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Existence results of quasilinear elliptic systems via Young measures

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Abstract

This paper is concerned with the following quasilinear elliptic system

$$\begin{cases} -\operatorname{div}(a(|Du|)Du) = f(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$. By means of the Young measure and the theory of Sobolev spaces, we obtain the existence of a weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Keywords: Quasilinear elliptic systems, Weak solutions, Young measures, Sobolev spaces
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1 Introduction

In this paper, we study the existence of weak solutions to the following boundary value system in the framework of Sobolev spaces

$$\begin{cases} -\operatorname{div}(a(|Du|)Du) = f(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $n \geq 2$. $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a function assumed to satisfy some assumptions (see the third section) and the function $a : (0,\infty) \to (0,\infty)$ belongs to $C^1(0,\infty)$ and satisfying

$$-1 < t_a =: \inf_{t>0} \frac{a'(t)t}{a(t)} \leq \sup_{t>0} \frac{a'(t)t}{a(t)} =: h_a < \infty. \tag{1.2}$$

Here, we denote by $\mathbb{M}^{m\times n}$ the set of $m \times n$ matrices with reduced $\mathbb{R}^{mn}$ topology, i.e., if $\gamma \in \mathbb{M}^{m\times n}$ then $|\gamma|$ is the norm of γ when regarded as a vector of $\mathbb{R}^{mn}$. We endow $\mathbb{M}^{m\times n}$ with the product

$$\gamma : \mu = \sum_{i,j} \gamma_{ij}\mu_{ij} \quad \text{for any } \gamma, \mu \in \mathbb{M}^{m\times n}.$$

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The study of partial differential equations involving the p-Laplacian generalised several types of problems not only in physics, but also in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction-diffusion system:

$$u_t = -\operatorname{div}\left(a|\nabla u|^{p-2}\nabla u\right) + f,$$

where $a \in \mathbb{R}^+$ is a positive constant, the function u generally describes the concentration, the term $\operatorname{div}\left(a|\nabla u|^{p-2}\nabla u\right)$ corresponds to the diffusion with coefficient $D(u) = a|\nabla u|^{p-2}$, and f is the reaction term related to source and loss processes. In general, the reaction term has a polynomial form with respect to the concentration u.
Because of the importance of this kind of problems, many authors have investigated the existence and uniqueness of their different types of solutions. When $a(|Du|) = |Du - \Theta(u)|^{p-2}$ and $f \in W^{-1,p'}(\Omega;\mathbb{R}^m)$, the authors studied the existence of weak solutions to problem (1.1) by using Young measures and without any Leray-Lions type growth conditions. In the case of $a(|Du|) = |Du - \Theta(u)|^{p-2}$ and f depends on u and Du, the existence of weak solutions has been established in [5] under some conditions on the function $f(x, u, Du)$. Cianchi and Maz'ya in their papers [8, 9] proved global Lipschitz regularity and obtained a sharp estimate for the decreasing of the length of the gradient for Dirichlet and Neumann problems associated to $-\operatorname{div}\left(|Du|^{p-2}Du\right) = f$ in Ω. For more topics, the reader can see [15, 17] and references therein. In [12], Hungerbühler proved some existence result by using the tool of Young measures and weak monotonicity over σ for the following quasilinear elliptic system

$$-\operatorname{div} \sigma(x,\ u,\ Du) = f \quad in\ \Omega. \tag{1.3}$$

E. Azroul F. Balaadich in [3], treated the following elliptic problem

$$\begin{cases} -\operatorname{div}(a(|Du|)Du) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

they proved the existence of weak solutions (i.e., in the sense of Definition 3.1) under some conditions on the function $a : (0, \infty) \to (0, \infty)$.
A large number of papers were devoted to the study of the existence of solutions of the elliptic problem of the type (1.3) under classical monotone methods developed by [7, 13, 14, 18].

The main goal of the paper is to prove the existence of a weak solution for a class of quasilinear elliptic types, we extend the result of established in [4] by considering a general source term by using the Galerkin's method to construct the approximating solutions and the theory of Young measures to identify weak limits when passing to the limit.

This paper is organized as follow. In Sec. 2 we introduce the basic assumptions and we recall some definitions, basic properties of Sobolev spaces we summarize some useful properties about the tool of Young measures. In Sec. 3 we establish some general convergence results for functions a obtained from the constructed approximating solutions by the Galerkin scheme, and we get the weak solutions by passing to the limit.

2 Preliminaries

In this section, we recall the properties of Lebesgue and Sobolev spaces which shall be used in the sequel. Let Ω be a bounded open domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$.

2.1 Lebesgue and Sobolev spaces

We define the Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p dx < \infty \right\},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

We denote by $L^{p'}(\Omega)$ the dual space of $L^p(\Omega)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1$$

The classical Sobolev space is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \text{ and } |\nabla u| \in L^p(\Omega)\},$$

with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p \quad \forall u \in W^{1,p}(\Omega).$$

For $1 < p < \infty$, $W^{1,p}(\Omega)$ is a reflexive Banach space. The space $W_0^{1,p}(\Omega)$ is well defined as the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm $\|u\|_{1,p}$. We can identify the dual of $W_0^{1,p}(\Omega)$ to a subspace of the space of distributions $\mathcal{D}'(\Omega)$ by:

$$W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)' \quad \left(p' = \frac{p}{p-1}\right).$$

In the manipulation of Sobolev spaces, very often one uses certain so-called Sobolev injections. We recall one of these injections given by the Rellich-Kondrachov theorem.

Proposition 2.1. Assume Ω of class $\mathcal{C}^\infty$ and $p < N$. Then

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \forall q \in [1, p^*[\text{ with } p^* = \frac{Np}{N-p}.$$

In particular, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for all $p \in [1, +\infty)$.

In the sequel, the Hölder inequality and the following Poincaré inequality (see [12], Lemma 2.2]); there exists a positive constant α such that

$$\|u\|_p \leq \alpha \|Du\|_p, \quad \forall u \in W_0^{1,p}(\Omega; \mathbb{R}^m) \tag{2.1}$$

are central to establish the required estimates to prove the desired results.

2.2 A review on Young measures

As mentioned before, we will use the tool of Young measure to prove our main result. In the following, we briefly summarize some useful properties needed in the sequel.

In the following, $C_0(\mathbb{R}^m)$ denotes the set of functions $\varphi \in C(\mathbb{R}^m)$ satisfying $\lim_{|\lambda|\to\infty} \varphi(\lambda) = 0$. We know that $(C_0(\mathbb{R}^m))' = \mathcal{M}(\mathbb{R}^m)$ and the duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ by

$$\langle \nu, \ \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

If $\varphi(\lambda) = \lambda$, note that $\langle \nu, \ \lambda \rangle = \int_{\mathbb{R}^m} \lambda d\nu(\lambda)$.

Lemma 2.2. ([11]). Let $(z_k)_k$ *be a bounded sequence in* $L^\infty(\Omega; \mathbb{R}^m)$. Then there exists a subsequence (denoted again by (z_k)) and a Borel probability measure ν_x on $\mathbb{R}^m$ for *a.e.* $x \in \Omega$, such that for each $\varphi \in C_0(\mathbb{R}^m)$ we have, $\varphi(z_k) \rightarrow^* \overline{\varphi}$ weakly in $L^\infty(\Omega; \mathbb{R}^m)$, where $\overline{\varphi}(x) = \langle \nu_x, \ \varphi \rangle$ for *a.e.* $x \in \Omega$.

Definition 2.3. We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated to (z_k) . In [6], it is shown that if for all $R > 0$

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R(0) \ : \ |z_k(x)| \geq L\}| = 0,$$

then the Young measure ν_x generated by z_k is a probability measure, i.e., $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$. The following properties build the basic tools used in the sequel. If $|\Omega| < \infty$, then there holds

$$z_k \to z \textit{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \textit{ for a.e. } x \in \Omega. \tag{2.2}$$

If we choose $z_k = Dw_k$ for $w_k : \Omega \to \mathbb{R}^m$, the above results remain valid.

Lemma 2.4. [1]. Assume that Dw_k is bounded in $L^p(\Omega; \mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dw_k satisfies:

1. ν_x is a probability measure.
2. The weak L^1- limit of Dw_k is given by $\langle \nu_x,\ id\rangle$.
3. The identification $\langle \nu_x,\ id\rangle = Dw(x)$ holds for $a.e.\ x \in \Omega$.

We conclude this section by recalling the following Fatou-type inequality.

Lemma 2.5. [10]. Let $\varphi : \mathbb{M}^{m\times n} \to \mathbb{R}$ be a continuous function and $w_k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that Dw_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$. Then

$$\liminf_{k\to\ \infty} \int_\Omega \varphi(Dw_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \varphi(\lambda)d\nu_x(\lambda)dx$$

provided that the negative part of $\varphi(Dw_k)$ is equiintegrable.

3 Main result

In this section we give the notion of a weak solution for the quasilinear elliptic system (1.1) and we state the main result of this paper. Firstly, we suppose the following assumptions

(A_0) There exists $\alpha_1 > 0$ such that

$$|a(|\xi|)\xi| \leq \alpha_1|\xi|^{p-1},$$

(A_1) There exists $\alpha_0 > 0$ and $d_1 \in L^1(\Omega)$ such that

$$a(|\xi|)\xi : \xi \geq \alpha_0|\xi|^p - d_1(x), \quad \forall \xi \in \mathbb{M}^{m\times n},$$

(A_2) The function a satisfies one of the following conditions:

(i) There exists a convex and C^1-function $b : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$a(|\xi|)\xi = \frac{\partial b(\xi)}{\partial \xi} := D_\xi b(\xi)\ .$$

(ii) a is striclty p-quasimonotone, i.e.,

$$\int_{\mathbb{M}^{m\times n}} (a(|\lambda|)\lambda - a(|\overline{\lambda}|)\overline{\lambda}) : (\lambda - \overline{\lambda})d\nu_x(\lambda) > 0$$

for $\overline{\lambda} = \langle \nu_x,\ id\rangle$, where $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L^p(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

(iii) a is striclty quasimonotone, i.e. there exist constants $c > 0$ and $r > 0$ such that

$$\int_\Omega (a(|Du|)Du - a(|Dv|)Dv) : (Du - Dv)dx \geqslant c\int_\Omega |Du - Dv|^r dx$$

for all $u, v \in W_0^{1,p}(\Omega)$.

(A_3) $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ is a Carathéodory function (i.e. $x \mapsto f(x, s, \xi)$ is measurable for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(s, \xi) \mapsto f(x, s, \xi)$ is continuous for almost every $x \in \Omega$.
Moreover, we assume that one of the following two additional conditions hold:

a) For constants $0 < \gamma < p - 1, 0 \leq \mu < p - 1$ and a function $d \in L^{p'}(\Omega)$ there holds

$$|f(x, s, \xi)| \leq d(x) + |s|^\gamma + |\xi|^\mu$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

b) In addition to (a), the function f is independent of the third variable, or, for almost $x \in \Omega$ and all $u \in \mathbb{R}^m$, the mapping $\xi \mapsto f(x, s, \xi)$ is linear.

A particular case when $a(t) = t^{p-2}$ is reduced to a p-Laplacian system

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = d(x) + |u|^{\gamma} + |Du|^{\mu} \quad \text{in } \Omega, \tag{3.1}$$

where $d \in L^{p'}(\Omega)$, $0 < \gamma < p-1$ and $0 \leq \mu < p-1$.
As usual, the solutions of (1.1) are taken in a weak meaning.

Definition 3.1. A function $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ is said to be a weak solution of (1.1) if

$$\int_\Omega a(|Du|)Du : D\varphi dx = \int_\Omega f(x,u,Du)\cdot\varphi dx,$$

holds for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Adapting the methods used in [2] and [4], we will prove the following main theorem for system (1.1).

Theorem 3.2. Assume that (1.2), $(A_0)-(A_3)$ hold. Then the Dirichlet problem (1.1) has a weak solution in the sense of Definition 3.1.

The proof of Theorem 3.2 is divided into several steps.
Step 1: Convergence results for a

We present a general result for functions a in this step (see $(H_0)-(H_2)$ below), which will be proved in Step 2. As a matter of fact, an elliptic div-curl inequality (see Lemma 3.3) is the key ingredient to prove that one can pass to the limit in the approximating equations. The following assumptions are assumed.

(H_0) The sequence (u_k) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ for some $p > 1$, thus a subsequence converges weakly in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ to an element denoted by u.

(H_1) The sequence $a_k(x) := a(|Du_k|)Du_k$ is uniformly bounded in $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and hence equiintegrable.

(H_2) The sequence $(a_k(x) : Du_k)^-$ is equiintegrable. Moreover, there exists a sequence (v_k) such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and

$$\int_\Omega a_k(x) : (Du_k - Dv_k)dx \to 0 \ as\ k \to \infty.$$

By (H_0) and Lemma 2.2, it follows the existence of a Young measure ν_x generated by Du_k in $L^p(\Omega;\mathbb{M}^{m\times n})$. Moreover, ν_x satisfies the properties of Lemma 2.4. Now, we can state

and prove the following div-curl inequality.

Lemma 3.3. Assume that $(H_0)-(H_2)$ hold. Then ν_x satisfies

$$\int_\Omega\int_{\mathbb{M}^{m\times n}} (a(|\lambda|)\lambda - a(|Du|)Du) : (\lambda - Du)d\nu_x(\lambda)dx \leq 0.$$

Proof . Consider the sequence

$$\begin{aligned} A_k &:= (a(|Du_k|)Du_k - a(|Du|)Du) \ : \ (Du_k - Du) \\ &= a(|Du_k|)Du_k : (Du_k - Du) - a(|Du|)Du : (Du_k - Du) =: A_{k,1} + A_{k,2}. \end{aligned}$$

On the one hand, by the growth condition in (A_0) we get

$$\int_\Omega |a(|Du|)Du|^{p'}dx \leq c\int_\Omega |Du|^p dx < \infty \tag{3.2}$$

for arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, thus $a(|Du|)Du \in L^{p'}(\Omega;\mathbb{M}^{m\times n})$. According to a weak convergence described in Lemma2.4, it follows that

$$\liminf_{k\to\infty}\int_\Omega A_{k,2}dx = \int_\Omega a(|Du|)Du \ : \ (\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du)dx = 0. \tag{3.3}$$

On the other hand, the growth condition in (A_0) implies that $(a(|Du_k|)Du_k : Du)$ is equi-integrable. This together with (H_2) $(a_k(x) : (Du_k - Du))$ is equiintegrable, and by virtue of Lemma 2.5, we get

$$A := \liminf_{k\to\ \infty} \int_\Omega A_k dx = \liminf_{k\to\ \infty} \int_\Omega A_{k,1} dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : (\lambda - Du) d\nu_x(\lambda) dx.$$

It is sufficient then to show that $A \leq 0$. By virtue of the second part of (H_2), we can write

$$\begin{aligned} A &= \liminf_{k\to\ \infty} \int_\Omega a_k(x) : (Du_k - Du)dx \\ &= \liminf_{k\to\ \infty} (\int_\Omega a_k(x)\ :\ (Du_k - Dv_k)dx + \int_\Omega a_k(x)\ :\ (Dv_k - Du)dx) \\ &= \liminf_{k\to\ \infty} \int_\Omega a_k(x) : (Dv_k - Du)dx \\ &\leq \liminf_{k\to\ \infty} \||a_k|\|_{p'}\ \|v_k - u\|_{1,p} = 0 \text{ (by Hölder's inequality).} \end{aligned}$$

This together with (3.3), our inequality follows. $\square$

Remark 3.4. An intermediary result is the following inequality:

$$\liminf_{k\to\infty} \int_\Omega \big(a(|Du_k|)Du_k - a(|Du|)Du\big) : (Du_k - Du)dx \leq 0.$$

To see this, the equiintegrability of the sequence $a(|Du_k|)Du_k : (Du_k - Du)$ (by (H_2) and (A_0)) together with (3.3) imply

$$\begin{aligned} &\liminf_{k\to\infty} \int_\Omega \big(a(|Du_k|)Du_k - a(|Du|)Du\big) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : (Du_k - Du)dx \\ &\leq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : (\lambda - Du) d\nu_x(\lambda) dx \leq 0, \end{aligned}$$

by Lemmas 2.5 and 3.3.

The proof of the following lemma can be found in [2], but for completeness of the paper, we give its proof.

Lemma 3.5. Assume that (1.2) holds, then a is monotone, i. e.,

$$(a(|\xi|)\xi - a(|\eta|)\eta) : (\xi - \eta) \geq 0\ \forall \xi,\ \eta \in \mathbb{M}^{m\times n}.$$

Proof . Let $t \in [0, 1]$ and take $\theta_t = t\xi + (1 - t)\eta$ for all $\xi,\ \eta \in \mathbb{M}^{m\times n}$. We have

$$\begin{aligned} (a(|\xi|)\xi - a(|\eta|)\eta) : (\xi - \eta) &= (\int_0^1 \frac{d}{dt}[a(|\theta_t|)\theta_t]dt) : (\xi - \eta) \\ &= (\int_0^1 [a'(|\theta_t|)|\theta_t| + a(|\theta_t|)]dx) : (\xi - \eta)^2 \\ &= (\int_0^1 a(|\theta_t|)[\frac{a'(|\theta_t|)|\theta_t|}{a(|\theta_t|)} + 1]dx) : (\xi - \eta)^2 \geq 0 \end{aligned}$$

by the equation (1.2). Thus a is monotone. $\square$

From Lemma 3.3 and 3.5 we can derive the following property:

$$(a(|\lambda|)\lambda - a(|Du|)Du)\ :\ (\lambda - Du) = 0 \ on\ \mathrm{supp}\nu_x. \tag{3.4}$$

We have the following convergence results for a_k.

Proposition 3.6. Assume that $(H_0)-(H_2)$ hold. Then (up to a subsequence) the sequence a_k converges weakly in $L^1(\Omega;\mathbb{M}^{m\times n})$ as $k\to\infty$ and the weak limit $\overline{a}$ is given by $\overline{a}\equiv a(|Du|)Du$.

Proof . Consider first the condition (A_2)(i). Let show that for a.e. $x\in\Omega$, the support of ν_x is in the set where b agrees with the supporting hyper-plane $L\equiv\{(\lambda,\ b(Du)+a(|Du|)Du:(\lambda-Du))\}$, i.e., we want to show that

$$\text{supp}\nu_x\subset\{\lambda\in\mathbb{M}^{m\times n}:b(\lambda)=b(Du)+a(|Du|)Du:(\lambda-Du)\}=:K_x.$$

Let $\lambda\in\text{supp}\nu_x$, then by (3.4)

$$(1-t)(a(|\lambda|)\lambda-a(|Du|)Du)\ :\ (\lambda-Du)=0\ \forall t\in[0,1]. \tag{3.5}$$

According to Lemma 3.5 and (3.5), for all $t\in[0,1]$

$$\begin{aligned}&0\le(1-t)(a(|\lambda|)\lambda-a(|Du+t(\lambda-Du)|)(Du+t(\lambda-Du))):(\lambda-Du)\\&=(1-t)(a(|Du|)Du-a(|Du+t(\lambda-Du)|)(Du+t(\lambda-Du))):(\lambda-Du).\end{aligned} \tag{3.6}$$

By monotonicity, we can write

$$(a(|Du|)Du-a(|Du+t(\lambda-Du)|)(Du+t(\lambda-Du)))\ :\ t(Du-\lambda)\ge0$$

which implies, since $t\in[0,1]$, that

$$(a(|Du|)Du-a(|Du+t(\lambda-Du)|)(Du+t(\lambda-Du))):(1-t)(Du-\lambda)\ge0. \tag{3.7}$$

From (3.6) and (3.7) we deduce

$$(a(|Du|)Du-a(|Du+t(\lambda-Du)|)(Du+t(\lambda-Du)))\ :\ (\lambda-Du)=0$$

for $t\in[0,1]$. It results from the above equation and $a(|\xi|)\xi=(\partial b(\xi))/(\partial\xi)$ that

$$\begin{aligned}b(\lambda)&=b(Du)+\int_0^1a(|Du|)Du\ :\ (\lambda-Du)dt\\&=b(Du)+a(|Du|)Du\ :\ (\lambda-Du)\ .\end{aligned}$$

Therefore $\lambda\in K_x$. By virtue of the convexity of b we can write

$$b(\lambda)\ge b(Du)+a(|Du|)Du\ :\ (\lambda-Du)=:\sigma(\lambda)\ ,\ \forall\lambda\in\mathbb{M}^{m\times n}.$$

Since b is a C^1-function, it follows for $t\in\mathbb{R}$ that

$$\frac{b(\lambda+t\xi)-b(\lambda)}{t}\ge\frac{\sigma(\lambda+t\xi)-\sigma(\lambda)}{t}\ if\ t>0$$

and

$$\frac{b(\lambda+t\xi)-b(\lambda)}{t}\le\frac{\sigma(\lambda+t\xi)-\sigma(\lambda)}{t}\ if\ t<0,$$

where $\xi\in\mathbb{M}^{m\times n}$ Consequently $D_\lambda b=D_\lambda\sigma$, i.e.,

$$\sigma(|\lambda|)\lambda=a(|Du|)Du\ for\ all\ \lambda\in K_x\supset\text{supp}\nu_x. \tag{3.8}$$

Therefore

$$\begin{aligned}\overline{a}(x)&=\int_{\mathbb{M}^{m\times n}}a(|\lambda|)\lambda d\nu_x(\lambda)=\int_{\text{sup p}\nu_x}a(|\lambda|)\lambda d\nu_x(\lambda)\\&=\int_{\text{sup p}\nu_x}a(|Du|)Dud\nu_x(\lambda)\\&=a(|Du|)Du.\end{aligned} \tag{3.9}$$

In the last equality, we have used (3.8) and $\|\nu_x\|_{\mathcal{M}} = 1$. The continuous function

$$\Phi(\xi) = |a(|\xi|)\xi - \overline{a}(x)|, \ \forall \xi \in \mathbb{M}^{m\times n}$$

is equiintegrable by that of $(a(|Du_k|)Du_k)$. For simplicity we denote $\Phi_k(x) \equiv \Phi(Du_k)$, and its weak L^1-limit is given by

$$\overline{\Phi}(x) = \int_{\mathbb{M}^{m\times n}} \Phi(\lambda)d\nu_x(\lambda) = \int_{\text{sup p}\nu_x} |a(|\lambda|)\lambda - \overline{a}(x)|d\nu_x(\lambda) = 0$$

by (3.8) and (3.9) . As a matter of fact, the above convergence is strong in $L^1(\Omega; \mathbb{M}^{m\times n})$, since $\Phi_k \geq 0$. Hence Proposition 3.6 follows in this case.

Now, consider the case (A_2)(ii) and suppose that ν_x is not a Dirac measure on a set $x \in \Omega'$ of positive measure $|\Omega'| > \infty$. Since $\overline{\lambda} = \langle \nu_x, \ id\rangle = Du(x)$ for a.e. $x \in \Omega$, we remark first that

$$\begin{aligned} &\int_{\mathbb{M}^{m\times n}} a(|\overline{\lambda}|)\overline{\lambda} : (\lambda - \overline{\lambda})d\nu_x(\lambda) \\ &= a(|\overline{\lambda}|)\overline{\lambda} : \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda} \int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) = 0. \end{aligned}$$

Therefore, the strict p-quasimonotone implies

$$\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} d\nu_x(\lambda) \ .$$

After integrating the above inequality over Ω, we obtain by Lemma 3.3 the following contradiction:

$$\begin{aligned} \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) &> \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} d\nu_x(\lambda) \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) \ . \end{aligned}$$

Hence ν_x is a Dirac measure, i.e., $\nu_x = \delta_{g(x)}$ for a.e. $x \in \Omega$. We have

$$g(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{g(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = Du(x) \ \text{ for a.e. } x \in \Omega.$$

Therefore $\nu_x = \delta_{Du(x)}$. By virtue of (2.2) , $Du_k \to Du$ in measure and almost everywhere (up to a subsequence). By the continuity of function a, $a(|Du_k|)Du_k \to a(|Du|)Du$ almost everywhere. Since, by assumption (H_1), $a_k(x)$ is equiintegrable, it follows from the Vitali convergence theorem that

$$a(|Du_k|)Du_k \to a(|Du|)Du \quad \text{in} \quad L^1(\Omega; \mathrm{M}^{m\times n}) \ ,$$

and Proposition 3.6 follows also in this case.

For the remaining case (A_2)(iii) it follows from the strictly quasimonotone and Remark 3.4 that

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^r dx = 0.$$

Therefore $Du_k \to Du$ in measure and almost everywhere as $k \to \infty$ (for a subsequence). We follow then the proof of the previous case and the proof is complete. □

Remark 3.7. If $(A_2)(i)$ or $(A_2)(ii)$ holds, then

$$a_k(x) \to \overline{a}(x) \ in \ L^1(\Omega; \mathbb{M}^{m\times n}) \ .$$

In cases $(A_2)(ii)$ and $(A_2)(iii)$, we have in addition (up to a subsequence) $Du_k \to Du$ in measure and a.e. in Ω.

Step 2: Existence of weak solution

In this step we use the well-known Galerkin method to construct approximating solutions. After that, we will verify the conditions $(H_0) - (H_2)$ of the previous step, for the constructed solutions. To this purpose, consider the mapping $T : W_0^{1,p}(\Omega; \mathbb{R}^m) \to W^{-1,p'}(\Omega; \mathbb{R}^m)$ given for arbitrary $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ by

$$\langle T(u), \varphi\rangle = \int_\Omega a(|Du|)Du : D\varphi dx - \int_\Omega f(x, u, Du) \cdot \varphi dx.$$

Lemma 3.8. $T(u)$ is well defined, linear and bounded.

Proof . For arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear and (without loss of generality, we may assume that $\gamma = p-1 = \mu$) and to according to (3.2),

$$\begin{aligned}
&|\langle T(u),\ \varphi\rangle| = |\int_\Omega a(|Du|)Du : D\varphi dx - \int_\Omega f(x,u,Du)\cdot\varphi dx| \\
&\leq \int_\Omega |a(|Du|)Du||D\varphi|dx + \int_\Omega |f(x,u,Du)||\varphi|dx \\
&\leq \Big(\int_\Omega |a(|Du|)Du|^{p'}dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega |D\varphi|^p dx\Big)^{\frac{1}{p}} + \big(\|d\|_{p'} + \|u\|_p^{p-1} + \|Du\|_p^{p-1}\big)\|\varphi\|_p \\
&\leq c(\int_\Omega |Du|^p dx)^{\frac{1}{p'}}\|D\varphi\|_p + \big(\|d\|_{p'} + \|u\|_p^{p-1} + \|Du\|_p^{p-1}\big)\|\varphi\|_p \\
&\leq (c\|Du\|_p^{p-1} + \|d\|_{p'} + \|u\|_p^{p-1} + \|Du\|_p^{p-1})\|\varphi\|_{1,p} \\
&\leq C\|D\varphi\|_p.
\end{aligned}$$

Where we have used the Poincaré inequality and the following inequality

$$|a+b|^p \leq 2^{p-1}\left(|a|^p + |b|^p\right) \quad \text{for} \quad p>1. \tag{3.10}$$

Thus $T(u)$ is is well defined and bounded. □

Lemma 3.9. The restriction of T to a finite linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.

Proof . Let X be a finite subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ with dim $X = r$ and $(w_i)_{i=1,\cdots,r}$ a basis of W. Let $\left(u_k = a_k^i w_i\right)$ be a sequence in X which converges to $u = a^i w_i$ in X (with conventional summation). Then $u_k \to u$ and $Du_k \to Du$ almost everywhere for a subsequence still denoted by (u_k). On the one hand, the continuity of a and f implies that

$$a\left(|Du_k|\right)Du_k \to a\left(|Du|\right)Du$$

and

$$f\left(x,u_k,Du_k\right) \to f(x,u,Du)$$

almost everywhere. On the other hand, since $u_k \to u$ strongly in X

$$\int_\Omega |u_k - u|^p\,dx \to 0 \quad \text{and} \quad \int_\Omega |Du_k - Du|^p\,dx \to 0.$$

Thus, there exists a subsequence of (u_k) still denoted by (u_k) and $g_1, g_2 \in L^1(\Omega)$ such that $|u_k - u|^p \leq g_1$ and $|Du_k - Du|^p \leq g_2$. According to (3.10), we get

$$\begin{aligned}
|u_k|^p = |u_k - u + u|^p &\leq 2^{p-1}\left(|u_k-u|^p + |u|^p\right) \\
&\leq 2^{p-1}\left(g_1 + |u|^p\right).
\end{aligned}$$

Similarly

$$|Du_k|^p \leq 2^{p-1}\left(g_2 + |Du|^p\right).$$

Consequently, $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded by a constant C. Now, in order to apply the Vitali Theorem, we show that the sequences $\left(a\left(|Du_k|\right)Du_k : D\varphi\right)$ and $\left(f\left(x,u_k,Du_k\right)\cdot\varphi\right)$ are equi-integrable. To do this, we take $\Omega' \subset \Omega$ to be measurable, we have (without loss of generality, we may assume that $\gamma = p-1 = \mu$)

$$\int_{\Omega'} \left|a\left(|Du_k|\right)Du_k : D\varphi\right|dx \leq C\|Du\|_p^{p-1}\left(\int_{\Omega'} |D\varphi|^p dx\right)^{\frac{1}{p}}$$

and

$$\int_{\Omega'} \left|f\left(x,u_k,Du_k\right)\cdot\varphi\right|dx \leq \alpha(\|d\|_{p'} + \underbrace{\|u_k\|_p^{p-1}}_{\leq C} + \underbrace{\|Du_k\|_p^{p-1}}_{\leq C})\left(\int_{\Omega'} |D\varphi|^p dx\right)^{\frac{1}{p}}.$$

Since $\int_{\Omega'} |D\varphi|^p dx$ is arbitrary small if the measure of Ω' is chosen small enough, then the equiintegrability of $\left(a\left(|Du_k|\right)Du_k : D\varphi\right)$ and $\left(f\left(x,u_k,Du_k\right)\cdot\varphi\right)$ follows. By virtue of the Vitali Theorem, T is continuous. □

Lemma 3.10. The operator T defined above is coercive.

Proof . By taking $\varphi = u$ in the definition of T, we have

$$\langle T(u), u\rangle = \int_\Omega a(|Du|)Du : Dudx - \int_\Omega f(x, u, Du) \cdot udx. \tag{3.11}$$

To prove the coercivity of T we argue as follows:
By the Hölder inequality, (2.1) and the condition (A_3)(a), we get

$$\begin{aligned} \left|\int_\Omega f(x, u, Du) \cdot udx\right| &\leq \int_\Omega d(x)|u|dx + \int_\Omega |u|^\gamma |u|dx + \int_\Omega |Du|^\mu |u|dx \\ &\leq \|d\|_{p'}\|u\|_p + \|u\|^\gamma_{\gamma p'}\|u\|_p + \|Du\|^\mu_{\mu p'}\|u\|_p \\ &\leq \alpha\|d\|_{p'}\|Du\|_p + (\alpha)^{\gamma+1}\|Du\|^{\gamma+1}_p + \alpha\|Du\|^{\mu+1}_p. \end{aligned}$$

Consequently, owing to (3.11) and (2.1), we obtain

$$\begin{aligned} \langle T(u)\ ,\ u\rangle &= \int_\Omega a(|Du|)Du : Dudx - \int_\Omega f(x, u, Du) \cdot udx \\ &\geq \int_\Omega (\alpha_0|Du|^p - d_1(x))dx - \alpha\|d\|_{p'}\|Du\|_p - (\alpha)^{\gamma+1}\|Du\|^{\gamma+1}_p - \alpha\|Du\|^{\mu+1}_p \end{aligned}$$

From the above estimation it follows that

$$\langle T(u), u\rangle \to \infty \quad \text{as} \quad \|u\|_{1,p} \to \infty$$

since $p > \max\{1, \gamma + 1, \mu + 1\}$. □

In what follows, let us fix some k and assume that X_k has the dimension r and $e_1, \ \ldots, \ e_r$ is a basis of X_k. We define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r$$
$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(\beta^i e_i), e_1\rangle \\ \langle T(\beta^i e_i), e_2\rangle \\ \vdots \\ \langle T(\beta^i e_i), e_r\rangle \end{pmatrix}$$

Lemma 3.11. *G is continuous and $G(\beta).\beta \to \infty$ as $\|\beta\|_{\mathbb{R}^r} \to \infty$, where $\beta = (\beta^1, \ \ldots, \ \beta^r)^t$ and the dot is the inner product of two vectors of $\mathbb{R}^r$.*

Proof . Let $u_j = \beta^j_i e_i \in X_k, \ u_0 = \beta^0_i e_i \in X_k$. Then $\|\beta^j\|_{\mathbb{R}^r}$ is equivalent to $\|u_j\|_{1,p}$ and $\|\beta^0\|_{\mathbb{R}^r}$ is equivalent to $\|u_0\|_{1,p}$ and

$$G(\beta) \cdot \beta = \langle T(u), u\rangle.$$

Lemma3.10 gives $G(\beta).\beta \to \infty$ when $\|\beta\|_{\mathbb{R}^r} \to \infty$. □

Lemma 3.12. For all $k \in \mathbb{N}$ there exists $u_k \in X_k$ such that

$$\langle T(u_k)\ ,\ \varphi\rangle = 0 \ for \ all \ \varphi \in X_k. \tag{3.12}$$

and there is a constant $R > 0$ such that

$$\|u_k\|_{1,p} \leq R \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.13}$$

Proof . From Lemma 3.11, it follows the existence of a constant $R > 0$ such that for any $\beta \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(\beta).\beta > 0$ and the topological argument [16] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Therefore, for each $k \in \mathbb{N}$ there exists $u_k \in X_k$ such that (3.12) holds. □

Before we pass to the limit and so to prove Theorem3.2, we verify first that the conditions $(H_0) - (H_2)$ hold for the Galerkin approximations solutions u_k constructed above. As in the proof of the Lemma 3.11, we have $\langle T(u)\ ,$

$u\rangle \to \infty$ as $\|u\|_{1,p} \to \infty$.
Hence, there exists $R > 0$ with the property, that $\langle T(u)\ ,\ u\rangle > 1$ whenever $\|u\|_{1,p} > R$. Consequently, for the sequence of Galerkin approximations $u_k \in X_k$ which satisfy (3.12) with φ replaced by u_k, we get that (u_k) is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Thus, (H_0) holds.
To verify (H_1) we use the growth condition in (A_0)

$$\int_\Omega |a_k(x)|^{p'}dx = \int_\Omega |a(|Du_k|)Du_k|^{p'}dx \le c\int_\Omega |Du_k|^p < \infty$$

by the boundedness of (u_k) in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Hence $a_k(x)$ is uniformly bounded.
The first part of (H_2) can be deduced easily by the coercivity condition in (A_1) . Indeed, for any measurable subset Ω' of Ω

$$\int_{\Omega'} |\min(a_k(x)\ :\ Du_k, 0)|dx \le \int_{\Omega'} |d_1(x)|dx < \infty.$$

For the second part of (H_2), we choose a subsequence v_k which belongs to the same finite dimensional space X_k as u_k such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$.
By testing the equation (3.12) with $u_k - v_k$ as a test function and using the Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega a(|Du_k|)Du_k\ :\ (Du_k - Dv_k)dx &= \int_\Omega f(x,u_k,Du_k)(u_k - v_k)dx\\ &\le \|f(x,u_k,Du_k)\|_{p'}\|u_k - v_k\|_p\\ &\le C\|u_k - v_k\|_p.\end{aligned}$$

By the choice of (v_k) , the right hand side of the above equality converges to zero since

$$u_k - v_k \to 0 \quad \text{in}\ \ W_0^{1,p}(\Omega;\mathbb{R}^m).$$

Hence the second part of (H_2) follows.
Now, we have all ingredients to pass to the limit and so to prove Theorem 3.2. Since the constructed approximating solutions (u_k) satisfy $(H_0)-(H_2)$, it follows by Proposition 3.6 that

$$\lim_{k\to\infty}\int_\Omega a\left(|Du_k|\right)Du_k : D\varphi dx = \int_\Omega a\left(|Du|\right)Du : D\varphi dx \quad \forall \varphi \in \cup_{k\ge 1}X_k.$$

Now, we consider $E_{k,\epsilon} = \{x \in \Omega : |u_k(x) - u(x)| \ge \epsilon\}$ for ϵ positive. By (3.13), we have (for a subsequence) $u_k \to u$ in $L^p\left(\Omega;\mathbb{R}^m\right)$.
Since

$$\int_\Omega |u_k - u|^p\, dx \ge \int_{E_{k,\epsilon}} |u_k - u|^p\, dx \ge \epsilon^p\, |E_{k,\epsilon}|$$

it follows that

$$|E_{k,\epsilon}| \le \frac{1}{\epsilon^p}\int_\Omega |u_k - u|^p\, dx \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $u_k \to u$ in measure for $k \to \infty$, and we may infer that, after extraction of a suitable subsequence, if necessary

$$u_k \to u \quad \text{almost everywhere for} \quad k \to \infty.$$

To pass to the limit on the source term, we need the convergence almost everywhere of Du_k. Similarly to $E_{k,\epsilon}$, we consider $F_{k,\epsilon} = \{x \in \Omega : |Du_k(x) - Du(x)| \ge \epsilon\}$, then $Du_k \to Du$ in measure for $k \to \infty$.
Thus (for a subsequence),

$$Du_k \to Du \quad \text{almost everywhere for}\ \ k \to \infty.$$

The continuity of f permit to deduce that

$$f\left(x, u_k, Du_k\right)\cdot \varphi \to f(x,u,Du)\cdot \varphi$$

for arbitrary $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. By using the growth condition on f, we have $(f(x,u_k,Du_k)\cdot\varphi(x))$ is equiintegrable (see Lemma 3.9 if necessary), thus $f(x,u_k,Du_k)\cdot\varphi(x) \to f(x,u,Du)\cdot\varphi(x)$ in $L^1(\Omega)$ by the Vitali Convergence Theorem. This implies

$$\lim_{k\to\infty}\int_\Omega f(x,u_k,Du_k)\cdot\varphi(x)dx = \int_\Omega f(x,u,Du)\cdot\varphi(x)dx, \quad \forall\varphi\in\bigcup_{k\geq 1}X_k.$$

In the case where f is independent of the third variable, we easily verify that

$$f(x,u_k)\to f(x,u) \quad \text{in} \quad L^{p'}(\Omega)$$

In the other situation, we have that, for a.e. $x\in\Omega$ and all $u\in\mathbb{R}^m$, the mapping $A\mapsto f(x,u,A)$ is linear. Here we argue as follows to identify the weak limit of $f(x,u_k,Du_k)$:

$$\begin{aligned} f(x,u_k,Du_k) \rightharpoonup \langle\nu_x, f(x,u,.)\rangle &= \int_{\mathbb{M}^{m\times n}} f(x,u,\lambda)d\nu_x(\lambda) \\ &= f(x,u,.)o\underbrace{\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda)}_{=:Du(x)} \\ &= f(x,u,Du), \end{aligned}$$

since $f(x,u_k,Du_k)$ is equiintegrable.

Conclusion

For every $\varphi\in W_0^{1,p}(\Omega;\mathbb{R}^m)$, since $\bigcup_{k\geq 1}X_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, there exists a sequence $\{\varphi_k\}\subset\bigcup_{k\geq 1}X_k$ such that $\varphi_k\to\varphi$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ as $k\to\infty$. We can now pass to the limit in the Galerkin equations:

$$\begin{aligned} &\langle T(u_k),\varphi_k\rangle - \langle T(u),\varphi\rangle \\ &= \int_\Omega a(|Du_k|)Du_k : D\varphi_k dx - \int_\Omega a(|Du|)Du : D\varphi dx \\ &- \int_\Omega f(x,u_k,Du_k)\cdot\varphi_k dx + \int_\Omega f(x,u,Du)\cdot\varphi dx \\ &= \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - D\varphi)dx + \int_\Omega (a(|Du_k|)Du_k - a(|Du|)Du) : D\varphi dx \\ &- \int_\Omega f(x,u_k,Du_k)\cdot(\varphi_k-\varphi)dx - \int_\Omega (f(x,u_k,Du_k) - f(x,u,Du))\cdot\varphi dx. \end{aligned}$$

The right-hand side of the above equation tends to zero as k tends to infinity by the previous results. By virtue of Lemma 3.9, it follows that $\langle T(u),\varphi\rangle = 0$ for all $\varphi\in W_0^{1,p}(\Omega;\mathbb{R}^m)$ as desired.

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