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A new approach to generalize metric functions

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Abstract

This article consists of a new concept of generalized metric space, called ϕ-metric space which is developed by making a suitable modification in the 'triangle inequality. The notion of ϕ-metric generalizes the concept of some existing metrizable generalized spaces like S-metric, b-metric, etc. It is shown that one can easily construct a ϕ-metric from those generalized metric functions and the notion of convergence of a sequence on those generalized metric spaces are identical with the respective induced ϕ-metric spaces. Moreover, ϕ-metric space is metrizable and its properties are pretty similar to the metric functions. So ϕ-metric functions substantially may play the role of metric functions. Also, the structure of ϕ-metric space is studied and some well-known fixed point theorems are established.

Keywords: ϕ-metric, ϕ-metric spaces, generalized distance function, metrizability
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1 Introduction

In modern mathematics metric spaces and topological spaces are two widely used concepts. Metric spaces are considered as a particular case of topological spaces. The notion of metric was developed by Frechet[1] and later Hausdorff [2] presented axiomatically. The proposed three metric axioms are very fundamental and geometrically appreciable. The properties of metric spaces are easier to check than general topological spaces. Because of this reason, metrizability is an interesting topic for topological spaces. Unfortunately, there are spaces which are not metrizable. So researchers try to develop functions which are more general than metric spaces. In 1963, Gahler[3] introduced 2-metric and later n-metric which are not metrizable. In another process of generalization, Dhage introduced D-metric[4], but it was a defective structure. To rectify the error in D-metric, Mustafa and Sims introduced a new concept called G-metric space [5], Sedghi introduced S-metric space[6]. There is another generalized metric called b-metric, introduced by Czerwik [7, 8].

In 2013, Chaipuniya and Kumam introduced the notion of g-3ps [9] and claimed that it is the general structure of the distance between three points. They proved the G-metric and S-metric spaces are also g-3ps and a g-3ps is not metrizable in general. But it has been proved that b-metric, G-metric and S-metric spaces are metrizable. So we think that instead of g-3ps, which is much more general in nature, a general structure closer to those distance functions may be a better alternative.

Following the different approaches of generalized metric spaces as mention above by several authors, in this paper we introduce a new notion of generalized metric which is called ϕ-metric. In this approach, we only change the 'triangle

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inequality' by adding a suitable non-negative real valued function in the right hand side. We show the b-metric, S-metric spaces are particular examples of ϕ-metric spaces. We study its topological properties and also prove that ϕ-metric spaces are metrizable. We have mentioned earlier that S-metric, b-metric spaces are also metrizable but in the line of proof of metrizability for ϕ-metric, b-metric or S-metric shows that one can not construct the respective metric easily whose metric topology is identical with the corresponding topology of each such generalized metric spaces. In this manuscript, we study some properties of ϕ-metric spaces and we observe that those are similar to usual metric. Since ϕ-metric can be induced from a b-metric or S-metric, so it is easy to study the topological properties of such spaces with the help of the respective induced ϕ-metric space. Another important advantage of ϕ-metric is that we can define ϕ-metric in l_p-space for $0 < p < 1$ which is not possible for usual metric. We also develop a parallel study on some elementary results of completeness, compactness, totally boundedness, etc. like metric spaces. Lastly we establish the famous Banach type, Kannan type and Edelstein type fixed point results in this new setting. All the result are justified by suitable examples also.
We refer some recently published papers[18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] in metric spaces which help us to develop the concept of generalized metric spaces.

The organization of this article is as follows. Section 2 provides some preliminary results. In Section 3, ϕ-metric space is defined, illustrated by examples and some elementary topological properties are studied. In Section 4, results on compactness, completeness, and totally boundedness are discussed. Section 5 consists of some fixed point theorems in ϕ-metric spaces. The straightforward proofs are omitted.

2 Preliminaries

In this section, we recollect some preliminary results which are used in this paper.

Definition 2.1. [3] Let X be a non-empty set. Then (X, D) is called a 2-metric space if the function $D : X \times X \times X \to \mathbb{R}$, named 2-metric satisfies the following conditions:

(i) For every $a, b \in X$ with $a \neq b$ there exists $c \in X$ such that $D(a, b, c) \neq 0$;
(ii) If at least two of $a, b, c \in X$ are the same, then $D(a, b, c) = 0$;
(iii) For all $a, b, c \in X$, $D(a, b, c) = D(a, c, b) = D(b, c, a) = D(b, a, c) = D(c, a, b) = D(c, b, a)$;
(iv) The rectangle inequality: for all $a, b, c, d \in X$, $D(a, b, c) \leq D(a, b, d) + D(b, c, d) + D(c, a, d)$.

Definition 2.2. [7] Let X be a nonempty set and $d : X \times X \to \mathbb{R}_{\geq 0}$ be a function for all $x, y, z \in X$ which satisfies the following conditions:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq 2[d(x, y) + d(y, z)]$.

Then d is called a b-metric and the pair (X, d) is called a b-metric space.

Later in 1998, Czerwik[8] modified this notion of b-metric replacing 2 by a constant $K \geq 1$ in the condition (b3) of Definition 2.1. Khamsi and Hussian [10] took a step further and considered the constant $K > 0$ and they named it as metric-type space. Another generalization of b-metric is Strong b-metric space, which was introduced by Kirk and Shahzad [11].

Definition 2.3. [11] Let X be a nonempty set and $K \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is called a strong b-metric if for all $x, y, z \in X$ it satisfies the following conditions:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq K d(x, y) + d(y, z)$.

Then (X, d, K) is called a strong b-metric space.

Definition 2.4. [6] Let X be a nonempty set. A function $S : X \times X \times X \to \mathbb{R}$ is called an S-metric if it satisfies the following properties:

(S1) $S(x, y, z) \geq 0$ for all $x, y, z \in X$;
(S2) $S(x, y, z) = 0$ if and only if $x = y = z$;
(S3) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$ for all $x, y, z, w \in X$.

The pair (X, S) is called an S-metric space.

Proposition 2.5. [12] Let (X, d) be a b-metric space.

(i) A subset A of X is called open if for any $a \in A$, there exist $t > 0$ such that $B(a, t) \subset A$ where

$$B(a, t) = \{y \in X : d(a, y) < t\}.$$

(ii) If τ is the collection of all open balls of (X, d) then τ defines a topology on X.

Proposition 2.6. [11] Every open ball $B(a, r) = \{x \in X : d(a, x) < r\}$ in a strong b-metric space (X, d, K) is open.

Definition 2.7. [6] Let (X, S) be an S-metric space. Then for any $x \in X$ and $t > 0$, open ball and closed ball are defined by

$$B_S(x, t) = \{y \in X : S(y, y, x) < t\} \text{ and } B_S[x, t] = \{y \in X : S(y, y, x) \leq t\}$$

respectively.

So far, we have discussed some generalized metric spaces. In Section 3, we introduce ϕ-metric which is a generalization of many established distance functions. Later, we involve ourselves to study the metrizability and topological properties of the discussed generalized spaces including ϕ-metric spaces. In this aspect, some topological definitions and results on topological spaces and other generalized metric spaces are given below.

Definition 2.8. [13] Let X be a topological space and $B = \{B_s : s \in S\}$ be a family of subsets of X. Then

(i) B is called locally finite if for each $x \in X$ there exists a neighborhood U of x such that the set $\{s \in S : A_s \cap U \neq \phi\}$ is finite.
(ii) B is called discrete if for each $x \in X$, there exists a neighborhood U of x such that the set $B_s \cap U \neq \phi$ for at most one $s \in S$.
(iii) B is called σ-locally finite if $B = \underset{i \in \mathbb{N}}{\cup} B_i$ where every B_i is locally finite.
(iv) B is called σ-discrete if $B = \underset{i \in \mathbb{N}}{\cup} B_i$ where every B_i is discrete.
(v) B is called a cover of X if $\underset{s \in S}{\cup} B_s = X$.
(vi) A cover $A = \{A_i : i \in I\}$ of subsets of X is called a refinement of the cover B_i if for each $i \in I$ there exists $s \in S$ such that $A_i \subset B_s$.

Definition 2.9. [13] Let (X, τ) be a topological space. Then X is said to be a

(i) regular space if for any closed subsets $A \subset X$ and for $x \in X \setminus A$ there exists two disjoint open sets U and V containing A and x respectively.
(ii) normal space if for any two disjoint closed subsets A and B of X there exists two disjoint open sets U and V containing A and B respectively.

Definition 2.10. [14, 15] Let (X, τ) be a topological space.

(a) A subset U of X is called sequentially open if each sequence $\{x_n\} \subset X$ converging to a point $x \in U$ then there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.
(b) A subset U of X is called sequentially closed if no sequence in U converges to a point not in U.
(c) X is called semi-metrizable if there exists a function $d : X \times X \to [0, \infty)$ such that for all $x, y \in X$
    (i) $d(x, y) = 0$ if and only if $x = y$;
    (ii) $d(x, y) = d(y, x)$;
    (iii) $x \in \overline{A}$ if and only if $d(x, A) = inf\{d(x, y) : y \in A\} = 0$ for any subset A of X.
(d) (X, τ) is said to be metrizable if there exists a metric on X whose topology is same as the topology τ.

Theorem 2.11. [13] (The Stone Theorem) Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete.

Theorem 2.12. [16] (The Bing Metrization Theorem) A topological space is metrizable if and only if it is regular and has a σ-discrete base.

Recently, N. V. Dung et al.[12] prove the following results about semi-metrizability, metrizability, and some other properties of b-metric spaces.

Theorem 2.13. [12] Every b-metric space (X, d) is a semi-metrizable space.

Theorem 2.14. [12] Let (X, d) be a b-metric space and d is continuous in one variable. Then

(i) X is regular.
(ii) Every open cover of X has an open refinement which is both locally finite and σ-discrete.
(iii) X has a σ-discrete base.
(iv) X is metrizable.

An important Corollary of Theorem 2.14 is given below.

Corollary 2.15. [12] Every S-metric space is metrizable.

3 Introduction to ϕ-metric

Recall that b-metric and S-metric spaces are metrizable (see Theorem 2.14, Corollary 2.15). However, the role of induced metrics in such spaces is implicit in nature. Thus it is difficult to construct a metric for a given S-metric or b-metric so that topology remains the same. In this connection, we introduce ϕ-metric in a new approach that helps to study the metrizability of S-metric spaces, b-metric spaces, etc.

Definition 3.1. Let X be a nonempty set. A function $d_\phi : X \times X \to \mathbb{R}_{\geq 0}$ is said to be a ϕ-metric if it satisfies the following conditions:

$(d_\phi 1)$ $d_\phi(x, y) = 0$ if and only if $x = y$;
$(d_\phi 2)$ $d_\phi(x, y) = d_\phi(y, x)$;
$(d_\phi 3)$ $d_\phi(x, y) \leq d_\phi(x, z) + d_\phi(z, y) + \phi(x, y, z)$;

for all $x, y, z \in X$ where $\phi : X \times X \times X \to \mathbb{R}_{\geq 0}$ is a function satisfying

$(\phi 1)$ $\phi(x, y, z) = 0$ if $x = z$ or $y = z$;
$(\phi 2)$ $\phi(x, y, z) = \phi(y, x, z)$;
$(\phi 3)$ for all $\epsilon > 0$ there exists $\delta > 0$ such that $\phi(x, y, z) < \epsilon$ whenever $d_\phi(x, z) < \delta$ or $d_\phi(y, z) < \delta$;

for all $x, y, z \in X$. A set X together with the function d_ϕ, (X, d_ϕ) is called a ϕ-metric space.

Example 3.2. Let (X, d) be a metric space and define a function on X by $d_\phi(x, y) = (d(x, y))^2$ for all $x, y \in X$. Then clearly $d_\phi(x, y) = 0$ if and only if $x = y$ and $d_\phi(x, y) = d_\phi(y, x)$ for all $x, y \in X$. Now for any $x, y, z \in X$,

$$\begin{aligned} & d(x, y) \leq d(x, z) + d(z, y) \\ \text{or} \quad & (d(x, y))^2 \leq (d(x, z))^2 + (d(z, y))^2 + 2d(x, z)d(y, z). \end{aligned}$$

This implies $d_\phi(x, y) \leq d_\phi(x, z) + d_\phi(z, y) + \phi(x, y, z)$ where $\phi(x, y, z) = 2\sqrt{d_\phi(x, z)d_\phi(z, y)}$ for all $x, y, z \in X$. Hence (X, d_ϕ) is a ϕ-metric space.

Remark **3.3.** Every metric space is a ϕ-metric space but not conversely. To justify, in Example 3.2, we take $X = l_p,\ 0 < p < 1$ then the distance function d_ϕ in $l_p,\ 0 < p < 1$ is not a metric but it satisfies the conditions of our new notion of distance.

Example 3.4. Let (X, d_ϕ) be a ϕ-metric space and define a function on X by $D_{\phi_1}(x, y) = (d_\phi(x, y))^2$ for all $x, y \in X$. Then (X, D_{ϕ_1}) is a ϕ-metric space. $(d_{\phi 1})$ and $(d_{\phi 2})$ are obvious. So we only verify $(d_{\phi 3})$. For any $x, y, z \in X$,

$$d_\phi(x, y) \leq d_\phi(x, z) + d_\phi(z, y) + \phi(x, y, z)$$

or

$$(d_\phi(x, y))^2 \leq (d_\phi(x, z))^2 + (d_\phi(z, y))^2 + 2d_\phi(x, z)d_\phi(y, z) + (\phi(x, y, z))^2 + 2(d_\phi(x, z) + d_\phi(z, y))\phi(x, y, z)$$

which implies $D_{\phi_1}(x, y) \leq D_{\phi_1}(x, z) + D_{\phi_1}(z, y) + \phi_1(x, y, z)$ where $\phi_1(x, y, z) = 2d_\phi(x, z)d_\phi(z, y) + \phi^2(x, y, z) + 2\phi(x, y, z)[d_\phi(x, z) + d_\phi(z, y)]$ for all $x, y, z \in X$. Moreover if (X, d_{ϕ_i}), $i = 1, 2 \cdots n$ be ϕ-metric spaces, then (X, D_ϕ) is a ϕ-metric space where $D_\phi(x, y) = \prod_{i=1}^{n} d_{\phi_i}(x, y)$ for all $x, y \in X$. But in case of infinite product of ϕ-metrics that is $D_\phi(x, y) = \underset{n\to\infty}{lim} (d_\phi(x, y))^n$ for all $x, y \in X$ is a ϕ-metric only if d_ϕ is the discrete metric.

Example 3.5. Let (X, S) be a S-metric space. Define $d_\phi(x, y) = S(x, x, y)$ for all $x, y \in X$ and we choose $\phi(x, y, z) = d_\phi(x, z) + d_\phi(y, z)$ for all $x, y, z \in X$.
Then the function ϕ satisfies the first and second conditions. For the third condition we take $\alpha > 0$. If $d_\phi(x, z) < \frac{\alpha}{2}$ and $d_\phi(y, z) < \frac{\alpha}{2}$ then $\phi(x, y, z) < \alpha$. Thus for all $\alpha > 0$ there exists $\beta > 0$ such that $\phi(x, y, z) < \alpha$ whenever $d_\phi(x, z) < \beta$ and $d_\phi(y, z) < \beta$ where $\beta = \frac{\alpha}{2}$. Hence (X, d_ϕ) is ϕ-metric space.

Example 3.6. Let (X, B) be a b-metric space with constant coefficient $K(> 1)$. For all $x, y \in X$, define $d_\phi(x, y) = B(x, y)$ and we choose $\phi(x, y, z) = (K - 1)[d_\phi(x, z) + d_\phi(y, z)]$ for all $x, y, z \in X$.
To verify the third condition for ϕ function, take $\alpha > 0$. Now if $d_\phi(x, z) < \frac{\alpha}{2K}$ and $d_\phi(y, z) < \frac{\alpha}{2K}$ then $\phi(x, y, z) < \dfrac{(K-1)\alpha}{K} < \alpha$. Thus for all $\alpha > 0$ there exists $\beta > 0$ such that $\phi(x, y, z) < \alpha$ whenever $d_\phi(x, z) < \beta$ and $d_\phi(y, z) < \beta$ where $\beta = \frac{\alpha}{2K}$. Hence (X, d_ϕ) is ϕ-metric space.

Example 3.7. Let (X, B) be a strong b-metric space with constant coefficient $K(> 1)$. Define $d_\phi(x, y) = B(x, y)$ for all $x, y \in X$ and take $\phi(x, y, z) = (K - 1)[d_\phi(x, z) + d_\phi(y, z)]$ for all $x, y, z \in X$. Then (X, d_ϕ) is ϕ-metric space.

***Remark* 3.8.** We call the ϕ-metrics defined in the Example 3.5, Example 3.6 and Example 3.7 as ϕ-metric induced by S-metric, b-metric and strong b-metric respectively. So it is clear that one can easily construct a ϕ-metric from those generalized distance functions.

To study the topological structure of ϕ-metric spaces, we define open and closed balls as given below.

Definition 3.9. For $x \in X$ and $r > 0$ we define open ball and closed ball with radius r and center x respectively as:

$$B_\phi(x, r) = \{y \in X : d_\phi(x, y) < r\} \text{ and } B_\phi[x, r] = \{y \in X : d_\phi(x, y) \leq r\}.$$

Proposition 3.10. Let (X, d_ϕ) be a ϕ-metric space. Then for all $r, s > 0$ and for all $a \in X$,

(i) $r \leq s$ if and only if $B_\phi(a, r) \subseteq B_\phi(a, s)$.
(ii) $r \leq s$ if and only if $B_\phi[a, r] \subseteq B_\phi(a, s)$.
(iii) $B_\phi(a, r) \subseteq B_\phi[a, r]$.

Theorem 3.11. Let (X, d_ϕ) be a ϕ-metric space and define

$$\tau_\phi = \{G \subseteq X : \forall\ x \in G, \exists\ r > 0 \textit{ such that } B_\phi(x, r) \subseteq G\}.$$

Then τ_ϕ is a topology on X.

Proof . Obviously $\phi, X \in \tau_\phi$ and τ_ϕ is closed under arbitrary union. To check the closedness of τ_ϕ under finite intersection, let us consider $G_1, G_2 \in \tau_\phi$. We need to show $G_1 \cap G_2 \in \tau_\phi$. Take any $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$. So there exists $r_1, r_2 > 0$ such that $B_\phi(x, r_1) \subseteq G_1$ and $B_\phi(x, r_2) \subseteq G_2$.

Now if $r = \min\{r_1, r_2\}$ then $B_\phi(x, r) \subseteq B_\phi(x, r_1) \subseteq G_1$ and $B_\phi(x, r) \subseteq B_\phi(x, r_2) \subseteq G_2$. Thus $B_\phi(x, r) \subseteq G_1 \cap G_2$. So, $G_1 \cap G_2 \in \tau_\phi$. □

Definition 3.12. Let (X, d_ϕ) be a ϕ-metric space and $B \subseteq X$. Then

(i) B is said to be open set if $B \in \tau_\phi$.
(ii) B is said to be closed set if $X \setminus B$ is in τ_ϕ.
(iii) $x \in X$ is called a limit point of B if there exists $r > 0$ such that $(B(x,r) \setminus \{x\}) \cap B$ contains infinitely many points of B.
(iv) The set of all limit points of B is called the derived set of B denoted by B'.
(v) A set that contains the points of B as well as limit points of B is called the closure of the set B denoted by $\overline{B}$.

The following two propositions in ϕ-metric space are obvious.

Proposition 3.13. Let (X, d_ϕ) be a ϕ-metric space and $A \subseteq X$. Then

(i) $\overline{A}$ is a closed set.
(ii) A is a closed set if and only if $A = \overline{A}$.
(iii) $x \notin \overline{A}$ if and only if $d_\phi(x, a) > 0$ for all $a \in A$.

Proposition 3.14. In a ϕ-metric space (X, d_ϕ),

(i) arbitrary union of open set is open.
(ii) arbitrary intersection of closed set is closed.

***Remark* 3.15.** Arbitrary union (respectively intersection) of closed (respectively open) set is not closed (respectively open), which can be justified by examples of metric spaces.

Now we are interested to find a basis for τ_ϕ. In fact, we want to show that the set of all open balls form a basis. For the first step, we prove the next result.

Theorem 3.16. In a ϕ-metric space, every open ball is an open set.

Proof . For some $x \in X$ and $r > 0$, we consider the open ball $B_\phi(x,r)$ and choose $y \in B_\phi(x,r)$. Then $d_\phi(x,y) = r'(say) < r$. We need to find some $s > 0$ such that $B_\phi(y,s) \subset B_\phi(x,r)$. Again for $x \in X,\ y \in B_\phi(x,r)$ and $a \in X$ we have

$$d_\phi(x,a) \le d_\phi(x,y) + d_\phi(y,a) + \phi(x,a,y). \tag{3.1}$$

Now for $\frac{r-r'}{2}(> 0)$ there exists $t > 0$ such that $\phi(x,z,y) < \frac{r-r'}{2}$ whenever $d_\phi(z,y) < t$ and $z \in X$. Let $s = \min\{\frac{r-r'}{2}, t\}$. Let us choose $z \in B_\phi(y,s)$. Then $d_\phi(y,z) < s$ and hence $\phi(x,z,y) < \frac{r-r'}{2}$. Therefore from the inequality (3.1) we have,

$$\begin{aligned} d_\phi(x,z) \le& d_\phi(x,y) + d_\phi(y,z) + \phi(x,z,y) \\ <& r' + s + \frac{r-r'}{2} \\ \le& r' + 2(\frac{r-r'}{2}) = r. \end{aligned}$$

Thus, $d_\phi(x,z) < r$ whenever $z \in B_\phi(y,s)$ where $s = \min\{\frac{r-r'}{2}, t\}$. Hence $B_\phi(y,s) \subset B_\phi(x,r)$ and this proves that $B_\phi(x,r)$ is an open set. □

Consider the collection of open balls $\beta = \{B_\phi(x,r) : x \in X, r > 0\}$. Now we will show that it generates a topology on X.

Theorem 3.17. Let (X, d_ϕ) be a ϕ-metric space. Then β is a base for (X, τ_ϕ).

Proof . Let $x \in X$. Then for any $r > 0,\ x \in B_\phi(x,r)$. Next suppose for some x and y in X and for $r_1, r_2 > 0$ there is $z \in B_\phi(x,r_1) \cap B_\phi(y,r_2)$. Since $z \in B_\phi(x,r_1)$ so there exists $s_1 > 0$ such that $B_\phi(z,s_1) \subseteq B_\phi(x,r_1)$ and $z \in B_\phi(y,r_2)$ implies there exists $s_2 > 0$ such that $B_\phi(z,s_2) \subseteq B_\phi(x,r_2)$.

Now take $s = \min\{s_1, s_2\}$. Then $z \in B_\phi(z,s) \subset B_\phi(x,r_1) \cap B_\phi(y,r_2)$. □

Theorem 3.18. In a ϕ-metric space (X, d_ϕ) every closed ball is a closed set.

Proof . For any $x \in X$, and $r > 0$, consider the closed ball $B_\phi[x, r]$. To prove that $B_\phi[x, r]$ is closed, it is enough to show that $X \setminus B_\phi[x, r] = A$(say) is open. Choose $y \in A$. Then $d_\phi(x, y) = r'(say) > r$.

Now we need to find some $s > 0$ such that $B_\phi(y, s) \subset A$. For $(\frac{r'-r}{2}) > 0$, there exists $t > 0$ such that $\phi(x, y, z) < \frac{r'-r}{2}$ whenever $d_\phi(y, z) < t$ and $z \in X$. Let $s = \min\{\frac{r'-r}{2}, t\}$. Choose $a \in B_\phi(y, s)$. Then $d_\phi(a, y) < s$ and so $\phi(x, y, a) < \frac{r'-r}{2}$. So for $x \in X,\ y \in A$ and $a \in B_\phi(y, s)$, we have

$$\begin{aligned} d_\phi(x, y) &\leq d_\phi(x, a) + d_\phi(y, a) + \phi(x, y, a) \\ or \quad d_\phi(x, a) &\geq d_\phi(x, y) - d_\phi(y, a) - \phi(x, y, a). \end{aligned}$$

Hence,

$$d_\phi(x, a) > r' - s - (\frac{r'-r}{2}) \geq r' - 2(\frac{r'-r}{2}) = r.$$

Therefore, $d_\phi(x, a) > r$ whenever $a \in B_\phi(y, s)$ where $s = \min\{\frac{r'-r}{2}, t\}$, which implies that A is an open set and consequently $B_\phi[x, r]$ is a closed set. □

Theorem 3.19. Every ϕ-metric space is regular.

Proof . Let A be a closed set in (X, d_ϕ) and $x \in X \setminus A$. So $d_\phi(x, a) > 0$ for all $a \in A$. Let $3r = \inf\{d_\phi(x, a) : a \in A\}$. Consider the open ball $B(x, r) = V(say)$. Now for any $a \in A$ and $r > 0$ there exists $\beta > 0$ such that $\phi(x, a, y) < r$ whenever $d_\phi(a, y) < \beta$ and $y \in X$. Take $\min\{\beta, r\} = r^*(say)$ and consider the open set $U = \underset{a\in A}{\cup} B_\phi(a, r^*)$. Then $A \subset U$. We claim that $U \cap V = \phi$. If possible suppose there exists $c \in U \cap V$. Then for any $a \in A,\ d_\phi(a, c) < r^*$ and $\phi(x, a, c) < r$. Now for any $a \in A$,

$$d_\phi(x, a) \leq\ d_\phi(x, c) + d_\phi(c, a) + \phi(a, x, c) <\ r^* + r + r \leq 3r.$$

This is a contradiction to our assumption and hence U and V are two disjoint open sets in X containing A and x respectively. □

Theorem 3.20. Every ϕ-metric space is normal.

Proof . Let A and B be two closed disjoint sets in (X, d_ϕ). Then for any $a \in A$ and $b \in B$, $d_\phi(a, b) > 0$. Let $3r = \inf\{d_\phi(a, b) : a \in A, b \in B\}$. Consider the open set $V = \underset{b\in B}{\cup} B_\phi(b, r)$ containing B. For any $a \in A,\ b \in B$ and $r > 0$ there exists $\delta > 0$ such that $\phi(a, b, z) < r$ whenever $d_\phi(a, z) < \delta$ and $z \in X$. Let $r^* = \min\{\delta, r\}$ and $U = \underset{a\in A}{\cup} B_\phi(a, r^*)$. Then U is open and $A \subset U$. Next, we claim that U and V are disjoint. If not, then there exists $c \in U \cap V$. Then for all $a \in A$ and for all $b \in B,\ d_\phi(a, c) < r^*,\ d_\phi(b, c) < r$ and $\phi(a, b, c) < r$. So for $a \in A,\ b \in B$ and $c \in U \cap V$,

$$d_\phi(a, b) \leq\ d_\phi(a, c) + d_\phi(c, b) + \phi(a, b, c) <\ r^* + r + r \leq 3r.$$

This contradicts our assumption and hence the theorem is proved. □

Now we prove the Stone-type theorem in a ϕ-metric space and use Bing metrization theorem to obtain a sufficient condition of metrizability.

Theorem 3.21. (Stone-type theorem) In a $\phi-$metric space (X, d_ϕ) every open cover of X has an open refinement which is both σ-locally finite and σ-discrete.

Proof . Let $\{\mathcal{U}_s : s \in S\}$ be an open cover of X. By the Zermelo theorem on well-ordering[13], we can take a well-ordering relation $<$ on S. Define the families $\mathcal{V}_i = \{\mathcal{V}_{s,i} : s \in S\}$ of subsets of X by letting $\mathcal{V}_{s,i} = \underset{c\in\mathcal{C}}{\cup} B_\phi(c, \frac{1}{2^i})$ where $\mathcal{C}$ is the set of all points $c \in X$ satisfying following conditions:

(i) s is the smallest element of S such that $c \in \mathcal{U}_s$.

(ii) $c \notin \mathcal{V}_{t,j}$ for all $j < i$ and for all $t \in S$.

(iii) $B_\phi(c, \frac{5}{2^i}) \subset \mathcal{U}_s$.

Obviously the sets $\mathcal{V}_{s,i}$ are open and by condition (iii), we have $V_{s,i} \subset \mathcal{U}_s$. For each $x \in X$ take the smallest $s \in S$ such that $x \in \mathcal{U}_s$ and a natural number i such that $B_\phi(x, \frac{5}{2^i}) \subset \mathcal{U}_s$. It implies that $x \in \mathcal{C}$ if and only if $x \notin \mathcal{V}_{t,j}$ for all $j < i$ and for all $t \in S$. Then $x \in \mathcal{V}_{s,i}$. Thus we have either $x \in \mathcal{V}_{t,j}$ for all $j < i$ and for all $t \in S$ or $x \in \mathcal{V}_{s,i}$. This proves that $\mathcal{V} = \underset{i\in\mathbb{N}}{\cup} \mathcal{V}_i$ is an open refinement of the cover $\{\mathcal{U}_s : s \in S\}$.

Now for every $i \in \mathbb{N}$, let $x_1 \in \mathcal{V}_{s_1,i}$ and $x_2 \in \mathcal{V}_{s_2,i}$ with $s_1 \neq s_2$. Let us assume $s_1 < s_2$. By the definition of $V_{s_1,i}$ and $\mathcal{V}_{s_2,i}$ there exists $c_1,\ c_2 \in X$ satisfying conditions (i), (ii), (iii) and
$x_1 \in B_\phi(c_1, \frac{1}{2^i}),\ x_2 \in B_\phi(c_2, \frac{1}{2^i})$. Again we have $B_\phi(c_1, \frac{5}{2^i}) \subset \mathcal{U}_{s_1}$ and $c_2 \notin \mathcal{U}_{s_1}$ and this implies $d_\phi(c_1, c_2) \geq \frac{5}{2^i}$. But we have

$$d_\phi(c_1, c_2) \leq d_\phi(c_1, x_1) + d_\phi(x_1, x_2) + d_\phi(x_2, c_2) + \phi(c_1, c_2, x_1) + \phi(x_1, c_2, x_2)$$

which implies

$$d_\phi(x_1, x_2) \geq \frac{5}{2^i} - d_\phi(c_1, x_1) - d_\phi(x_2, c_2) - \phi(c_1, c_2, x_1) - \phi(x_1, c_2, x_2). \tag{3.2}$$

Again for $\frac{1}{2^{i+1}}(> 0)$ there exists $\beta_1,\ \beta_2(> 0)$ such that $\phi(c_1, c_2, x_1) < \frac{1}{2^{i+1}}$ whenever $d_\phi(c_1, x_1) < \beta_1$ and $\phi(c_1, c_2, x_2) < \frac{1}{2^{i+1}}$ whenever $d_\phi(c_2, x_2) < \beta_2$. Let $\min\{\beta_1,\ \beta_2,\ \frac{1}{2^i}\} = \beta$. Then $d_\phi(c_1, x_1) < \beta,\ d_\phi(c_2, x_2) < \beta$ and $\phi(c_1, c_2, x_1) < \frac{1}{2^{i+1}},\ \phi(c_1, c_2, x_2) < \frac{1}{2^{i+1}}$. Then (3.2) gives,

$$d_\phi(x_1, x_2) > \frac{5}{2^i} - 2\beta - 2 \times \frac{1}{2^{i+1}} \geq \frac{5}{2^i} - 2 \times \frac{1}{2^i} - \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$

To prove that the families V_i are σ-discrete, suppose there exists $x \in X$ such that $x_1, x_2 \in B_\phi(x, \frac{1}{2^{i+1}})$. Then we have $d_\phi(x, x_1) < \frac{1}{2^{i+1}},\ d_\phi(x, x_2) < \frac{1}{2^{i+1}}$ and

$$\frac{1}{2^{i-1}} < d_\phi(x_1, x_2) \leq d_\phi(x_1, x) + d_\phi(x, x_2) + \phi(x_1, x_2, x). \tag{3.3}$$

Now for $\frac{1}{2^{i+1}} > 0$ there exists $\beta' > 0$ such that $\phi(x_1, x_2, x) < \frac{1}{2^{i+1}}$ whenever $d_\phi(x_2, x) < \beta'$. If $\delta = \min\{\beta', \frac{1}{2^{i+1}}\}$ then $d_\phi(x_2, x) < \delta$ and $\phi(x_1, x_2, x) < \frac{1}{2^{i+1}}$. The inequality (3.3) gives,

$$\frac{1}{2^{i-1}} < d_\phi(x_1, x_2) < \frac{1}{2^{i+1}} + \delta + \frac{1}{2^{i+1}} \leq 2 \times \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} < \frac{1}{2^i} + \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$

This is a contradiction and hence it proves that each ball of radius $\frac{1}{2^{i+1}}$ meets at most one element of $\mathcal{V}_i$ that is $\mathcal{V} = \cup_{i\in\mathbb{N}}\mathcal{V}_i$ is σ-discrete. Let $i \in \mathbb{N}$ then for all $t \in S,\ i \geq j + k$ and $c \in \mathcal{C}$ implies $c \notin \mathcal{V}_{t,j}$. Now if $B_\phi(x, \frac{1}{2^{k-1}}) \subset \mathcal{V}_{t,j}$ then $c \notin B_\phi(x, \frac{1}{2^{k-1}})$ and $d_\phi(x, c) \geq \frac{1}{2^{k-1}}$. Again $j + k \geq k + 1$ and $i \geq k + 1$ implies $\frac{1}{2^{j+k}} \leq \frac{1}{2^{k+1}}$ and $\frac{1}{2^i} \leq \frac{1}{2^{k+1}}$. Next suppose there exists $y \in B_\phi(x, \frac{1}{2^{j+k}}) \cap B_\phi(c, \frac{1}{2^i})$. Then

$$d_\phi(x, c) \leq d_\phi(x, y) + d_\phi(y, c) + \phi(x, c, y). \tag{3.4}$$

For $\frac{1}{2^k} > 0$ there exists $\alpha > 0$ such that $\phi(x, c, y) < \frac{1}{2^k}$ whenever $d_\phi(x, y) < \alpha$. Let $\gamma = \min\{\alpha, \frac{1}{2^{j+k}}\}$. Then $d_\phi(x, y) < \gamma$ and $\phi(x, c, y) < \frac{1}{2^k}$. Therefore from (3.4) we obtain,

$$\frac{1}{2^{k-1}} \leq d_\phi(x, c) < \gamma + \frac{1}{2^i} + \frac{1}{2^k} \leq \frac{1}{2^{j+k}} + \frac{1}{2^{k+1}} + \frac{1}{2^k} \leq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \frac{1}{2^k} = \frac{1}{2^{k-1}}.$$

Which concludes $B_\phi(x, \frac{1}{2^{j+k}}) \cap B_\phi(c, \frac{1}{2^i}) = \phi$ and this implies $B_\phi(x, \frac{1}{2^{j+k}}) \cap \mathcal{V}_{s,i} = \phi$ for $i \geq j + k$ and $s \in S$ with $B_\phi(x, \frac{1}{2^{k-1}}) \subset \mathcal{V}_{t,j}$. Since $\mathcal{V}$ is a refinement of $\{\mathcal{U}_s : s \in S\}$, so for each $x \in X$, there exists l, j and t such that $B_\phi(x, \frac{1}{2^l}) \subset \mathcal{V}_{t,j}$ and thus there exists k, j and t such that $B_\phi(x, \frac{1}{2^{k-1}}) \subset \mathcal{V}_{t,j}$. Then the ball $B_\phi(x, \frac{1}{2^{j+k}})$ meets at most $(j + k - 1)$ members of $\mathcal{V}$. This proves that $\mathcal{V}_i$ is locally finite that is $\mathcal{V}$ is σ-locally finite. □

Corollary 3.22. Let (X, d_ϕ) be a ϕ-metric space. Then X has σ-discrete base.

Proof . For every $i \in \mathbb{N}$, let $\mathcal{A}_i = \{B_\phi(x, \frac{1}{i}) : x \in X\}$. Then $\mathcal{A}_i$ is an open cover of X. By Theorem 3.21, there exists an open σ-discrete refinement $\mathcal{B}_i$ of $\mathcal{A}_i$. Put $\mathcal{B} = \underset{i\in\mathbb{N}}{\cup} \mathcal{B}_i$. Then $\mathcal{B}$ is a σ-discrete base of X. □

Corollary 3.23. Every ϕ-metric space is metrizable.

Proof . From Theorem 3.19 and Corollary 3.22 it follows that X is regular space with $\sigma-$discrete base. Then from Bing metrization theorem(Theorem 2.12), X is metrizable. □

Corollary 3.24. Let (X, d_ϕ) be a ϕ-metric space and τ_ϕ is a topology on X. Then τ_ϕ is a Hausdorff topology on X.

Proof . Since (X, τ_ϕ) is a regular topological space, so it is Hausdorff. □

Remark **3.25.** Till now we have shown that ϕ-metric can be induced from a b-metric, S-metric, etc. From this, we deduce that open balls of S-metric(or b-metric) are the same as the open balls of induced ϕ-metric. Therefore topology generated by the open balls of S-metric(or b-metric) is identical to the topology generated by the open balls of respective induced ϕ-metric. Thus S-metric and b-metric spaces are ϕ-metrizable.

Next, we discuss the convergence of a sequence in ϕ-metric space including its basic properties.

Definition 3.26. A sequence $\{x_n\} \subseteq X$ is said to converge to x if for any $\epsilon > 0$ there exists a positive integer N such that

$$d_\phi(x_n, x) < \epsilon \text{ for all } \ n \geq N \quad \text{ that is } d(x_n, x) \to 0 \ \text{ as } n \to \infty.$$

We denote this by $\lim\limits_{n\to\infty} x_n = x$.

Proposition 3.27. In a ϕ-metric space (X, d_ϕ), every convergent sequence has unique limit.

Proof . Since (X, τ_ϕ) is a Hausdorff topological space, the result is obvious. □

Proposition 3.28. Let (X, d_ϕ) be a ϕ-metric space and d be the metric on X whose topology is identical to the ϕ-metric topology. Then for any sequence $\{x_n\} \subseteq X$ and $x \in X$,

$$\lim_{n\to\infty} d_\phi(x_n, x) = 0 \text{ if and only if } \lim_{n\to\infty} d(x_n, x) = 0.$$

Proof . First assume that $\lim\limits_{n\to\infty} d_\phi(x_n, x) = 0$. Then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_\phi(x_n, x) < \epsilon \text{ for all } n \geq N \ \text{ that is } \ x_n \in B_\phi(x, \epsilon) \ \text{ for all } n \geq N.$$

Hence, there exists $\delta(\epsilon) > 0$ such that $x_n \in B_\phi(x, \epsilon) \subset B(x, \delta)$ for all $n \geq N$. Therefore, $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

Conversely assume $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} d(x_n, x) &< \epsilon \ \text{ for all } n \geq N \\ \textit{or} \ \ x_n &\in B(x, \epsilon) \text{ for all } n \geq N. \end{aligned}$$

Hence there exists $\delta(\epsilon) > 0$ such that $x_n \in B(x, \epsilon) \subset B_\phi(x, \delta)$ for all $n \geq N$. Thus, $\lim\limits_{n\to\infty} d_\phi(x_n, x) = 0$. Hence the proof is complete. □

Proposition 3.29. Let (X, d_ϕ) be a ϕ-metric space and $\{x_n\}$ and $\{y_n\}$ be two sequences in X converging to x and y respectively. Then the sequence $\{d_\phi(x_n, y_n)\}$ converges to $d_\phi(x, y)$.

Proof . Let $\epsilon > 0$. We have,

$$\begin{aligned} d_\phi(x, y) &\leq d_\phi(x, x_n) + d_\phi(x_n, y) + \phi(x, y, x_n) \\ &\leq d_\phi(x, x_n) + d_\phi(x_n, y_n) + d_\phi(y_n, y) + \phi(x_n, y, y_n) + \phi(x, y, x_n) \end{aligned}$$

and

$$\begin{aligned} d_\phi(x_n, y_n) \leq& d_\phi(x_n, x) + d_\phi(x, y_n) + \phi(x_n, y_n, x) \\ \leq& d_\phi(x_n, x) + d_\phi(x, y) + d_\phi(y, y_n) + \phi(x, y_n, y) + \phi(x_n, y_n, x). \end{aligned}$$

Now for $\frac{\epsilon}{4}(>0)$ there exists $\beta_1, \beta_2 > 0$ such that $\phi(z, y, y_n) < \frac{\epsilon}{4},\ \phi(z, y_n, y) < \frac{\epsilon}{4}$ whenever $d_\phi(y_n, y) < \beta_1,\ z \in X$ and $\phi(x, w, x_n) < \frac{\epsilon}{4},\ \phi(x_n, w, x) < \frac{\epsilon}{4}$ whenever $d_\phi(x_n, x) < \beta_2,\ w \in X$. Let $\delta = \min\{\beta_1, \beta_2, \frac{\epsilon}{4}\}$. Then for $\delta > 0$ there exists $N_1, N_2 \in \mathbb{N}$ such that $d_\phi(x_n, x) < \delta$ for all $n \geq N_1$ and $d_\phi(y_n, y) < \delta$ for all $n \geq N_2$. Take $\max\{N_1, N_2\} = N(say)$. Then for all $n \geq N$, $d_\phi(x_n, x) < \delta$ and $d_\phi(y_n, y) < \delta$ implies $\phi(x, y, x_n) < \frac{\epsilon}{4}$, $\phi(x_n, y, y_n) < \frac{\epsilon}{4}$, $\phi(x_n, y_n, x) < \frac{\epsilon}{4}$ and $\phi(x, y_n, y) < \frac{\epsilon}{4}$. Thus for all $n \geq N$ we have,

$$d_\phi(x, y) < \delta + d_\phi(x_n, y_n) + \delta + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq 4 \cdot \frac{\epsilon}{4} + d_\phi(x_n, y_n) = \epsilon + d_\phi(x_n, y_n)$$

and

$$d_\phi(x_n, y_n) < \delta + d_\phi(x, y) + \delta + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq 4 \cdot \frac{\epsilon}{4} + d_\phi(x, y) = \epsilon + d_\phi(x, y).$$

Since $\epsilon > 0$ is arbitrary, by taking limit as $n \to \infty$ on both side, we obtain $d_\phi(x, y) \leq \lim_{n\to\infty} d_\phi(x_n, y_n)$ and $\lim_{n\to\infty} d_\phi(x_n, y_n) \leq d_\phi(x, y)$. Which implies $\lim_{n\to\infty} d_\phi(x_n, y_n) = d(x, y)$. □

Definition 3.30. In a ϕ-metric space (X, d_ϕ), a sequence $\{x_n\} \subseteq X$ is said to be Cauchy if for any $\epsilon > 0$, there exists a positive integer N such that

$$d_\phi(x_n, x_m) < \epsilon \text{ for all } m, n \geq N \quad \text{that is } d_\phi(x_n, x_m) \to 0 \text{ as } m, n \to \infty.$$

Proposition 3.31. In a ϕ-metric space (X, d_ϕ), every convergent sequence is Cauchy.

Proof . Let $\epsilon > 0$ and $\{x_n\} \subseteq X$ converges to x. Now,

$$d_\phi(x_m, x_n) \leq d_\phi(x_m, x) + d_\phi(x, x_n) + \phi(x_m, x_n, x) \text{ for all } m, n \in \mathbb{N}.$$

For $\frac{\epsilon}{3}$ there exists $\beta > 0$ such that $\phi(x_m, x_n, x) < \frac{\epsilon}{3}$ whenever $d_\phi(x_m, x) < \beta$. Let $\delta = \min\{\beta, \frac{\epsilon}{3}\}$. Again for $\delta > 0$ there exists a natural number N such that $d_\phi(x_n, x) < \delta$ for all $n \geq N$. So for all $m, n \geq N,\ d_\phi(x_m, x) < \delta, d_\phi(x_n, x) < \delta$ and $\phi(x_m, x_n, x) < \frac{\epsilon}{3}$. Thus for all $m, n \geq N$,

$$d_\phi(x_m, x_n) < \delta + \delta + \frac{\epsilon}{3} \leq 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. □

As in metric space, we can define the boundedness of a set in a ϕ-metric space.

Definition 3.32. Let (X, d_ϕ) be a ϕ-metric space. $A \subseteq X$ is said to be bounded if there exists a non-negative real number K such that $d_\phi(x, y) < K$ for all $x, y \in A$.

Proposition 3.33. Every convergent sequence a ϕ-metric space (X, d_ϕ) is bounded.

Proof . Let $\epsilon > 0$ and $\{x_n\}$ be a sequence in X converging to $x \in X$. Again we have,

$$d_\phi(x_n, x_m) \leq d_\phi(x_n, x) + d_\phi(x_m, x) + \phi(x_n, x_m, x) \text{ for all } m, n \in \mathbb{N}.$$

Let us choose $\epsilon = 1$. For $\epsilon = 1$ there exists $\beta > 0$ such that $\phi(x_n, x_m, x) < 1$ whenever $d_\phi(x_m, x) < \beta$. Let $\delta = \min\{\beta, 1\}$. Then for $\delta > 0$ there exists $N \in \mathbb{N}$ such that $d_\phi(x_m, x) < \delta$ for all $m \geq N$. So for $m, n \geq N,\ d_\phi(x_n, x) < \delta,\ d_\phi(x_m, x) < \delta$ and $\phi(x_n, x_m, x) < 1$. Then for all $m, n \geq N$,

$$d_\phi(x_n, x_m) < \delta + \delta + 1 \leq 3.$$

Now suppose $M = \max\{d_\phi(x_r, x_s) : 1 \leq r, s < N\}$. Then $d_\phi(x_n, x_m) \leq M$ for all $m, n < N$. If $K = \max\{M, 3\}$ then $d_\phi(x_n, x_m) \leq K$ for all $m, n \in \mathbb{N}$. This completes the proof. □

***Remark* 3.34.** The converse of Proposition 3.31 and Proposition 3.33 are not true in general, since converse result of those statements do not hold for metric spaces.

***Remark* 3.35.** As open ball in an S-metric (or b-metric) space is same as the open ball in induced ϕ-metric space, so the definition convergence of a sequence is also same in both cases.

4 Some basic properties of ϕ-metric spaces

In Example 3.4, we have shown that the product of a finite number of ϕ-metrics on a non-empty set is also a ϕ-metric on that set. Now if we consider the set of all ϕ-metrics on a non-empty set X with the binary operation multiplication$(\cdot)$ then we obtain an algebraic structure, say $(\mathbb{D}, \cdot)$. Let us discuss its structure in detail. Already we have shown $\mathbb{D}$ is closed under '$\cdot$' and obviously '$\cdot$' is both commutative and associative on $\mathbb{D}$. So $(\mathbb{D}, \cdot)$ forms a commutative semigroup. The discrete metric is the only idempotent element and there is no nilpotent element in $(\mathbb{D}, \cdot)$. Moreover $(\mathbb{D}, \cdot)$ is a commutative monoid as the discrete metric on X acts as the identity element. Next, we are interested to study the topology generated by a finite number of ϕ-metrics. It is enough to study for the product of only two ϕ-metrics.

Proposition 4.1. Consider the ϕ-metrics $D_\phi,\ d_{\phi_1},\ d_{\phi_2}$ where $D_\phi = d_{\phi_1} \cdot d_{\phi_2}$ and $\tau_\phi,\ \tau_{\phi_1},\ \tau_{\phi_2}$ are the topologies induced by the open balls of $D_\phi,\ d_{\phi_1}$ and d_{ϕ_2} respectively. Then $\tau_\phi = \tau_{\phi_1} \cap \tau_{\phi_2}$.

Proof . Let us denote the open balls of (X, d_{ϕ_i}) by $B_{\phi_i}(x, r),\ i = 1, 2$ and that of (X, D_ϕ) by $B_\phi(x, r)$, for some $x \in X,\ r > 0$. Choose $x \in X$ and $r > 0$ such that $y \in B_{\phi_1}(x, r) \cap B_{\phi_2}(x, r)$. Then, $d_{\phi_i}(x, y) < r$ for $i = 1, 2$. Therefore,

$$D_\phi(x, y) < r^2 \ \ \textit{that is} \ \ y \in B_\phi(x, r^2).$$

This implies $\tau_{\phi_1} \cap \tau_{\phi_2} \subseteq \tau_\phi$. Again if $\{x_n\}$ converges to x in (X, d_{ϕ_1}) then $\{x_n\}$ converges to x in (X, D_ϕ). So, for all $\epsilon > 0$ there exists two positive integers N_1, N_2 such that

$$\begin{aligned} & d_{\phi_1}(x_n, x) < \epsilon \text{ for all } n \geq N_1 \text{ implies } D_\phi(x_n, x) < \epsilon \text{ for all } n \geq N_2 \\ \textit{or} \ \ & \text{for all } n \geq N,\ x_n \in B_{\phi_1}(x, \epsilon) \text{ implies } x_n \in B_\phi(x, \epsilon) \ \ \text{where } N = \max\{N_1, N_2\} \end{aligned}$$

This implies $\tau_\phi \subseteq \tau_{\phi_1}$. Similarly, $\tau_\phi \subseteq \tau_{\phi_2}$. Therefore, $\tau_\phi \subseteq \tau_{\phi_1} \cap \tau_{\phi_2}$. This completes the proof. $\square$

In the previous section, we have defined a bounded set in which the distance between two elements is finite. This leads us to define the diameter of a set and encourages us to check the relation between the diameter of a set and its closure.

Definition 4.2. Let (X, d_ϕ) be a ϕ-metric space. Diameter of a set $F \subseteq X$ denoted by $\delta(F)$ and defined by $\delta(F) = \sup\limits_{x,y \in F} d_\phi(x, y)$. Therefore F is said to be bounded if $\delta(F) < \infty$, otherwise unbounded.

Theorem 4.3. For a subset A of a ϕ-metric space $(X, d_\phi),\ \delta(\overline{A}) = \delta(A)$ where $\overline{A}$ denotes closure of A.

Proof . Since $A \subseteq \overline{A}$,

$$\delta(A) \leq \delta(\overline{A}). \tag{4.1}$$

Next choose any $\epsilon > 0$ and $x, y \in \overline{A}$. Now for $x, y \in \overline{A}$ and $a, b \in X$,

$$\begin{aligned} d_\phi(x, y) &\leq d_\phi(x, a) + d_\phi(a, y) + \phi(x, y, a) \\ &\leq d_\phi(x, a) + d_\phi(a, b) + d_\phi(b, y) + \phi(a, y, b) + \phi(x, y, a). \end{aligned}$$

Again for $\frac{\epsilon}{4}$ there exists $\beta_1,\ \beta_2 > 0$ such that $\phi(a, y, b) < \frac{\epsilon}{4}$ whenever $d_\phi(b, y) < \beta_1$ and $\phi(x, y, a) < \frac{\epsilon}{4}$ whenever $d_\phi(a, x) < \beta_2$. Let $\gamma = \min\{\beta_1, \beta_2, \frac{\epsilon}{4}\}$. Since $x, y \in \overline{A}$, there exists $x_1 \in A \cap B_\phi(x, \gamma)$ and $y_1 \in A \cap B_\phi(y, \gamma)$. Therefore, $d_\phi(x, x_1) < \gamma,\ d_\phi(y_1, y) < \gamma$ and hence $\phi(x, y, x_1) < \frac{\epsilon}{4},\ \phi(x_1, y, y_1) < \frac{\epsilon}{4}$. Hence for $x, y \in \overline{A}$ and $x_1 \in A \cap B_\phi(x, \gamma),\ y_1 \in A \cap B_\phi(y, \gamma)$, we have

$$\begin{aligned} d_\phi(x, y) &\leq d_\phi(x, x_1) + d_\phi(x_1, y_1) + d_\phi(y_1, y) + \phi(x_1, y, y_1) + \phi(x, y, x_1) \\ &< \gamma + d_\phi(x_1, y_1) + \gamma + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &\leq 4 \cdot \frac{\epsilon}{4} + d_\phi(x_1, y_1) \leq \epsilon + \sup_{a,b \in A} d_\phi(a, b) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\sup_{a,b \in \overline{A}} d_\phi(a, b) \leq \sup_{a,b \in A} d_\phi(a, b),$$

that is

$$\delta(\overline{A}) \leq \delta(A). \tag{4.2}$$

The relations (4.1) and (4.2) together gives $\delta(\overline{A}) = \delta(A)$. $\square$

We now discuss compactness, the most useful notion of a topological space including completeness.

Theorem 4.4. A ϕ-metric space (X, d_ϕ) is compact if and only if it is sequentially compact.

Proof . Since (X, d_ϕ) is metrizable, there exists a metric on X, say d whose topology is identical with the ϕ-metric topology. Then (X, d_ϕ) is compact if and only if (X, d) is compact if and only if (X, d) is sequentially compact if and only if (X, d_ϕ) is sequentially compact. $\square$

Theorem 4.5. Every compact ϕ-metric space (X, d_ϕ) is closed and bounded.

Proof . If possible, suppose X is not closed. So there exists a sequence of points $\{x_n\}$ such that $x_n \in X$ converges to a point $x \notin X$. Since X is compact, $\{x_n\}$ has a subsequence which converges to a point in X. But subsequence must converge to x which does not belong to X. This contradicts the compactness of X. Hence X is closed. Next, we prove that X is bounded. If possible suppose that X is unbounded and choose $x_0 \in X$ any fixed element. Since X is unbounded, there exist $x_1 \in A$ such that $d_\phi(x_1, x_0) > 1$. Similarly, there exists $x_2 \in X$ such that $d_\phi(x_2, x_0) > 2$. Continuing in this way, there exists $x_n \in X$ such that $d_\phi(x_n, x_0) > n$ for all $n \in \mathbb{N}$. Since X is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_{n_k} = x \in X$. But we have, $d_\phi(x_{n_k}, x_0) > n_k$. Again,

$$d_\phi(x_{n_k}, x_0) \leq d_\phi(x_{n_k}, x) + d_\phi(x, x_0) + \phi(x_{n_k}, x_0, x) \tag{4.3}$$

Let $\epsilon > 0$ be given. Now for $\epsilon > 0$ there exists $\delta > 0$ such that $\phi(x_l, x_0, x) < \frac{\epsilon}{2}$ whenever $d_\phi(x_l, x) < \delta$. Let $\beta = \min\{\frac{\epsilon}{2}, \delta\}$. Since $x_{n_k} \to x$ as $k \to \infty$, there exists $N \in \mathbb{N}$ such that $d_\phi(x_{n_k}, x) < \beta$ for all $k \geq N$.
Therefore for all $k \geq N$, $d_\phi(x_{n_k}, x) < \beta$ and $\phi(x_{n_k}, x_0, x) < \frac{\epsilon}{2}$. Thus the relation (4.3) gives, for all $k \geq N$,

$$n_k < d_\phi(x_{n_k}, x_0) < \beta + d_\phi(x_0, x) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + d_\phi(x_0, x) + \frac{\epsilon}{2} = \epsilon + d_\phi(x_0, x).$$

Taking limit as $k \to \infty$ on both sides of the above inequality, we obtain $\infty \leq d_\phi(x, x_0)$. This contradicts that d_ϕ is a real valued function. Hence X is bounded. $\square$

***Remark* 4.6.** Since each metric space is also a ϕ-metric space and the converse result of Theorem 4.5 does not hold in metric space, thus the converse result of Theorem 4.5 may not be true.

Definition 4.7. A ϕ-metric space (X, d_ϕ) is said to be complete if every Cauchy sequence in X converges to some point in X.

Theorem 4.8. Every compact ϕ-metric space (X, d_ϕ) is complete.

Proof . Let $\epsilon > 0$ and $\{x_n\}$ be a Cauchy sequence in the compact ϕ-metric space (X, d_ϕ). So there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_{k_n} = x \in X$. Now we have,

$$\begin{aligned} d_\phi(x_n, x) &\leq d_\phi(x_n, x_l) + d_\phi(x_l, x) + \phi(x_n, x, x_l) \\ &\leq d_\phi(x_n, x_l) + d_\phi(x_{k_m}, x_l) + d_\phi(x_{k_m}, x) + \phi(x_l, x, x_{k_m}) + \phi(x_n, x, x_l). \end{aligned}$$

For $\frac{\epsilon}{5} > 0$ there exists $\delta_1,\ \delta_2 > 0$ such that $\phi(x_n, x, x_l) < \frac{\epsilon}{5}$ whenever $d_\phi(x_n, x_l) < \delta_1$ and $\phi(x_l, x, x_{k_m}) < \frac{\epsilon}{5}$ whenever $d_\phi(x_{k_m}, x) < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2, \frac{\epsilon}{5}\}$. Since $\{x_n\}$ is a Cauchy sequence, for $\delta > 0$ there exists a positive integer n_0 such that

$$d_\phi(x_n, x_m) < \delta \ \text{ for all } n, m \geq n_0.$$

In particular,

$$d_\phi(x_n, x_{n_0}) < \delta \ \text{ for all } n \geq n_0. \tag{4.4}$$

Again $x_{k_n} \to x$ as $n \to \infty$ implies there exists $m \in \mathbb{N}$ such that

$$d_\phi(x_{k_m}, x) < \delta \text{ for all } m \geq n_0. \tag{4.5}$$

Since $k_m \geq m \geq n_0$, from (4.4),

$$d_\phi(x_{k_m}, x_{n_0}) < \delta. \tag{4.6}$$

Therefore for $n \geq n_0$, $d_\phi(x_n, x_{n_0}) < \delta$, $d_\phi(x_{k_n}, x) < \delta$ and $\phi(x_n, x, x_{n_0}) < \frac{\epsilon}{5}$, $\phi(x_{n_0}, x, x_{k_m}) < \frac{\epsilon}{5}$. So for all $n \geq n_0$,

$$\begin{aligned} d_\phi(x_n, x) \leq \; & d_\phi(x_n, x_{n_0}) + d_\phi(x_{k_n}, x_{n_0}) + d_\phi(x_{k_n}, x) + \phi(x_{n_0}, x, x_{k_n}) + \phi(x_n, x, x_{n_0}) \\ < \; & \delta + \delta + \delta + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\ \leq \; & \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + 2\frac{\epsilon}{5} = \epsilon. \end{aligned}$$

Hence the Cauchy sequence $\{x_n\}$ converges to $x \in X$ and this proves that X is complete. □

***Remark* 4.9.** Since each metric space is also a ϕ-metric space and the converse result of Theorem 4.8 does not hold in metric space, thus the converse result of Theorem 4.8 may not be true.

Cantor's intersection theorem ensures the completeness of a metric space. Our next theorem is the generalization of such theorem in a ϕ-metric space.

Theorem 4.10. A necessary and sufficient condition that the ϕ-metric space (X, d_ϕ) be complete is that every nested sequence of non-empty closed subsets $\{F_i\}$ with $\delta(F_i) \to 0$ as $i \to \infty$ be such that $F = \cap_{i=1}^{\infty} F_i$ contains exactly one point.

Proof . First suppose that X is complete. Consider a sequence of closed subsets $\{F_i\}$ such that $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\delta(F_i) \to 0$ as $i \to \infty$. For all $n \in \mathbb{N}$ choose $a_n \in F_n$. Hence we generate a sequence $\{a_n\}$ in X. We verify that the sequence $\{a_n\}$ is a Cauchy sequence. Now for some $n \in \mathbb{N}$, $a_n \in F_n$ implies $a_{n+p} \in F_{n+p} \subset F_n$ for all $p = 1, 2, \cdots$. So, for all $p = 1, 2, \cdots$, $d_\phi(a_n, a_{n+p}) \leq \delta(F_n)$ for all $n \in \mathbb{N}$ which implies $\lim\limits_{n\to\infty} d_\phi(a_n, a_{n+p}) = 0$. Hence $\{a_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $a \in X$ such that $a_n \to a$ as $n \to \infty$. For a fixed positive integer k, consider the subset F_k. Then each $a_k, a_{k+1}, a_{k+2}, \cdots \in F_k$. Since F_k is closed, $a \in F_k$. Now k being an arbitrary positive integer, so we can conclude $a \in \bigcap\limits_{i\in\mathbb{N}} F_i$. Finally we show that a is unique. For, let there exists $b (\neq a) \in \bigcap\limits_{i\in\mathbb{N}} F_i$. Then for each $k \in \mathbb{N}$,

$$a, b \in F_k \text{ that is } d_\phi(a, b) \leq \delta(F_k).$$

Therefore, $d_\phi(a, b) = 0$, since $\delta(F_k) \to 0$ as $k \to \infty$ and hence $a = b$.

Conversely suppose that the condition of the theorem holds. To show that X is complete, take a Cauchy sequence $\{x_n\}$ in X. Let $F_n = \{x_n, x_{n+1}, x_{n+2}, \cdots\}$ for all $n \in \mathbb{N}$. If we choose any $\epsilon > 0$, then there exists a positive integer n_0(say) such that

$$\begin{aligned} & d_\phi(x_n, x_m) < \epsilon \quad \text{for all } n > m \geq n_0 \\ \text{or} \quad & \delta(F_n) \leq \epsilon \text{ for all } n \geq n_0 \\ \text{or} \quad & \delta(\overline{F_n}) \leq \epsilon \text{ for all } n \geq n_0 \\ \text{or} \quad & \delta(\overline{F_n}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Clearly $F_{n+1} \subset F_n$ for each n and thus $\overline{F_{n+1}} \subset \overline{F_n}$ for each n. So $\{\overline{F_n}\}$ constitutes a closed, nested sequence of non-empty sets in X whose diameter tends to zero. By hypothesis, there exists a unique point $x \in \bigcap\limits_{n\in\mathbb{N}} \overline{F_n}$. Now for each $n = 1, 2, \cdots$, $x_n \in F_n \subseteq \overline{F_n}$ implies

$$d_\phi(x_n, x) \leq \delta(\overline{F_n}) \text{ for all } n \in \mathbb{N}.$$

Therefore, $d_\phi(x_n, x) \to 0$ as $n \to \infty$. This shows that the Cauchy sequence $\{x_n\}$ converges to $x \in X$ and hence X is complete. □

Now we introduce an idea which is stronger than boundedness.

Definition 4.11. Let (X, d_ϕ) be a ϕ-metric space and $B \subseteq X$. B is said to be totally bounded if for every $\epsilon > 0$ there exists a finite subset A_ϵ of X such that $B = \underset{a \in A_\epsilon}{\cup} B_\phi(a, \epsilon)$.

Theorem 4.12. Every totally bounded subset of a ϕ-metric space (X, d_ϕ) is bounded.

Proof . Let $\epsilon > 0$, and B be a totally bounded subset of (X, d_ϕ). For any $\alpha, \beta \in B$ and for any $x, y \in X$,

$$\begin{aligned} d_\phi(\alpha,\beta) &\le d_\phi(\alpha,x) + d_\phi(x,\beta) + \phi(\alpha,\beta,x) \\ &\le d_\phi(\alpha,x) + d_\phi(x,y) + d_\phi(y,\beta) + \phi(x,\beta,y) + \phi(\alpha,\beta,x). \end{aligned}$$

We choose $\epsilon = 1$. Then for $\epsilon = 1$ there exists $\delta_1, \delta_2 > 0$ such that for any $\alpha, \beta \in B$, $\phi(x, \beta, y) < 1$ whenever $d_\phi(y, \beta) < \delta_1,\ x, y \in X$ and $\phi(\alpha, \beta, x) < 1$ whenever $d_\phi(x, \alpha) < \delta_2,\ x \in X$. Let $\delta = \min\{1, \delta_1, \delta_2\}$. Since B is a totally bounded set, so for $\delta > 0$, there exists a finite subset $S = \{x_1, x_2, \cdots, x_n\}$ of X such that $B = \cup_{x_i=1}^{n} B_\phi(x_i, \delta)$. Choose any $\alpha, \beta \in B$. Then there exists $x_i, x_j \in S$ such that $\alpha \in B_\phi(x_i, \delta)$ and $\beta \in B_\phi(x_j, \delta)$. Hence we obtain $d_\phi(\alpha, x_i) < \delta,\ d_\phi(x_j, \beta) < \delta$ and $\phi(x_i, \beta, x_j) < 1,\ \phi(\alpha, \beta, x_i) < 1$. Suppose $K = \max\{d_\phi(x_i, x_j) : x_i, x_j \in S\}$. Therefore for any $x_i, x_j \in S$ and $\alpha, \beta \in B$,

$$\begin{aligned} d_\phi(\alpha,\beta) &< \delta + d_\phi(x_i,x_j) + \delta + 1 + 1 \\ &\le 1 + \max\{d_\phi(x_i,x_j) : x_i, x_j \in S\} + 1 + 2 = \ 4 + K. \end{aligned}$$

Since $\alpha, \beta \in B$ are arbitrary, $d_\phi(\alpha, \beta) \le K + 4$ for all $\alpha, \beta \in B$. This proves that B is bounded. □

Theorem 4.13. In a totally bounded ϕ-metric space (X, d_ϕ), every sequence has a Cauchy subsequence.

Proof . Let $\{x_n\}$ be a sequence in X. Since X is totally bounded so it can be covered by a finite number of open balls of any radius. Let us consider balls of radius 1. Then at least one of these open balls, say A_1 contains infinitely many elements of the sequence. Choose $x_{k_1} \in A_1$ for some $k_1 \in \mathbb{N}$. Similarly, A_1 being totally bounded can be covered by a finite number of open balls each of radius $\frac{1}{2}$. Then at least one of these open balls, say A_2 contains infinitely many elements of the sequence. We choose $x_{k_2} \in A_2$ for some $k_1 < k_2 \in \mathbb{N}$. Continuing in this way we obtain a sequence $\{A_n\}$ of open balls with radius $\frac{1}{n}$ such that $A_1 \supset A_2 \supset \cdots$ and $x_{k_n} \in A_{k_n}$ with $k_1 < k_2 \cdots$. Clearly $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$. Choose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. Now for all $r, s \ge N,\ x_{k_r}, x_{k_s} \in A_N$ and hence $d_\phi(x_{k_r}, x_{k_s}) < \frac{2}{N} < \epsilon$ which implies $\{x_{k_n}\}$ is a Cauchy sequence in X. □

Theorem 4.14. Every compact ϕ-metric space (X, d_ϕ) is totally bounded.

Proof . From the compactness of X it follows that, for every $\epsilon > 0$, $\beta = \{B_\phi(a, \epsilon) : a \in X\}$ is an open cover of X and there exists a finite subset of β which covers X. Therefore X is totally bounded. □

Remark **4.15.** The converse of the Theorem 4.12 and Theorem 4.14 do not hold in general. Since metric spaces are also ϕ-metric spaces, so this can be justified by the examples of metric spaces.

But totally boundedness and completeness together force the ϕ-metric space to be compact. We prove this in our next theorem.

Theorem 4.16. Let (X, d_ϕ) be a ϕ-metric space. If X is totally bounded and complete then X is compact.

Proof . Let $\{x_n\}$ be a Cauchy sequence in X. So totally boundedness of X implies that $\{x_n\}$ has a Cauchy subsequence, say $\{x_{k_n}\}$. Since X is complete, thus $\{x_{k_n}\}$ converges in X. Therefore X is sequentially compact and hence compact. □

5 Some fixed point theorems in ϕ-metric spaces

In this section, we establish the existence of a fixed point for the Banach type and the Kannan type contraction principle and also develop the Edelstein theorem in ϕ-metric spaces.
Before going to the main results, we prove a useful lemma.

Lemma 5.1. Let (X, d_ϕ) be a ϕ-metric space. If $\{x_n\}_{n=0}^{\infty}$ be a sequence in X which satisfies

$$d_\phi(x_n, x_{n+1}) \leq k d_\phi(x_{n-1}, x_n),\ n = 1, 2, \cdots$$

where $0 < k < 1$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof . Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X which satisfies the mentioned conditions. Let $\epsilon > 0$. Now for all $n = 1, 2, \cdots$,

$$d_\phi(x_n, x_{n+1}) \leq k d_\phi(x_{n-1}, x_n)$$

implies

$$\begin{aligned} d_\phi(x_n, x_{n+1}) &\leq k^2 d_\phi(x_{n-2}, x_{n-1}) \\ &\leq k^3 d_\phi(x_{n-3}, x_{n-2}) \leq \cdots \leq k^n d_\phi(x_0, x_1). \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} d_\phi(x_n, x_{n+1}) = 0$ since $0 < k < 1$. Again for $m > n$ we have,

$$\begin{aligned} d_\phi(x_m, x_n) &\leq d_\phi(x_m, x_{n+1}) + d_\phi(x_{n+1}, x_n) + \phi(x_m, x_n, x_{n+1}) \\ &\leq \{d_\phi(x_m, x_{n+2}) + d_\phi(x_{n+2}, x_{n+1}) + \phi(x_m, x_{n+1}, x_{n+2})\} + d_\phi(x_n, x_{n+1}) + \\ &\quad \phi(x_m, x_n, x_{n+1}) \\ &\leq \{d_\phi(x_m, x_{m-1}) + ... + d_\phi(x_{n+1}, x_{n+2}) + d_\phi(x_{n+1}, x_n)\} + \\ &\quad \{\phi(x_m, x_n, x_{n+1}) + \phi(x_m, x_{n+1}, x_{n+2}) + ... + \phi(x_m, x_{m-2}, x_{m-1})\} \\ &\leq \{k^{m-1} + k^{m-2} + ... + k^{n+1} + k^n\} d_\phi(x_1, x_0) + \sum_{i=n}^{m-2} \phi(x_m, x_i, x_{i+1}) \\ &\leq k^n\{1 + k + k^2 + \cdots\} d_\phi(x_1, x_0) + \sum_{i=n}^{m-2} \phi(x_m, x_i, x_{i+1}). \end{aligned}$$

So,

$$d_\phi(x_m, x_n) \leq \frac{k^n}{1-k} d_\phi(x_1, x_0) + \sum_{i=n}^{m-2} \phi(x_m, x_i, x_{i+1}) \text{ for all } m > n. \tag{5.1}$$

Now for $\epsilon > 0$ there exists $\delta > 0$ such that $\phi(x_t, x_l, x_{l+1}) < \epsilon$ whenever $d_\phi(x_l, x_{l+1}) < \delta$. Let $\beta = \min\{\epsilon, \delta\}$. Then since $\lim\limits_{i\to\infty} d_\phi(x_i, x_{i+1}) = 0$ so for that $\beta > 0$ there exists a positive integer N such that $d_\phi(x_i, x_{i+1}) < \beta$ for all $i \geq N$.

Thus for all $m > n \geq N$ we have $d_\phi(x_n, x_{n+1}) < \beta$ and $\phi(x_m, x_n, x_{n+1}) < \epsilon$. Since $\epsilon > 0$ arbitrarily chosen, so $\lim\limits_{n\to\infty} d_\phi(x_n, x_{n+1}) = 0$ implies $\lim\limits_{m,n\to\infty} \phi(x_m, x_n, x_{n+1}) = 0$. Hence the relation (5.1) gives $\mathop{lim}\limits_{m,n\to\infty} d_\phi(x_n, x_m) = 0$ which implies $\{x_n\}$ is a Cauchy sequence in X. $\square$

Theorem 5.2. (Banach type contraction) Let (X, d_ϕ) be a complete ϕ-metric space and T be a self-mapping on X satisfying

$$d_\phi(Tx, Ty) \leq k d_\phi(x, y)$$

for all $x, y \in X$ where $k \in (0, 1)$. Then T has a unique fixed point in X.

Proof . If any fixed point of T exists then uniqueness directly follows from the contraction condition. Here we only prove the existence of a fixed point. For, consider an iterative sequence, $x_0,\ x_1 = Tx_0,\ x_2 = Tx_1, ..., x_{n+1} = Tx_n, \cdots$, for some fixed $x_0 \in X$. Then,

$$d_\phi(x_{n+1}, x_n) = d_\phi(Tx_n, Tx_n - 1) \leq k d_\phi(x_n, x_{n-1}) \quad \text{for all } n = 1, 2, \cdots.$$

Hence by Lemma 5.1, we can conclude $\{x_n\}$ is a Cauchy sequence in (X, d_ϕ). Since X is complete, so $\{x_n\}$ converges to some $\zeta \in X$. Lastly, we will prove that ζ is a fixed point for T. Now

$$d_\phi(T\zeta, \zeta) = \lim_{n\to\infty} d_\phi(T\zeta, x_n) \leq k \lim_{n\to\infty} d_\phi(\zeta, x_{n-1})$$

which implies $d_\phi(T\zeta, \zeta) = 0$ that is $T\zeta = \zeta$. $\square$

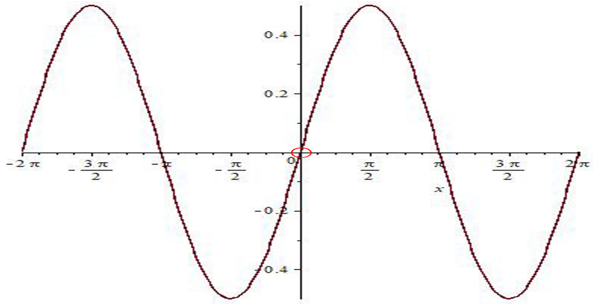

Figure 1: $Tx = \dfrac{\sin x}{2} \quad x \in \mathbb{R}$

Example 5.3. Consider the complete ϕ-metric space (X, d_ϕ) where $X = \mathbb{R}$ and the ϕ-metric defined by $d_\phi(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Let us define a self-mapping T on $\mathbb{R}$ defined by $Tx = \frac{\sin x}{2}$ for all $x \in \mathbb{R}$. Then

$$d_\phi(Tx, Ty) = \frac{1}{4}(\sin x - \sin y)^2 = [\cos(\frac{x+y}{2}) \cdot \sin(\frac{x-y}{2})]^2 \le \frac{1}{4}(x-y)^2 = \frac{1}{4}d_\phi(x, y).$$

Thus T satisfies the Banach type contraction for $\frac{1}{4} \le k < 1$ and $x = 0$ is the unique fixed point for T.

Theorem 5.4. (Kannan type contraction) Let (X, d_ϕ) be a complete ϕ-metric space and T be a self-mapping on X satisfying

$$d_\phi(Tx, Ty) \le k[d_\phi(x, Tx) + d_\phi(y, Ty)]$$

for all $x, y \in X$, where $k \in (0, \frac{1}{2})$. Then T has a unique fixed point in X.

Proof . For some fixed $x_0 \in X$, consider the iterative sequence, $x_0,\ x_1 = Tx_0,\ x_2 = Tx_1, \cdots, x_{n+1} = Tx_n, \cdots$. Then,

$$\begin{aligned} &d_\phi(x_2, x_1) = d_\phi(Tx_1, Tx_0) \le k[d_\phi(x_1, x_2) + d_\phi(x_0, x_1)] \\ \text{or}\quad &d_\phi(x_2, x_1) \le \alpha d_\phi(x_1, x_0) \quad \text{where} \quad \alpha = \frac{k}{1-k} \quad \text{and } 0 < \alpha < 1 \end{aligned}$$

Proceeding in this way, we can write $d_\phi(x_{n+1}, x_n) \le \alpha d_\phi(x_n, x_{n-1})$ for all $n \in \mathbb{N}$ where $0 < \alpha < 1$. Applying Lemma 5.1, we can conclude $\{x_n\}$ is a Cauchy sequence in (X, d_ϕ) and since X is complete, so $\{x_n\}$ converges to some $\zeta \in X$. Now,

$$\begin{aligned} &d_\phi(T\zeta, \zeta) = \lim_{n\to\infty} d_\phi(T\zeta, x_n) \le k \lim_{n\to\infty}[d_\phi(\zeta, T\zeta) + d_\phi(x_{n-1}, x_n)] \\ \text{or}\quad &d_\phi(T\zeta, \zeta) \le k d_\phi(T\zeta, \zeta) \end{aligned}$$

Therefore, $d_\phi(T\zeta, \zeta) = 0$ as $0 < k < \frac{1}{2}$ which implies $T\zeta = \zeta$. Hence ζ is a fixed point of T and uniqueness easily follows from the contraction condition. $\square$

Example 5.5. Define a function d_ϕ on a set $X = [-2, 2]$ by $d_\phi(x, y) = (x - y)^2$ for all $x, y \in [-2, 2]$. Then (X, d_ϕ) is a complete ϕ-metric space. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{10} & \text{when } -2 \le x < 1 \\ \frac{x}{5} & \text{when } \quad 1 \le x \le 2 \end{cases}$$

For all $x, y \in \mathbb{R}$, $\quad |x|^2 + |y|^2 \ge 2|x||y|$. Hence we obtain,

$$\begin{aligned} |x - y|^2 &\le [|x| + |y|]^2 \\ &= |x|^2 + |y|^2 + 2|x||y| \\ &\le (|x|^2 + |y|^2) + (|x|^2 + |y|^2) \quad \text{for all } x, y \in \mathbb{R}. \end{aligned}$$

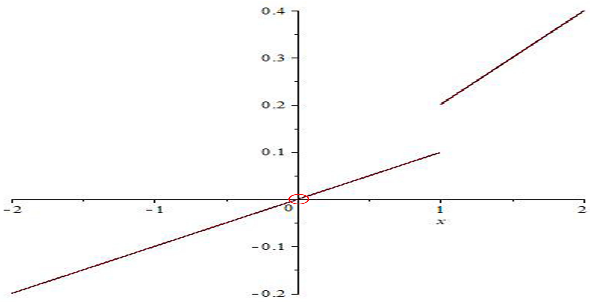

Figure 2: $Tx = \frac{x}{10}$ if $-2 \leq x < 1$; $\frac{x}{5}$ if $1 \leq x \leq 2$

That is

$$|x - y|^2 \leq 2(|x|^2 + |y|^2) \text{ for all } x, y \in \mathbb{R}. \tag{5.2}$$

Case:I Let $x, y \in [-2, 1)$. Then

$$d_\phi(x, Tx) + d_\phi(y, Ty) = (x - \frac{x}{10})^2 + (y - \frac{y}{10})^2 = \frac{81}{100}[|x|^2 + |y|^2]$$

and

$$\begin{aligned} d_\phi(Tx, Ty) &= \frac{1}{100}(x - y)^2 \\ &\leq \frac{2}{100}[|x|^2 + |y|^2], \text{ using the relation (5.2)} \\ &= \frac{2}{81}[d_\phi(x, Tx) + d_\phi(y, Ty)]. \end{aligned}$$

Case:II Let $x, y \in [1, 2]$. Then

$$d_\phi(x, Tx) + d_\phi(y, Ty) = (x - \frac{x}{5})^2 + (y - \frac{y}{5})^2 = \frac{16}{25}[|x|^2 + |y|^2]$$

and

$$\begin{aligned} d_\phi(Tx, Ty) &= \frac{1}{25}(x - y)^2 \\ &\leq \frac{2}{25}[|x|^2 + |y|^2], \text{ using the relation (5.2)} \\ &= \frac{1}{8}[d_\phi(x, Tx) + d_\phi(y, Ty)]. \end{aligned}$$

Case:III Let $x \in [-2, 1)$, $y \in [1, 2]$. Then

$$d_\phi(x, Tx) + d_\phi(y, Ty) = (x - \frac{x}{10})^2 + (y - \frac{y}{5})^2 = \frac{81}{100}|x|^2 + \frac{16}{25}|y|^2$$

and

$$\begin{aligned} d_\phi(Tx, Ty) &= (\frac{x}{10} - \frac{y}{5})^2 \\ &\leq 2[\frac{|x|^2}{100} + \frac{|y|^2}{25}], \text{ using the relation (5.2)} \\ &= \frac{2}{81} \cdot \frac{81}{100}|x|^2 + \frac{2}{16} \cdot \frac{16}{25}|y|^2 \\ &< \frac{1}{8}[d_\phi(x, Tx) + d_\phi(y, Ty)]. \end{aligned}$$

Therefore for all $x, y \in [-2, 2]$, $d_\phi(Tx, Ty) \leq k[d_\phi(x, Tx) + d_\phi(y, Ty)]$ where $k = \frac{1}{8}$.

Theorem 5.6. Let (X, d_ϕ) be a ϕ-metric space and T be a self-mapping on X satisfying

$$d_\phi(Tx, Ty) < d_\phi(x, y)$$

for all $x, y \in X$. If there exists $x \in X$ such that the sequence $\{T^n x\}$ has a subsequence converging to ζ then ζ is the unique fixed point of T.

Proof . Let $\{T^{n_i} x\}$ be a subsequence of $\{T^n x\}$ converging to ζ. If $T^k x = T^{k+1} x$ for some $k \in \mathbb{N}$, then ζ is a fixed point of T. If $T^k x \neq T^{k+1} x$ for any $k \in \mathbb{N}$ and $T\zeta \neq \zeta$ then

$$d_\phi(T\zeta, T^2\zeta) < d_\phi(\zeta, T\zeta). \tag{5.3}$$

For $x \in X$ and for fixed n_l, we have for all $n > n_l + 1$,

$$d_\phi(T^n x, T^{n+1} x) < d_\phi(T^{n_l+1} x, T^{n_l+2} x).$$

Clearly $d_\phi(\zeta, T\zeta)$ is a limit point of the sequence $\{d_\phi(T^n x, T^{n+1} x)\}$ and so

$$d_\phi(\zeta, T\zeta) \leq d_\phi(T^{n_l+1} x, T^{n_l+2} x) \tag{5.4}$$

In equation (5.4) letting $l \to \infty$, we get $d_\phi(\zeta, T\zeta) \leq d_\phi(T\zeta, T^2\zeta)$ which contradicts equation (5.3). So either $T^k x = T^{k+1} x$ for some $k \in \mathbb{N}$ or $T\zeta = \zeta$ or both. Hence ζ is a fixed point of T which is unique also. $\square$

Example 5.7. Consider the ϕ-metric space (X, d_ϕ) where $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the ϕ-metric defined by $d_\phi(x, y) = (x-y)^2$ for all $x, y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Let us define a self-mapping T on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by $Tx = \tan^{-1} x - x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then,

$$\begin{aligned}
d_\phi(Tx, Ty) =& |\tan^{-1} x - x - \tan^{-1} y + y|^2 \\
=& |(y - x) - (\tan^{-1} y - \tan^{-1} x)|^2 \\
=& |(y - x) - (\frac{y - x}{1 + \eta^2})|^2, \ \min\{x, y\} < \eta < \max\{x, y\} \text{ (by Lagrange's mean value theorem)} \\
=& |y - x|^2 \frac{\eta^2}{1 + \eta^2} \\
<& (x - y)^2 = d_\phi(x, y).
\end{aligned}$$

Thus T satisfies contractive condition and for $x = 0$, all subsequence of $\{T^n 0\} = \{0\}$ converges to 0. Hence from the Theorem 5.6 we conclude that $x = 0$ is the unique fixed point for T.

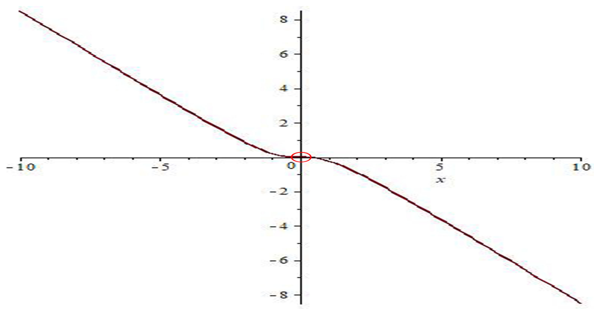

Figure 3: $Tx = \tan^{-1} x - x \ \ x \in [-\frac{\pi}{2}.\frac{\pi}{2}]$

Moreover, from the given self-mapping it is clear that $x = 0$ is the unique fixed point for T.

***Remark* 5.8.** (a) If we take S-metric space (X, S) then $d_\phi(x, y) = S(x, x, y)$ for all $x, y \in X$ defines a ϕ-metric on X. Then Theorem 5.2 and Theorem 5.6 reduce to Theorem 3.1 and Theorem 3.3 respectively of Section 3 of Sedghi et al. [6].

(b) If we consider b-metric space (X, B) then $d_\phi(x, y) = B(x, y)$ for all $x, y \in X$ defines a ϕ-metric on X. Then Theorem 5.2 and Theorem 5.4 reduce to Theorem 1 and Theorem 2 of the main results of respectively Mehmet Kir et al. [17].

6 Conclusion

In this article, we introduce a notion of generalized metric function called ϕ–metric, which generalizes the concept of many existing metric functions such as S-metric, b-metric etc. In the definition of ϕ–metric, the 'triangle inequality' of metric axioms has been modified by adding a suitable function. We also study the notion of convergence of a sequence, some elementary topological properties and some well known fixed point theorems in this new setting. We also have established the metrizability of ϕ-metric space.

ϕ-metric is not only for seeking generalization but also it helps to study S-metric, b-metric spaces and may play the role of metrics in many scenarios. As we have mentioned earlier, though S-metric and b-metric spaces are metrizable, it is very troublesome to find a metric for S-metric and b-metric spaces. This is true for ϕ-metric space also. However, one can easily construct a ϕ-metric from S-metric and b-metric which may help to study the topological properties for such spaces. We hope our developments will motivate researchers to work further on S-metric, b-metric, strong b-metric spaces with the help of ϕ-metric space. There is a lot of scope to exercise on metric fixed point theory and its applications on ϕ-metric spaces. We think one most important applicable direction of ϕ-metric space is that this idea may be applied to calculate distance between any two points on non-planer surfaces. We know that 'triangle inequality' is not necessarily affirm on non-planer surfaces, so usual metric is not suitable to develop topological properties in such spaces. In that case, ϕ-metric may play a pivotal role to study several properties of non-planer surfaces. In our subsequent work, we have a plan to study the topological properties of non-Euclidean geometry using ϕ-metric.

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