u, v) - \left(\frac{L}{5}, \frac{L}{5}\right) \right\| \\ &\ge (\lambda - 1)m\|(u, v)\| - \left\| \left(\frac{L}{5}, \frac{L}{5}\right) \right\| \\ &\ge \epsilon m\|(u, v)\| - \frac{L}{5} \\ &= \epsilon m r - \frac{L}{5} \\ &> \frac{L}{5}, \end{aligned}$$

which is a contradiction. Thus, the condition (**i**) of Theorem 2.6 holds.

2. Now, assume that there are $\lambda \ge 1 + \epsilon$, $(u, v) \in \partial U_3$ and $\lambda(u, v) \in \Omega$ so that

$$S_3(u, v) = (I - T_1)(\lambda u, \lambda v).$$

As above,

$$\begin{aligned} \|S_2(u, v)\| &\ge (\lambda - 1)m\|(u, v)\| - \left\| \left(\frac{L}{5}, \frac{L}{5}\right) \right\| \\ &\ge \epsilon m\|(u, v)\| - \frac{L}{5} \\ &= \epsilon m R_1 - \frac{L}{5} \\ &> \epsilon m r - \frac{L}{5} \\ &> \frac{L}{5}, \end{aligned}$$

which is a contradiction. Hence, the condition (**iii**) of Theorem 2.6 holds.

3. Assume that for any $(w_0, z_0) \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $(u, v) \in \partial\mathcal{P}_L \cap (\Omega + \lambda(w_0, z_0))$ such that

$$S_3(u, v) = (I - T_1)((u, v) - \lambda(w_0, z_0)).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u, v) - \lambda(w_0, z_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(u, v) = \lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2(u, v)\| = \left\|\lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This is a contradiction. Form here, the condition (**ii**) of Theorem 2.6 holds.

Now, by Theorem 2.6, it follows that the IVP (1.1) has at least three classical solutions. □

5 An illustrative example

Below, we will illustrate our main results. Let

$$h(s) = \log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore,

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty}(1 + s + s^2)h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty}(1 + s + s^2)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2 + s^3 + s^4 + s^5 + s^6)\left(\frac{1}{44\sqrt{2}}\log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s \to \pm 1} l(s) = \frac{\pi}{2}$ and by [12] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}}\log\frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t, x) = Q(t)Q(x), \quad t \in [0, \infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ such that

$$8(1+t)^2\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g_1(t_1,x_1)dx_1\right|dt_1 \leq C_2, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x)=\frac{A}{C_2}g_1(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Then

$$8(1+t)^2\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \leq A, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

i.e., $(H2)$ holds. Now, consider the initial value problem

$$\begin{aligned} u_t-uu_x-v_x+2u_{xx} &= 0, \quad t\in(0,\infty), \quad x\in\mathbb{R},\\ v_t-(uv)_x-2v_{xx} &= 0, \quad t\in(0,\infty), \quad x\in\mathbb{R},\\ u(0,x) &= \tfrac{1}{1+x^2+x^4+3x^6}, \quad x\in\mathbb{R},\\ v(0,x) &= \tfrac{1}{1+3x^2}, \quad x\in\mathbb{R}, \end{aligned} \tag{5.1}$$

so that $(H1)$ holds, with $B=10$, for example. Here, $\alpha=2$, take

$$B=10, \quad \text{and } A=\frac{1}{10^4}.$$

Then

$$B_1=\max\{2B, B^2+B+|\alpha|B, 2B^2+|\alpha|B\}=\max\{20,130,220\}=220$$

and

$$AB_1=\frac{220}{10^4}<B.$$

So, Condition $(H3)$ is fulfilled. Thus, the conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied. Hence, by Theorem 4.1, it follows that IVP (5.1) has at least one nonnegative solution $(u,v)\in\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))\times\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$.

In the sequel, take

$$R_1=B=10, \quad L=5, \quad \alpha=2, \quad r=4, \quad m=10^{50}, \quad A=\epsilon=\frac{1}{10^4}.$$

Clearly,

$$r<L<R_1\leq B, \quad \epsilon>0, \quad R_1>\left(\frac{2}{5m}+1\right)L, \quad AB_1<\frac{L}{5}.$$

i.e., $(H4)$ holds. Thus, the conditions $(H1)$, $(H2)$, and $(H4)$ are satisfied. Hence, by Theorem 4.4, it follows that the IVP (5.1) has at least two nonnegative solutions $(u_1,v_1),(u_2,v_2)\in\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))\times\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$. Moreover,

$$\epsilon rm=4\cdot 10^{50}\cdot\frac{1}{10^4}>2=\frac{2L}{5},$$

i.e., $(H5)$ holds. Thus, the conditions $(H1)$, $(H2)$, $(H4)$ and $(H5)$ are satisfied. Hence, by Theorem 4.5, it follows that the IVP (5.1) has at least three nonnegative solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)\in\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))\times\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}))$.

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