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Fixed point theorems in intuitionistic fuzzy b-metric like spaces

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Abstract

The aim of this manuscript is to introduce the concept of intuitionistic fuzzy b-metric-like spaces and discuss some fixed point results to certify the existence and uniqueness of a fixed point. Non-trivial examples are imparted to illustrate the viability of the proposed method.

Keywords: intuitionistic fuzzy b-metric-like spaces, fixed point theorem, unique solution
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1 Introduction

The concept of fuzzy sets (FS) was initiated by Zadeh [11], which gave a new aspect to research activity leading to the improvement of fuzzy system. Afterwards, several researchers contributed towards some significant results in FS.

Kramosil and Michalek [7] introduced the concept of fuzzy metric spaces by generalizing the concepts of probabilistic metric spaces to fuzzy metric spaces. George and Veeramani [1] derived a Hausdorff topology initiated by fuzzy metric to modify the concept of fuzzy metric spaces. Later on, the existence theory of fixed point in fuzzy metric was enriched with a number of different generalizations. Garbiec [14] displayed the fuzzy version of Banach contraction principle in fuzzy metric spaces. For some necessary definitions, examples and basic results, we refer to [13, 6, 24, 10, 20, 23] and the references herein.

As we know, fixed point theory plays a crucial role in proving the existence and solutions for different mathematical models and has a wide range of applications in different fields related to mathematics. This theory has intrigued many researchers and recently, Harandi [2] initiated the concept of metric-like spaces, which generalizes the notion of metric spaces in a nice way. Alghamdi *et al.* [12] used the concept of metric like spaces to introduce the notion of b-metric-like spaces (BMLS). In this sequel, Shukla and Abbas [21] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces (FMLS). The approach of intuitionistic fuzzy metric spaces was tossed by Park [8] and Konwar [17] initiated the notion of intuitionistic fuzzy b-metric space (IFBMS). For some necessary definitions, we refer [5, 9, 3, 15, 22, 25]. Saleem et al. [18, 19] established several fixed point results for contraction mappings. Delfani et al. [16] proved sevral fixed point results in the context of b-metric spaces.

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In this article, our aim is to generalize the concept of IFBMS by introducing the concept of Intuitionistic fuzzy b-metric-like spaces (IFBMLS) and prove some related fixed point results in this framework. We also furnish this work with examples.

Some following notations used throughout this paper, as CTN for a continuous triangular norm and CTCN for a continuous triangular co-norm.

2 Preliminaries

In this section, we provide serval notions from the existing literature.

Definition 2.1. [4] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is known as CTN if the following axioms are satisfied:
(a1) $*$ is associative and commutative,
(a2) $*$ is continuous,
(a3) $a * 1 = a$, for all $a \in [0,1]$,
(a4) If $a \leq b$ and $c \leq d$ with $a, b, c, d \in [0,1]$,then $a * c \leq b * d$.

Definition 2.2. [4] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is known as CTCN if the following axioms are satisfied:
(a1) $\circ$ is associative and commutative,
(a2) $\circ$ is continuous,
(a3) $a \circ 1 = 1$, for all $a \in [0,1]$,
(a4) If $a \leq b$ and $c \leq d$ with $a, b, c, d \in [0,1]$,then $a \circ c \leq b \circ d$.

Definition 2.3. [12] A BMLS on a set $X \neq \emptyset$ is a function $\sigma : X \times X \to [0, +\infty)$ such that for all $e, o, z \in X$ and $b \geq 1$, it satisfies the following conditions:

1. If $\sigma(e, o) = 0$, then $e = o$;
2. $\sigma(e, o) = \sigma(o, e)$;
3. $\sigma(e, o) \leq b[\sigma(e, z) + \sigma(z, o)]$.

The pair (X, σ) is called a BMLS.

Example 2.4. [12] Let $X = [0, \infty)$. Define $\sigma : X \times X \to [0, +\infty)$ by $\sigma(e, o) = (e + o)^2$. Then (X, σ) is a BMLS with $b = 2$.

Example 2.5. [12] Let $X = [0, \infty)$. Define $\sigma : X \times X \to [0, +\infty)$ by $\sigma(e, o) = (\max\{e, o\})^2$. Then (X, σ) is a BMLS with $b = 2$.

Definition 2.6. [21] A 3-tuple $(X, M, \star)$ is said to be an FMLS if $X \neq \emptyset$ is a random set, $\star$ is a CTN and M is an FS on $X \times X \times (0, \infty)$ such that for all $e, o, z \in X$, $t, s > 0$,

$FL1)$ $M(e, o, t) > 0$;

$FL2)$ If $M(e, o, t) = 1$, then $e = o$;

$FL3)$ $M(e, o, t) = M(o, e, t)$;

$FL4)$ $M(e, z, t + s) \geq M(e, o, t) \star M(o, z, s)$;

$FL5)$ $M(e, o, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.7. [21] Let $X = \mathbb{R}^+$, $q \in \mathbb{R}^+$ and $m > 0$. Define CTN by $g \star h = gh$ and M by $M(e, o, t) = \frac{qt}{qt + m(\max\{e, o\})}$ for all $e, o \in X$, $t > 0$. Then $(X, M, \star)$ is an FMLS.

Definition 2.8. [17] Suppose $X \neq \emptyset$. A five tuple $(X, M_b, N_b, \star, \circ)$ is said to be an intuitionistic fuzzy b-metric, where $\star$ is a CTN, $\circ$ is a CTCN, $b \geq 1$ and M_b, N_b are FSs on $X \times X \times (0, \infty)$, if it satisfies the following, for all $e, o \in X$ and $t, s > 0$,

(I) $M_b(e, o, t) + N_b(e, o, t) \leq 1$,

(II) $M_b(e,o,t) > 0$;

(III) $M_b(e,o,t) = 1$ if and only if $e = o$;

(IV) $M_b(e,o,t) = M_b(o,e,t)$;

(V) $M_b(e,z,b(t+s)) \geq M_b(e,o,t) \star M_b(o,z,s)$;

(VI) $M_b(e,o,\cdot)$ is non-decreasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} M_b(e,o,t) = 1$;

(VII) $N_b(e,o,t) > 0$;

$(VIII)$ $N_b(e,o,t) = 0$ if and only if $e = o$;

(IX) $N_b(e,o,t) = N_b(o,e,t)$;

(X) $N_b(e,z,b(t+s)) \leq N_b(e,o,t) \circ N_b(o,z,s)$;

(XI) $N_b(e,o,\cdot)$ is non-increasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} N_b(e,o,t) = 0$.

Then $(X, M_b, N_b, \star, \circ)$ is called an IFBMS.

3 Main Results

In this section, we introduce the notion of IBMLS and prove some fixed point results.

Definition 3.1. Suppose $X \neq \emptyset$. For a five tuple $(X, M_{bl}, N_{bl}, \star, \circ)$, where $\star$ is a CTN, $\circ$ is a CTCN, $b \geq 1$ and M_{bl}, N_{bl} are FS on $X \times X \times (0,\infty)$, assume that $(X, M_{bl}, N_{bl}, \star, \circ)$ satisfies the following, for all $e, o \in X$ and $t, s > 0$,

(i) $M_{bl}(e,o,t) + N_{bl}(e,o,t) \leq 1$;

(ii) $M_{bl}(e,o,t) > 0$;

(iii) $M_{bl}(e,o,t) = 1 \implies e = o$;

(iv) $M_{bl}(e,o,t) = M_{bl}(o,e,t)$;

(v) $M_{bl}(e,z,b(t+s)) \geq M_{bl}(e,o,t) \star M_{bl}(o,z,s)$;

(vi) $M_{bl}(e,o,\cdot)$ is non-decreasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} M_{bl}(e,o,t) = 1$;

(vii) $N_{bl}(e,o,t) > 0$;

$(viii)$ $N_{bl}(e,o,t) = 0 \implies e = o$;

(ix) $N_{bl}(e,o,t) = N_{bl}(o,e,t)$;

(x) $N_{bl}(e,z,b(t+s)) \leq N_{bl}(e,o,t) \circ N_{bl}(o,z,s)$;

(xi) $N_{bl}(e,o,\cdot)$ is non-increasing function of $\mathbb{R}^+$ and $\lim_{t\to\infty} N_{bl}(e,o,t) = 0$.

Then $(X, M_{bl}, N_{bl}, \star, \circ)$ is called an IFBMLS.

Remark 3.1. In the above definition, assume that a set X is an IFBMLS with a CTN $(\star)$ and CTCN $(\circ)$. Then the IFBMLS X does not satisfy (II) and $(VIII)$ of IFBMS, that is, the self-distance may not be equal to 1 and 0, i.e., $M_{bl}(e,e,t) \neq 1$ and $N_{bl}(e,e,t) \neq 0$ for all $t > 0$ or may be for all $e \in X$. But all other conditions are the same.

Example 3.2. Let $X = (0,\infty)$. Define a CTN by $g \star h = gh$ and a CTCN by $g \circ h = \max\{g,h\}$ and also define M_{bl} and N_{bl} by

$$M_{bl}(e,o,t) = [e^{\frac{(e+o)^2}{t}}]^{-1},$$

$$N_{bl}(e,o,t) = 1 - [e^{\frac{(e+o)^2}{t}}]^{-1}$$

for all $e, o \in X, t > 0$. Then it is an IFBMLS. But it is not an IFBMS.

Proof . (i)–(iv), (vi)–(ix) and (xi) are obvious.

Now we prove (v) and (x). Since

$$(e+z)^2 \leq (\frac{b(t+s)}{t})(e+o)^2 + (\frac{b(t+s)}{s})(o+z)^2,$$

where b is an arbitrary integer, we have

$$\frac{(e+z)^2}{b(t+s)} \leq \frac{(e+o)^2}{t} + \frac{(o+z)^2}{s}.$$

That is,

$$e^{\frac{(e+z)^2}{b(t+s)}} \leq e^{\frac{(e+o)^2}{t}} \cdot e^{\frac{(o+z)^2}{s}}.$$

Since e^e is an increasing function, for $e > 0$, we have

$$\begin{aligned} [e^{\frac{(e+z)^2}{b(t+s)}}]^{-1} &\geq [e^{\frac{(e+o)^2}{t}}]^{-1} \cdot [e^{\frac{(o+z)^2}{s}}]^{-1}, \\ M_{bl}(e,z,b(t+s)) &\geq M_{bl}(e,o,t) \star M_{bl}(o,z,s) \end{aligned}$$

for all $e, o, z \in X,\ t, s > 0$. Hence (v) holds.

Since

$$(e+z)^2 \geq \max\{(e+o)^2, (o+z)^2\},$$
$$\frac{(e+z)^2}{b(t+s)} \leq \max\{\frac{(e+o)^2}{t}, \frac{(o+z)^2}{s}\},$$

where b is an arbitrary integer. Since e^e is an increasing function, for $e > 0$, we have

$$e^{\frac{(e+z)^2}{b(t+s)}} \leq \max\{e^{\frac{(e+o)^2}{t}}, e^{\frac{(o+z)^2}{s}}\}.$$

So

$$[e^{\frac{(e+z)^2}{b(t+s)}}]^{-1} \geq \max\{[e^{\frac{(e+o)^2}{t}}]^{-1}, [e^{\frac{(o+z)^2}{s}}]^{-1}\}.$$

That is,

$$1 - [e^{\frac{(e+z)^2}{b(t+s)}}]^{-1} \leq \max\{1 - [e^{\frac{(e+o)^2}{t}}]^{-1}, 1 - [e^{\frac{(o+z)^2}{s}}]^{-1}\}.$$

Hence

$$N_{bl}(e,z,b(t+s)) \leq N_{bl}(e,o,t) \circ N_{bl}(o,z,s)$$

for all $e, o, z \in X,\ t, s > 0$. This implies that (x) holds.

Now, we have to prove that $(X, M_{bl}, N_{bl}, \star, \circ)$ is not an IFBMS. For this purpose, we investigate the self-distance. Since

$$M_{bl}(e,e,t) = [e^{\frac{(e+e)^2}{t}}]^{-1} = \frac{1}{e^{\frac{(e+e)^2}{t}}} \neq 1$$

and

$$N_{bl}(e,e,t) = 1 - [e^{\frac{(e+e)^2}{t}}]^{-1} = 1 - \frac{1}{e^{\frac{(e+e)^2}{t}}} \neq 0$$

for all $t > 0, e \in X$. Hence $(X, M_{bl}, N_{bl}, \star, \circ)$ is not an IFBMS. $\square$

Remark 3.2. The above example shows that IFBMLS need not be an IFBMS. Also every IFBMS must be an IFBMLS.

The following example shows that an IFBMLS need not be continuous.

Example 3.3. Let $X = [0, \infty)$, $M_{bl}(e,o,t) = (e^{-\frac{\sigma(e,o)}{t}})$ and $M_{bl}(e,o,t) = 1 - (e^{-\frac{\sigma(e,o)}{t}})$ for all $e, o \in X, t > 0$ and

$$\sigma(e,o) = \begin{cases} 0, \text{ if } e = o \\ 2\left(e+o\right)^2, \text{ if } e, o \in [0,1] \\ \frac{1}{2}\left(e+o\right)^2, \text{ otherwise.} \end{cases}$$

Define a CTN as $g \star h = gh$ and a CTCN as $g \circ h = \max\{g, h\}$. Then $(X, M_{bl}, N_{bl}, \star, \circ)$ is an IFBMLS with a coefficient $b = 4$. To illustrate the discontinuity, we have

$$\lim_{n\to\infty} M_{bl}(0, 1 - \frac{1}{n}, t) = \lim_{n\to\infty} e^{-2(1-\frac{1}{n})^2} = e^{-2} = M_{bl}(0,1,t)$$

and

$$\lim_{n\to\infty} N_{bl}(0, 1-\frac{1}{n}, t) = 1 - \lim_{n\to\infty} e^{-2(1-\frac{1}{n})^2} = 1 - e^{-2} = N_{bl}(0, 1, t).$$

However, since

$$\lim_{n\to\infty} M_{bl}(1, 1-\frac{1}{n}, t) = \lim_{n\to\infty} e^{-2(2-\frac{1}{n})^2} = e^{-8} \neq 1 = M_{bl}(1, 1, t)$$

and

$$\lim_{n\to\infty} N_{bl}(1, 1-\frac{1}{n}, t) = 1 - \lim_{n\to\infty} e^{-2(2-\frac{1}{n})^2} = 1 - e^{-8} \neq 0 = N_{bl}(1, 1, t),$$

$M_{bl}(e, o, t)$ and $N_{bl}(e, o, t)$ are not continuous.

Proposition 3.4. Let (X, σ) be a BMLS. If we take CTN and CTCN as in Example 3.2, then $(X, M_{bl}, N_{bl}, \star, \circ)$ is an IFBMLS defined as

$$M_{bl}(e, o, t) = e^{\frac{-\sigma(e,o)}{t^n}}$$

and

$$N_{bl}(e, o, t) = 1 - e^{\frac{-\sigma(e,o)}{t^n}}$$

for all $t > 0$ and all $e, o \in X, n \in \mathbb{N}$.

Proof . $(i)-(iv), (vi)-(ix)$ and (xi) are obvious.

Now, we prove (v) and (x). Since

$$\sigma(e, z) \leq b[\sigma(e, o) + \sigma(o, z)],$$
$$\frac{\sigma(e, z)}{(t+s)^n} \leq \frac{b[\sigma(e, o) + \sigma(o, z)]}{(t+s)^n}.$$

Thus

$$\frac{\sigma(e, z)}{b(t+s)^n} \leq \frac{\sigma(e, o)}{t^n} + \frac{\sigma(o, z)}{s^n}.$$

That is,

$$e^{\frac{\sigma(e,z)}{b(t+s)^n}} \geq e^{\frac{\sigma(e,o)}{t^n}} \cdot e^{\frac{\sigma(o,z)}{s^n}}$$

Hence

$$M_{bl}(e, z, b(t+s)) \geq M_{bl}(e, o, t) \star M_{bl}(o, z, s).$$

This says that (v) holds.

Since

$$\sigma(e, z) \geq \max\{\sigma(e, o), \sigma(o, z)\},$$
$$\frac{\sigma(e, z)}{b(t+s)} \leq \max\{\frac{\sigma(e, o)}{t}, \frac{\sigma(e, z)}{s}\},$$

where b is an arbitrary integer. Since e^e is an increasing function, for $e > 1$, we have

$$e^{\frac{\sigma(e,z)|}{b(t+s)}} \leq \max\{e^{\frac{\sigma(e,o)}{t}}, e^{\frac{\sigma(o,z)}{s}}\}.$$

Thus

$$e^{-\frac{\sigma(e,z)}{b(t+s)}} \geq \max\{e^{-\frac{\sigma(e,o)}{t}}, e^{-\frac{\sigma(o,z)}{s}}\}.$$

That is,

$$1 - e^{-\frac{\sigma(e,z)}{b(t+s)}} \leq \max\{1 - e^{-\frac{\sigma(e,o)}{t}}, 1 - e^{-\frac{\sigma(e,z)}{s}}\}.$$

Hence

$$N_{bl}(e, z, b(t+s)) \leq N_{bl}(e, o, t) \circ N_{bl}(o, z, s)$$

for all $e, o, z \in X$ and $t, s > 0$. This says that (x) holds. Hence $(X, M_{bl}, N_{bl}, \star, \circ)$ is an IFBMLS. □

Remark 3.3. Note that the above proposition also holds for $g \star h = \min\{g, h\}$ and $g \circ h = \max\{g, h\}$.

Proposition 3.5. Let (X,σ) be a BMLS. Then a five triple $(X, M_{bl}, N_{bl}, \star, \circ)$ is an IFBMLS, where $\star$ is defined by $g \star h = gh$ and $\circ$ is defined by $g \circ h = \max\{g,h\}$ and fuzzy sets M_{bl} and N_{bl} are given by

$$M_{bl}(e,o,t) = \frac{ot^n}{ot^n + m\sigma(e,o)} \text{ for all } e,o \in X,\ t > 0$$

and

$$N_{bl}(e,o,t) = \frac{m\sigma(e,o)}{ot^n + m\sigma(e,o)} \text{ for all } e,o \in X,\ t > 0.$$

Here $o \in \mathbb{R}^+, m > 0$ and $n \geq 1$.

Remark 3.4. Note that the above proposition also holds for CTN $g \star h = \min\{g,h\}$ and $g \circ h = \max\{g,h\}$.

Remark 3.5. Proposition 3.5 shows that every BMLS induces an IFBMLS. For $o = n = m = 1$ the induced IFBMLS $(X, M_{bl}, N_{bl}, \star, \circ)$ is called the standard IFBMLS, where $o \in \mathbb{R}^+$

$$M_{bl}(e,o,t) = \frac{ot}{ot + \sigma(e,o)} \text{ for all } e,o \in X, t > 0$$

and

$$N_{bl}(e,o,t) = \frac{\sigma(e,o)}{ot + \sigma(e,o)} \text{for all } e,o \in X, t > 0.$$

Example 3.6. Let $X = \mathbb{R}^+$,$o \in \mathbb{R}^+$and $m > 0$. Define $\star$ by $g \star h = gh$ and $\circ$ by $g \circ h = \max\{g,h\}$ and FS M_{bl} and N_{bl} in $X \times X \times (0,\infty)$ by

$$M_{bl}(e,o,t) = \frac{ot}{ot + m(\max\{e,o\}^2)} \text{ for all } e,o \in X, t > 0$$

and

$$N_{bl}(e,o,t) = \frac{m(\max\{e,o\}^2)}{ot + m(\max\{e,o\}^2)} \text{ for all } e,o \in X, t > 0.$$

Since $\sigma(e,o) = \max\{e,o\}^2$ for all $e,o \in X$ is a BMLS on X, by Proposition 3.5, $(X, M_{bl}, N_{bl}, \star, \circ)$ is an IFBMLS, but it is not an IFBMS, since

$$M_{bl}(e,e,t) = \frac{ot}{ot + me^2} \neq 1 \text{ for all } e,o \in X, t > 0$$

and

$$N_{bl}(e,e,t) = \frac{me^2}{ot + me^2} \neq 0 \text{ for all } e,o \in X, t > 0.$$

Remark 3.6. In IFBMLS, the limit of a convergent sequence may not be unique, for instance, for an IFBMLS $(X, M_{bl}, N_{bl}, \star, \circ)$ given in Proposition 3.4 with $\sigma(e,o) = \max\{e,o\}^2$ and $n = 1$. Define a sequence $\{e_n\}$ in X by $e_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. If $e \geq 1$, then

$$\lim_{n\to\infty} M_{bl}(e_n,e,t) = \lim_{n\to\infty} e^{\frac{-\max e_n, e^2}{t}} = e^{-\frac{e^2}{t}} = M_{bl}(e,e,t) \text{ for all } t > 0$$

and

$$\lim_{n\to\infty} N_{bl}(e_n,e,t) = \lim_{n\to\infty} (1 - e^{\frac{-\max\{e_n,e\}^2}{t}}) = 1 - e^{-\frac{e^2}{t}} = N_{bl}(e,e,t) \text{ for all } t > 0.$$

Therefore, the sequence $\{e_n\}$ converge to all $e \in X$ with $e \geq 1$.

Remark 3.7. In an IFBMLS, a convergent sequence may not be Cauchy. Assume an IFBMLS $(X, M_{bl}, N_{bl}, \star, \circ)$ is in Remark 3.6. Define a sequence $\{e_n\}$ in X by $e_n = 1 + (-1)^n$ for all $n \in \mathbb{N}$. If $e \geq 2$, then

$$\lim_{n\to\infty} M_{bl}(e_n,e,t) = \lim_{n\to\infty} e^{\frac{-\max e_n, e^2}{t}} = e^{-\frac{e^2}{t}} = M_{bl}(e,e,t), \forall t > 0$$

and

$$\lim_{n\to\infty} N_{bl}(e_n,e,t)=\lim_{n\to\infty}(1-e^{\frac{-\max\{e_n,e\}^2}{t}})=1-e^{-\frac{e^2}{t}}=N_{bl}(e,e,t),\forall t>0.$$

Therefore, a sequence $\{e_n\}$ converges to all $e\in X$ with $e\geq 2$, but it is not a Cauchy sequence, since $\lim_{n\to\infty} M_{bl}(e_n,e_{n+p},t)$ and $\lim_{n\to\infty} N_{bl}(e_n,e_{n+p},t)$ do not exist.

Definition 3.7. A sequence $\{e_n\}$ in an IFBMLS $(X,M,N,\star,\circ)$ is said to be convergent to $e\in X$ if $\lim_{n\to\infty} M_{bl}(e_n,e,t)=M_{bl}(e,e,t)$ for all $t>0$ and $\lim_{n\to\infty} N_{bl}(e_n,e,t)=N_{bl}(e,e,t)$ for all $t>0$.

Definition 3.8. A sequence $\{e_n\}$ in an IFBMLS $(X,M_{bl},N_{bl},\star,\circ)$ is said to be a Cauchy sequence if $\lim_{n\to\infty} M_{bl}(e_n,e_{n+p},t)$ and $\lim_{n\to\infty} N_{bl}(e_n,e_{n+p},t)$ exist and are finite for all $t\geq 0,p\geq 1$.

Definition 3.9. An IFBMLS $(X,M_{bl},N_{bl},\star,\circ)$ is said to be complete if every Cauchy sequence $\{e_n\}$ in X converges to some $e\in X$ such that $\lim_{n\to\infty} M_{bl}(e_n,e,t)=M_{bl}(e,e,t)=\lim_{n\to\infty} M_{bl}(e_n,e_{n+p},t)$ for all $t\geq 0,p\geq 1$ and

$$\lim_{n\to\infty} N_{bl}(e_n,e,t)=N_{bl}(e,e,t)=\lim_{n\to\infty} N_{bl}(e_n,e_{n+p},t)$$

for all $t\geq 0,p\geq 1$.

Theorem 3.10. Let $(X,M_{bl},N_{bl},\star,\circ)$ be a complete IFBMLS such that $\lim_{t\to\infty} M_{bl}(e,o,t)=1$ and $\lim_{t\to\infty} N_{bl}(e,o,t)=0$ for all $e,o\in X,t>0$ and $T:X\to X$ be a mapping satisfying the conditions

$$M_{bl}(Te,To,\alpha t)\geq M_{bl}(e,o,t) \text{ and } N_{bl}(Te,To,\alpha t)\leq N_{bl}(e,o,t) \tag{3.1}$$

for all $e,o\in X,t>0$, where $\alpha\in(0,1)$. Then T has a unique fixed point $w\in X$ and $M_{bl}(w,w,t)=1,N_{bl}(w,w,t)=0$, for all $t>0$.

Proof . Let $(X,M_{bl},N_{bl},\star,\circ)$ be a complete IFBMLS. For a given element $e_0\in X$, define a sequence $\{e_n\}$ in X by

$$e_1=Te_0,e_2=T^2e_0=Te_1,\dots,e_n=T^ne_0=Te_{n-1} \text{ for all } n\in\mathbb{N}$$

If $e_n=e_{n-1}$ for some $n\in\mathbb{N}$, then e_n is a fixed point of T. We assume that $e_n\neq e_{n-1}$ for all $n\in\mathbb{N}$. For $t>0$ and $n\in\mathbb{N}$, we get from (3.1) that

$$M_{bl}(e_n,e_{n+1},t)\geq M_{bl}(e_{n+1},e_n,\alpha t)=M_{bl}(Te_n,Te_{n-1},\alpha t)\geq M(e_n,e_{n-1},t)$$

and

$$N_{bl}(e_n,e_{n+1},t)\leq N_{bl}(e_{n+1},e_n,\alpha t)=N_{bl}(Te_n,Te_{n-1},\alpha t)\leq N_{bl}(e_n,e_{n-1},t)$$

for all $n\in\mathbb{N}$ and $t>0$. Therefore, by applying the above expression, we can deduce that

$$\begin{aligned} M_{bl}(e_{n+1},e_n,t)&\geq M_{bl}(e_{n+1},e_n,\alpha t)=M_{bl}(Te_n,Te_{n-1},\alpha t)\geq M_{bl}(e_n,e_{n-1},t)\\ &=M_{bl}(Te_{n-1},Te_{n-2},t)\geq M_{bl}(e_{n-1},e_{n-2},\frac{t}{\alpha})\geq\dots\geq M_{bl}(e_1,e_0,\frac{t}{\alpha^n})\end{aligned} \tag{3.2}$$

and

$$\begin{aligned} N_{bl}(e_{n+1},e_n,t)&\leq N_{bl}(e_{n+1},e_n,\alpha t)=N_{bl}(Te_n,Te_{n-1},\alpha t)\leq N_{bl}(e_n,e_{n-1},t)\\ &=N_{bl}(Te_{n-1},Te_{n-2},t)\leq N_{bl}(e_{n-1},e_{n-2},\frac{t}{\alpha})\leq\dots\leq N_{bl}(e_1,e_0,\frac{t}{\alpha^n})\end{aligned} \tag{3.3}$$

for all $n\in\mathbb{N}$, $p\geq 1$ and $t>0$. Thus we have

$$M_{bl}(e_n,e_{n+p},t)\geq M_{bl}(e_n,e_{n+1},\frac{t}{b})\star M_{bl}(e_{n+1},e_{n+p},\frac{t}{b})$$

and

$$N_{bl}(e_n,e_{n+p},t)\leq N_{bl}(e_n,e_{n+1},\frac{t}{b})\circ N_{bl}(e_{n+1},e_{n+p},\frac{t}{b}).$$

Continuing in this way, we get

$$M_{bl}(e_n, e_{n+p}, t) \geq M_{bl}(e_n, e_{n+1}, \frac{t}{b}) \star M_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \star \cdots \star M_{bl}(e_{n+p-1}, e_{n+p}, \frac{t}{b^{p-1}})$$

and

$$N_{bl}(e_n, e_{n+p}, t) \leq N_{bl}(e_n, e_{n+1}, \frac{t}{b}) \circ N_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \circ \cdots \circ N_{bl}(e_{n+p-1}, e_{n+p}, \frac{t}{b^{p-1}}).$$

Using (3.2) and (3.3) in the above inequality, we have

$$M_{bl}(e_n, e_{n+p}, t) \geq M_{bl}(e_0, e_1, \frac{t}{b\alpha^n}) \star M_{bl}(e_0, e_1, \frac{t}{b^2\alpha^{n+1}}) \star \cdots \star M_{bl}(e_0, e_1, \frac{t}{b^{p-1}\alpha^{n+p-1}}) \tag{3.4}$$

and

$$N_{bl}(e_n, e_{n+p}, t) \leq N_{bl}(e_0, e_1, \frac{t}{b\alpha^n}) \circ N_{bl}(e_0, e_1, \frac{t}{b^2\alpha^{n+1}}) \circ \cdots \circ N_{bl}(e_0, e_1, \frac{t}{b^{p-1}\alpha^{n+p-1}}) \tag{3.5}$$

Here b is an arbitrary positive integer.

We know that $\lim_{n\to\infty} M_{bl}(e, o, t) = 1$ and $\lim_{n\to\infty} M_{bl}(e, o, t) = 1$ for all $e, o \in X$ and $t > 0, \alpha \in (0, 1)$. It follows from (3.4) and (3.5) that

$$\lim_{n\to\infty} M_{bl}(e_n, e_{n+p}, t) = 1 \star 1 \star \cdots \star 1 = 1 \text{ for all } t > 0, p \geq 1$$

and

$$\lim_{n\to\infty} N_{bl}(e_n, e_{n+p}, t) = 0 \circ 0 \circ \cdots \circ 0 = 0 \text{ for all } t > 0, p \geq 1.$$

Hence $\{e_n\}$ is a Cauchy sequence. The completeness of the IFBMLS $(X, M_{bl}, N_{bl}, \star, \circ)$ ensures that there exists $w \in X$ such that

$$\lim_{n\to\infty} M_{bl}(e_n, w, t) = \lim_{n\to\infty} M_{bl}(e_n, e_{n+p}, t) = M_{bl}(w, w, t) = 1 \text{ for all } t > 0, p \geq 1 \tag{3.6}$$

and

$$\lim_{n\to\infty} N_{bl}(e_n, w, t) = \lim_{n\to\infty} N_{bl}(e_n, e_{n+p}, t) = N_{bl}(w, w, t) = 1 \text{ for all } t > 0,\ p \geq 1 \tag{3.7}$$

Now, we show that $w \in X$ is a fixed point of T. We have

$$\begin{aligned} M_{bl}(w, Tw, t) &\geq M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star M_{bl}(e_{n+1}, Tw, \frac{t}{2b}) \\ &= M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star M_{bl}(Te_n, Tw, \frac{t}{2b}) \\ &\geq M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star M_{bl}(e_n, w, \frac{t}{2b\alpha}) \end{aligned}$$

and

$$\begin{aligned} N_{bl}(w, Tw, t) &\leq N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ N_{bl}(e_{n+1}, Tw, \frac{t}{2b}) \\ &= N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ N_{bl}(Te_n, Tw, \frac{t}{2b}) \\ &\leq N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ N_{bl}(e_n, w, \frac{t}{2b\alpha}) \end{aligned}$$

for all $t > 0$. Taking the limit as $n \to +\infty$, and by (3.6) and (3.7), we get

$$M_{bl}(w, Tw, t) = 1 \star 1 = 1$$

and

$$N_{bl}(w, Tw, t) = 0 \circ 0 = 0.$$

Therefore, w is a fixed point of T and $M_{bl}(w, w, t) = 1$ and $N_{bl}(w, w, t) = 0$ for all $t > 0$. Now, we investigate the uniqueness of fixed point. For this, assume that v and w are two fixed points of T. Then by (3.1), we have

$$M_{bl}(w, v, t) = M_{bl}(Tw, Tv, t) \geq M_{bl}(w, v, \frac{t}{\alpha})$$

and

$$N_{bl}(w,v,t) = N_{bl}(Tw,Tv,t) \leq N_{bl}(w,v,\frac{t}{\alpha})$$

for all $t > 0$. Thus we obtain

$$M_{bl}(w,v,t) \geq M_{bl}(w,v,\frac{t}{\alpha^n}) \text{ for all } n \in \mathbb{N}$$

and

$$N_{bl}(w,v,t) \geq N_{bl}(w,v,\frac{t}{\alpha^n}) \text{ for all } n \in \mathbb{N}$$

Taking the limit as $n \to +\infty$ and using $\lim_{t\to\infty} M_{bl}(e,o,t) = 1$ and $\lim_{t\to\infty} N_{bl}(e,o,t) = 0$, we get $w = v$. Hence the fixed point is unique. □

Example 3.11. Let $X = [0,1]$ and the CTN and CTCN, respectively, be defined by $g \star h = gh$ and $g \circ h = \max\{a,b\}$. Also, M_{bl} and N_{bl} are defined by

$$M_{bl}(e,o,t) = e^{\frac{-(e+o)^2}{t}}$$

and

$$N_{bl}(e,o,t) = 1 - e^{\frac{-(e+o)^2}{t}} \text{ for all } e,o \in X, t > 0.$$

Then $(X, M_{bl}, N_{bl}, \star, \circ)$ is a complete IFBMLS. Define $T : X \to X$ by

$$Te = \begin{cases} 0, & e \in \left[0, \frac{1}{2}\right], \\ \frac{e}{6}, & e \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Then

$$\lim_{t\to\infty} M_{bl}(e,o,t) = \lim_{t\to\infty} e^{\frac{-(e+o)^2}{t}} = 1 \text{ and } \lim_{t\to\infty} M_{bl}(e,o,t) = \lim_{t\to\infty} (1 - e^{\frac{-(e+o)^2}{t}}) = 0.$$

For $\alpha \in [\frac{1}{2}, 1)$, we have four cases:

Case 1) If $e,o \in [0, \frac{1}{2}]$, then $Te = To = 0$.

Case 2) If $e \in [0, \frac{1}{2}]$ and $o \in (\frac{1}{2}, 1]$, then $Te = 0$ and $To = \frac{o}{6}$.

Case 3) If $e,o \in (\frac{1}{2}, 1]$, then $Te = \frac{e}{6}$ and $To = \frac{o}{6}$.

Case 4) If $e \in (\frac{1}{2}, 1]$ and $o \in [0, \frac{1}{2}]$, then $Te = \frac{e}{6}$ and $To = 0$.

From all 4 cases, we obtain that

$$M_{bl}(Te,To,\alpha t) \geq M_{bl}(e,o,t)$$

and

$$N_{bl}(Te,To,\alpha t) \leq N_{bl}(e,o,t).$$

Hence all the conditions of Theorem 3.10 are satisfied and 0 is the unique fixed point of T. Also,

$$M_{bl}(w,w,t) = M_{bl}(0,0,t) = e^0 = 1 \text{ for all } t > 0$$

and

$$N_{bl}(w,w,t) = N_{bl}(0,0,t) = 1 - e^0 = 0 \text{ for all } t > 0.$$

Definition 3.12. Let $(X, M_{bl}, N_{bl}, \star, \circ)$ be an IFBMLS. A mapping $T : X \to X$ is said to be IFBML contractive if there exists$q \in (0,1)$ such that

$$\frac{1}{M_{bl}(Te,To,t)} - 1 \leq q[\frac{1}{M_{bl}(e,o,t)} - 1] \text{ and } N_{bl}(Te,To,t) \leq qN_{bl}(e,o,t) \tag{3.8}$$

for all $e,o \in X$ and $t > 0$. Here q is called the IFBML contractive constant of T.

Theorem 3.13. Let $(X, M_{bl}, N_{bl}, \star, \circ)$ be a complete IFBMLS and $T : X \to X$ be a IFBML contractive mapping with an IFBML contractive constant q. Then T has a unique fixed point $w \in X$ such that $M_{bl}(w,w,t) = 1$ and $N_{bl}(w,w,t) = 0$ for all $t > 0$.

Proof . Let $(X, M_{bl}, N_{bl}, \star, \circ)$ be a complete IFBMLS. For a given element $e_0 \in X$, define a sequence $\{e_n\}$ in X by

$$e_1 = Te_0, e_2 = T^2 e_0 = Te_1, \ldots, e_n = T^n e_0 = Te_{n-1} \text{ for all } n \in \mathbb{N}$$

If $e_n = e_{n-1}$ for some $n \in \mathbb{N}$, then e_n is a fixed point of T. We assume that $e_n \neq e_{n-1}$ for all $n \in \mathbb{N}$. For $t > 0$ and $n \in \mathbb{N}$, we get from (3.8) that

$$\frac{1}{M_{bl}(e_n, e_{n+1}, t)} - 1 = \frac{1}{M_{bl}(Te_{n-1}, Te_n, t)} - 1 \leq q[\frac{1}{M_{bl}(e_{n-1}, e_n, t)} - 1].$$

Then we have

$$\begin{aligned} &\frac{1}{M_{bl}(e_n, e_{n+1}, t)} \leq \frac{q}{M_{bl}(e_{n-1}, e_n, t)} + (1-q) \\ &= \frac{q}{M_{bl}(Te_{n-2}, Te_{n-1}, t)} + (1-q) \leq \frac{q^2}{M_{bl}(e_{n-2}, e_{n-1}, t)} + q(1-q) + (1-q) \end{aligned}$$

for all $t > 0$. Continuing in this way, we get

$$\begin{aligned} \frac{1}{M_{bl}(e_n, e_{n+1}, t)} &\leq \frac{q^n}{M_{bl}(e_0, e_1, t)} + q^{n-1}(1-q) + q^{n-2}(1-q) + \ldots + q(1-q) + (1-q) \\ &\leq \frac{q^n}{M_{bl}(e_0, e_1, t)} + (q^{n-1} + q^{n-2} + \ldots + 1)(1-q) \\ &\leq \frac{q^n}{M_{bl}(e_0, e_1, t)} + (1-q^n). \end{aligned}$$

Thus

$$\frac{1}{\frac{q^n}{M_{bl}(e_0, e_1, t)} + (1-q^n)} \leq M_{bl}(e_n, e_{n+1}, t) \text{ for all } t > 0, n \in \mathbb{N} \tag{3.9}$$

and

$$\begin{aligned} N_{bl}(e_n, e_{n+1}, t) &= N_{bl}(Te_{n-1}, Te_n, t) \leq qN_{bl}(e_{n-1}, e_n, t) = qN_{bl}(Te_{n-2}, Te_{n-1}, t) \\ &\leq q^2 N_{bl}(e_{n-2}, e_{n-1}, t) \leq \cdots \leq q^n N_{bl}(e_0, e_1, t). \end{aligned} \tag{3.10}$$

Now, for $p \geq 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} M_{bl}(e_n, e_{n+p}, t) &\geq M_{bl}(e_n, e_{n+1}, \frac{t}{b}) \star M_{bl}(e_{n+1}, e_{n+p}, \frac{t}{b}) \\ &\geq M_{bl}(e_n, e_{n+1}, \frac{t}{b}) \star M_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \star M_{bl}(e_{n+2}, e_{n+p}, \frac{t}{b^2}). \end{aligned}$$

Continuing in this way, we get

$$M_{bl}(e_n, e_{n+p}, t) \geq M_{bl}(e_n, e_{n+1}, \frac{t}{b}) \star M_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \star \cdots \star M_{bl}(e_{n+p-1}, e_{n+p}, \frac{t}{b^{p-1}})$$

and

$$\begin{aligned} N_{bl}(e_n, e_{n+p}, t) &\leq N_{bl}(e_n, e_{n+1}, \frac{t}{b}) \circ N_{bl}(e_{n+1}, e_{n+p}, \frac{t}{b}) \\ &\leq N_{bl}(e_n, e_{n+1}, \frac{t}{b}) \circ N_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \circ N_{bl}(e_{n+2}, e_{n+p}, \frac{t}{b^2}). \end{aligned}$$

Continuing in this way, we get

$$N_{bl}(e_n, e_{n+p}, t) \leq N_{bl}(e_n, e_{n+1}, \frac{t}{b}) \circ N_{bl}(e_{n+1}, e_{n+2}, \frac{t}{b^2}) \circ \cdots \circ N_{bl}(e_{n+p-1}, e_{n+p}, \frac{t}{b^{p-1}}).$$

Using (3.9) and (3.10) in the above inequality, we have

$$\begin{aligned} M_{bl}(e_n, e_{n+p}, t) &\geq \frac{1}{\frac{q^n}{M_{bl}(e_0,e_1,\frac{t}{b})} + (1-q^n)} \star \frac{1}{\frac{q^{n+1}}{M_{bl}(e_0,e_1,\frac{t}{b^2})} + (1-q^{n+1})} \star \cdots \star \frac{1}{\frac{q^{n+p-1}}{M_{bl}(e_0,e_1,\frac{t}{b^{p-1}})} + (1-q^{n+p-1})} \\ &\geq \frac{1}{\frac{q^n}{M_{bl}(e_0,e_1,\frac{t}{b})} + 1} \star \frac{1}{\frac{q^{n+1}}{M_{bl}(e_0,e_1,\frac{t}{b^2})} + 1} \star \cdots \star \frac{1}{\frac{q^{n+p-1}}{M_{bl}(e_0,e_1,\frac{t}{b^{p-1}})} + 1} \end{aligned}$$

and

$$N_{bl}(e_n, e_{n+p}, t) \leq q^n N_{bl}(e_0, e_1, \frac{t}{b}) \circ q^{n+1} N_{bl}(e_1, e_2, \frac{t}{b^2}) \circ \cdots \circ q^{n+p-1} N_{bl}(e_0, e_1, \frac{t}{b^{p-1}}).$$

Here b is arbitrary positive integer and $q \in (0,1)$. So we deduce from the above expression that

$$\begin{aligned} \lim_{n\to\infty} M_{bl}(e_n, e_{n+p}, t) &= 1 \text{ for all } t > 0, p \geq 1, \\ \lim_{n\to\infty} N_{bl}(e_n, e_{n+p}, t) &= 0 \text{ for all } t > 0, p \geq 1. \end{aligned}$$

So $\{e_n\}$ is a Cauchy sequence in $(X, M_{bl}, N_{bl}, \star, \circ)$. By the completeness of $(X, M_{bl}, N_{bl}, \star, \circ)$, there is $w \in X$ such that

$$\lim_{n\to\infty} M_{bl}(e_n, w, t) = \lim_{n\to\infty} M_{bl}(e_n, e_{n+p}, t) = \lim_{n\to\infty} M_{bl}(w, w, t) = 1, \tag{3.11}$$

$$\lim_{n\to\infty} N_{bl}(e_n, w, t) = \lim_{n\to\infty} N_{bl}(e_n, e_{n+p}, t) = \lim_{n\to\infty} N_{bl}(w, w, t) = 0 \tag{3.12}$$

for all $t > 0, p \geq 1$.

Now, we prove that w is a fixed point for T. For this, we obtain from (3.8) that

$$\begin{aligned} \frac{1}{M_{bl}(Te_n, Tw, t)} - 1 &\leq q[\frac{1}{M_{bl}(e_n, w, t)} - 1] = \frac{q}{M_{bl}(e_n, w, t)} - q, \\ \frac{1}{\frac{q}{M_{bl}(e_n,w,t)} + 1 - q} &\leq M_{bl}(Te_n, Tw, t). \end{aligned}$$

Using the above inequality, we obtain

$$\begin{aligned} M_{bl}(w, Tw, t) &\geq M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star M_{bl}(e_{n+1}, Tw, \frac{t}{2b}) \\ &= M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star M_{bl}(Te_n, Tw, \frac{t}{2b}) \\ &\geq M_{bl}(w, e_{n+1}, \frac{t}{2b}) \star \frac{1}{\frac{q}{M_{bl}(e_n,w,\frac{t}{2b})} + 1 - q} \end{aligned}$$

and

$$\begin{aligned} N_{bl}(w, Tw, t) &\leq N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ N_{bl}(e_{n+1}, Tw, \frac{t}{2b}) = N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ N_{bl}(Te_n, Tw, \frac{t}{2b}) \\ &\leq N_{bl}(w, e_{n+1}, \frac{t}{2b}) \circ q N_{bl}(e_n, w, \frac{t}{2b}). \end{aligned}$$

Taking the limit as $n \to \infty$ and using (11) and (12) in the above expression, we get that $M_{bl}(w, Tw, t) = 1$ and $N_{bl}(w, Tw, t) = 0$, that is, $Tw = w$. Therefore, w is a fixed point of T and $M_{bl}(w, w, t) = 1$ and $N_{bl}(w, w, t) = 0$ for all $t > 0$.

Now we show the uniqueness of the fixed point w of T. Let v be another fixed point of T such that $M_{bl}(w, v, t) \neq 1$ and $N_{bl}(w, v, t) \neq 0$ for some $t > 0$. It follows from (3.1) that

$$\frac{1}{M_{bl}(w, v, t)} - 1 = \frac{1}{M_{bl}(Tw, Tv, t)} - 1 \leq q[\frac{1}{M_{bl}(w, v, t)} - 1] < \frac{1}{M_{bl}(w, v, t)} - 1$$

and so

$$N_{bl}(w, v, t) = N_{bl}(Tw, Tv, t) \leq q N_{bl}(w, v, t) < N_{bl}(w, v, t),$$

which is a contradiction. Therefore, we have $M_{bl}(w, v, t) = 1$ and $N_{bl}(w, v, t) = 0$ for all $t > 0$, and hence $w = v$. □

Corollary 3.14. Let $(X, M_{bl}, N_{bl}, \star, \circ)$ be a complete IFBMLS and $T : X \to X$ be a mapping satisfying

$$\frac{1}{M_{bl}(T^n e, T^n o, t)} - 1 \leq q[\frac{1}{M_{bl}(e, o, t)} - 1] \text{ and } M_{bl}(T^n e, T^n o, t) \leq q M_{bl}(e, o, t)$$

for some $n \in \mathbb{N}$ and all $e, o \in X, t > 0$, where $0 < q < 1$. Then T has a unique fixed point $w \in X$ and $M_{bl}(w, w, t) = 1, N_{bl}(w, w, t) = 0$ for all $t > 0$.

Proof . Let $w \in X$ be the unique fixed point of T^n given by Theorem 3.13 and $M_{bl}(w, w, t) = 1, N_{bl}(w, w, t) = 0$ for all $t > 0$. Then Tw is also a fixed point of T^n as $T^n(Tw) = Tw$ and by Theorem 3.13, $Tw = w$ and so w is the unique fixed point, since the unique fixed point of T is also the unique fixed point of T^n. □

Example 3.15. Let $X = [0, 2]$ and the CTN and CTCN, respectively, be defined by $g \star h = gh$ and $g \circ h = \max\{g, h\}$. Consider M_{bl} and N_{bl} as

$$M_{bl}(e, o, t) = e^{\frac{-(\max\{e,o\})^2}{t}} \text{ and } N_{bl}(e, o, t) = 1 - e^{\frac{-(\max\{e,o\})^2}{t}}$$

for all $e, o \in X$ and $t > 0$. Then $(X, M_{bl}, N_{bl}, \star, \circ)$ is a complete IFBMLS. Define $T : X \to X$ as

$$Te = \begin{cases} 0, \ e = 1 \\ \frac{e}{2}, \ e \in [0, 1) \\ \frac{e}{4}, \ e \in (1, 2]. \end{cases}$$

Then we have 9 cases:

Case 1) If $e = o = 1$, then $Te = To = 0$.

Case 2) If $e = 1$ and $o \in [0, 1$, then $Te = 0$ and $To = \frac{o}{2}$.

Case 3) If $e = 1$ and $o \in (1, 2]$, then $Te = 0$ and $To = \frac{o}{4}$.

Case 4) If $e \in [0, 1)$ and $o \in (1, 2]$, then $Te = \frac{e}{2}$ and $To = \frac{o}{4}$.

Case 5) If $e \in [0, 1)$ and $o \in [0, 1)$, then $Te = \frac{e}{2}$ and $To = \frac{o}{2}$.

Case 6) If $e \in [0, 1)$ and $o = 1$, then $Te = \frac{e}{2}$ and $To = 0$.

Case 7) If $e \in (1, 2]$ and $o = 1$, then $Te = \frac{e}{4}$ and $To = 0$.

Case 8) If $e \in (1, 2]$ and $o \in (1, 2]$, then $Te = \frac{e}{4}$ and $To = \frac{o}{4}$.

All the above cases satisfy the IFBML contraction:

$$\frac{1}{M_{bl}(Te, To, t)} - 1 \leq q[\frac{1}{M_{bl}(e, o, t)} - 1] \text{ and } N_{bl}(Te, To, t) \leq q N_{bl}(e, o, t)$$

with the IFBML contractive constant $q \in \left[\frac{1}{2}, 1\right)$. Hence T is an IFBML contractive mapping with $q \in \left[\frac{1}{2}, 1\right)$. All the conditions of Theorem 3.10 are satisfied. Also, 0 is the unique fixed point of T and $M_{bl}(0, 0, t) = 1$ and $N_{bl}(0, 0, t) = 0$ for all $t > 0$.

4 Conclusion and future work

In this paper, we introduced the concept of IFBML and established fixed point theorem in order to study the unique fixed point in the space. This work is the extended form of fuzzy b-metric like space [9]. This work provided a new motivation to the researchers to the develop the area of fixed point theory in a new manner. It would a very interesting topic for future to study this kind of work in soft set or rough set models and also to apply them in multi-criteria group decision making.

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