

## SOME PROPERTIES OF CONTINUOUS LINEAR OPERATORS IN TOPOLOGICAL VECTOR PN - SPACES

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**ABSTRACT.** The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance. Probabilistic Metric space was introduced by Karl Menger. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [1]. In this note we study the PN spaces which are topological vector spaces and the open mapping and closed graph Theorems in this spaces are proved.

### 1. INTRODUCTION AND PRELIMINARIES

Probabilistic normed spaces were first defined by Šerstnev in 1962 (see [16]). Their definition was generalized in [1]. We recall the definition of probabilistic normed space briefly as given in [1], together with the notation that will be needed (see [12]). We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left-continuous and non-decreasing functions from  $\overline{\mathbb{R}}$  into  $[0, 1]$  such that  $F(0) = 0$  and  $F(+\infty) = 1$ . Here as usual,  $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ . The spaces of these functions will be denoted by  $\Delta^+$ , while the subset  $D^+ \subseteq \Delta^+$  will denote the set of all proper distance d.f.'s, namely those for which  $\ell^-F(+\infty) = 1$ . Here  $\ell^-f(x)$  denotes the left limit of the function  $f$  at the point  $x$ . The space  $\Delta^+$  is partially ordered by the usual pointwise ordering of functions i.e.,  $F \leq G$  if and only if  $F(x) \leq G(x)$  for all  $x$  in  $\mathbb{R}$ . For any  $a \geq 0$ ,  $\varepsilon_a$  is the d.f. given by

$$\varepsilon_a = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

The space  $\Delta^+$  can be metrized in several ways [12], but we shall here adopt the Sibley metric  $d_S$ . If  $F, G$  are d.f.'s and  $h$  is in  $]0, 1[$ , let  $(F, G; h)$  denote the condition:

$$G(x) \leq F(x + h) + h \text{ for all } x \in \left]0, \frac{1}{h}\right[.$$

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*Date:* Received: March 2009; Revised: October 2009.

*2000 Mathematics Subject Classification.* 54E70.

*Key words and phrases.* continuous linear map, homomorphism, triangular function, topological vector space.

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Then the Sibley metric  $d_S$  is defined by

$$d_S(F, G) := \inf\{h \in ]0, 1[ : \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

In particular, under the usual pointwise ordering of functions,  $\varepsilon_0$  is the maximal element of  $\Delta^+$ . A triangle function is a binary operation on  $\Delta^+$ , namely a function  $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$  that is associative, commutative, nondecreasing in each place and has  $\varepsilon_0$  as identity, that is, for all  $F, G$  and  $H$  in  $\Delta^+$ :

$$(TF1) \quad \tau(\tau(F, G), H) = \tau(F, \tau(G, H)),$$

$$(TF2) \quad \tau(F, G) = \tau(G, F),$$

$$(TF3) \quad F \leq G \implies \tau(F, H) \leq \tau(G, H),$$

$$(TF4) \quad \tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F.$$

Moreover, a triangle function is *continuous* if it is continuous in the metric space  $(\Delta^+, d_S)$ .

Typical continuous triangle functions are

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

$$\tau_{T^*}(F, G) = \inf_{s+t=x} T^*(F(s), G(t)).$$

Here  $T$  is a continuous t-norm, i.e. a continuous binary operation on  $[0, 1]$  that is commutative, associative, nondecreasing in each variable and has 1 as identity;  $T^*$  is a continuous t-conorm, namely a continuous binary operation on  $[0, 1]$  which is related to the continuous t-norm  $T$  through  $T^*(x, y) = 1 - T(1 - x, 1 - y)$ . Let us recall among the triangular function one has the function defined via  $T(x, y) = \min(x, y) = M(x, y)$  and  $T^*(x, y) = \max(x, y)$  or  $T(x, y) = \pi(x, y) = xy$  and  $T^*(x, y) = \pi^*(x, y) = x + y - xy$ .

**Definition 1.1.** A *Probabilistic Normed space* (briefly, PN space) is a  $(V, \nu, \tau, \tau^*)$ , where  $V$  is a real vector space,  $\tau$  and  $\tau^*$  are continuous triangle functions with  $\tau \leq \tau^*$  and  $\nu$  is a mapping (the *probabilistic norm*) from  $V$  into  $\Delta^+$ , such that for every choice of  $p$  and  $q$  in  $V$  the following hold:

$$(N1) \quad \nu_p = \varepsilon_0 \text{ if, and only if, } p = \theta \text{ (}\theta \text{ is the null vector in } V\text{);}$$

$$(N2) \quad \nu_{-p} = \nu_p;$$

$$(N3) \quad \nu_{p+q} \geq \tau(\nu_p, \nu_q);$$

$$(N4) \quad \nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \text{ for every } \lambda \in [0, 1].$$

A PN space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right),$$

holds for every  $\alpha \neq 0 \in \mathbb{R}$  and  $x > 0$ . There is a natural topology in a PN-space  $(V, \nu, \tau, \tau^*)$ , called *strong topology*; it is defined for  $p \in V$  and  $t > 0$ , by the neighborhoods

$$N_p(t) := \{q \in V; \nu_{q-p}(t) > 1 - t\} = \{q \in V; d_S(\nu_{q-p}, \varepsilon_0) < t\}.$$

A set  $A$  in the PN space  $(V, \nu, \tau, \tau^*)$  is said to be  $D$ -bounded if its *probabilistic radius*,  $R_A$  belongs to  $D^+$ . The probabilistic radius of  $A$  is defined by

$$R_A(x) := \begin{cases} \ell^- \inf\{\nu_p(x) : p \in A\}, & x \in [0, +\infty) \\ 1, & x = +\infty. \end{cases}$$

Of course, if  $V$  is a normed space under the norm  $\|\cdot\|$ , then the set  $A$  may be bounded when regarded as a subset of the normed space  $(V, \|\cdot\|)$ . The two notions need not coincide (see [8]).

**Definition 1.2.** A subset  $A$  of a PN space which is a TV space is said to be *bounded* if for every  $m \in \mathbb{N}$  there exists a  $k \in \mathbb{N}$  such that

$$A \subset k\mathcal{N}_\theta(1/m).$$

The papers [4, 17, 6, 5] on the relationship between the two types of boundedness ought also to be kept in mind.

The concept of paranorm is a generalization of that of absolute value. The paranorm of  $x$  may be thought of as the distance from  $x$  to 0.

**Definition 1.3.** [18] A paranorm is a real function  $p : V \rightarrow \mathbb{R}$  where  $V$  is a vector space, and satisfying conditions (i) through (v) for all vectors  $a, b$  in  $V$ .

- (i)  $p(\theta) = 0$
- (ii)  $p(a) \geq 0$
- (iii)  $p(-a) = p(a)$
- (iv)  $p(a + b) \leq p(a) + p(b)$
- (v) If  $t_n$  is a sequence of scalars with  $t_n \rightarrow t$  and  $u_n$  is a sequence of vectors with  $u_n \rightarrow u$ , then  $p(t_n u_n - tu) \rightarrow 0$  (continuity of multiplication).

A paranorm  $p$  for which  $p(a) = 0$  implies  $a = \theta$  will be called *total*.

One owes the following result to Alsina, Schweizer & Sklar ([2]).

**Theorem 1.4.** *Every PN space  $(V, \nu, \tau, \tau^*)$ , when it is endowed with the strong topology induced by the probabilistic norm  $\nu$ , is a topological vector space if, and only if, for every  $p \in V$  the map from  $\mathbb{R}$  into  $V$  defined by*

$$\lambda \mapsto \lambda p \tag{1.1}$$

*is continuous.*

It was proved in [2, Theorem 4], that, if the triangle function  $\tau^*$  is Archimedean, i.e., if  $\tau^*$  admits no idempotents other than  $\epsilon_0$  and  $\epsilon_\infty$  ([12]), then the mapping (1.1) is continuous and, as a consequence, the PN space  $(V, \nu, \tau, \tau^*)$  is a TV space.

## 2. SOME PROPERTIES OF CONTINUOUS LINEAR OPERATOR IN TOPOLOGICAL VECTOR PN SPACES

**Definition 2.1.** A Probabilistic Total Paranormed space (briefly P tp-N space) is a triple  $(V, \nu, \tau)$ , where  $V$  is real vector space,  $\tau$  is a continuous triangle function and  $\nu$  is a mapping (the probabilistic total paranorm) from  $V$  in to  $\Delta^+$ , such that for every choise of  $p$  and  $q$  in  $V$  the following hold:

- (p1)  $\nu_p = \varepsilon_0$  if, and only if,  $p = \theta$  ( $\theta$  is the null vector in  $V$ );  
 (p2)  $\nu_{-p} = \nu_p$ ;  
 (p3)  $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ ;  
 (p4) If  $u_n$  and  $\alpha_n$  be two sequence of vectors and scalars respectively and  $u_n \rightarrow u$ ,  
 $\alpha_n \rightarrow \alpha$ ; Then  $\nu_{\alpha_n u_n - \alpha u} \rightarrow \varepsilon_0$ .

**Example 2.2.** Suppose  $p$  is a total paranorm on the vector space  $V$  and let  $\tau$  be a triangle function such that  $\tau(\epsilon_a, \epsilon_b) \leq \epsilon_{a+b}$  and  $\nu_a = \epsilon_{p(a)}$ . Then  $(V, \nu, \tau)$  is a P tp-N space.

*Proof.* (p1)  $\nu_a = \epsilon_0 \Leftrightarrow \epsilon_{p(a)} = \epsilon_0 \Leftrightarrow a = 0$ .

(p2)  $\nu_a = \epsilon_{p(a)} = \epsilon_{p(-a)} = \nu_{-a}$ .

(p3)  $\nu_{a+b} = \epsilon_{p(a+b)} \geq \epsilon_{p(a)+p(b)} \geq \tau(\epsilon_{p(a)}, \epsilon_{p(b)})$ .

(p4) Suppose  $u_n$  and  $\alpha_n$  be two sequences of vectors and scalars respectively and  $u_n \rightarrow u$ ,  $\alpha_n \rightarrow \alpha$ . Then  $\nu_{\alpha_n u_n - \alpha u} = \epsilon_{p(\alpha_n u_n - \alpha u)} \geq \epsilon_{\alpha_n(u_n - u) + p(\alpha_n - \alpha)} \rightarrow \epsilon_0$ .  $\square$

**Theorem 2.3.** [3] *A PN-space is a topological vector space if, and only if, it is a P tp-N space.*

**Definition 2.4.** A continuous linear map on  $V$  into  $W$ , where  $V$  and  $W$  are topological vector space over  $\mathbb{R}$ , is called topological homomorphism, if each open subset  $G \subset V$ , the image  $u(G)$  is an open subset of  $u(V)$ .

**Definition 2.5.** If  $X$  is a topological space, a set  $E \subset X$  is meager set if  $E$  is a countable union of nowhere dense sets.

**Theorem 2.6.** *Let  $(V, \nu_1, \tau_1)$  and  $(W, \nu_2, \tau_2)$  be two P tp-N spaces.  $T : V \rightarrow W$  be a continuous map of  $V$  with range dense in  $W$ . Then either  $T(V)$  is meager in  $W$  or else  $T(V) = W$ . Also  $T$  is topological homomorphism.*

*Proof.* Let  $(V, \nu_1, \tau_1)$  be a P tp-N space. By theorem 2.3,  $(V, \nu_1, \tau_1)$  is topological vector space and the set  $N_\theta(\frac{1}{n}), n \in N$  is a  $\theta$ -neighborhood base. Let  $N_\theta(\frac{1}{n_i}), i \in N$  be a base of circled  $\theta$ -neighborhoods satisfying

$$N_\theta(\frac{1}{n_{i+1}}) + N_\theta(\frac{1}{n_{i+1}}) \subset N_\theta(\frac{1}{n_i}) \quad n \in N \quad (2.1)$$

For each nonempty finite subset  $K$  of  $N$ , define the circled  $\theta$ -neighborhood  $V_K$  by  $V_K = \sum_{n \in K} N_\theta(\frac{1}{n})$  and the real number  $P_K$  by  $P_K = \sum_{n \in K} 2^{-n}$ . We define the real valued function  $x \rightarrow |x|$  on  $V$  by  $|x| = 1$  if  $x$  is not contained in any  $V_K$  and otherwise by

$$|x| = \inf_K \{P_K, x \in V_K\}.$$

Then  $|\cdot|$  is a pseudo-norm on  $V$ , similarly we can find a pseudo-norm  $|\cdot|$  on  $W$ . The family  $\{S_r, r > 0\}$  is a  $\theta$ -neighborhood in  $V$ . For fixed  $r$ , let  $U_1 = S_r$  and  $U_2 = S_{\frac{r}{2}}$ . Then  $U_2 + U_2 \subset U_1$  and  $T(V) = \cup_1^\infty nT(U_2)$ .

since by assumption  $T(V)$  is a complete metric space there exist  $n \in N$  such that  $\overline{[nT(U_2)]}$  has an interior point; Hence  $\overline{[T(U_2)]}$  has an interior point. Now

$$\overline{[T(U_2)]} + \overline{[T(U_2)]} \subset \overline{[T(U_2) + T(U_2)]}.$$

Hence there exist  $\zeta > 0$  such that  $T(V) \cap S_\zeta \subset T(S_{r+\varepsilon})$  for every  $\varepsilon > 0$ . Thus  $T$  is a topological homomorphism and  $T(V) = W$ .  $\square$

**Corollary 2.7.** *A continuous linear map  $T$  of a  $P$   $tp$ - $N$  space  $(V, \nu_1, \tau_1)$  in to another such space  $(W, \nu_2, \tau_2)$  is a topological homomorphism iff  $T(V)$  is closed in  $W$ .*

*Proof.*  $T(V)$  is isomorphic with  $\frac{V}{T^{-1}(0)}$  and hence in  $W$ . conversely if  $T(V)$  is closed in  $W$ , then it is  $P$   $tp$ - $N$  space and hence can replace  $W$  in preceding theorem.  $\square$

**Theorem 2.8.** *If  $(V, \nu_1, \tau_1)$  and  $(W, \nu_2, \tau_2)$  are two  $P$   $tp$ - $N$  spaces.  $T : V \rightarrow W$  is a continuous surjection and  $G$  is open in  $V$  then  $T(G)$  is open in  $W$ .*

*Proof.* We first show for every  $r > 0$ ;

$$o \in \overline{\text{int}[T(N_\theta(r))]} \quad (2.2)$$

Not that because  $T$  is surjective we have,

$$W = \overline{\cup_{k=1}^{\infty} [T(N_\theta(k\frac{r}{2}))]} = \overline{\cup_{k=1}^{\infty} k[T(N_\theta(\frac{r}{2}))]}.$$

By the Baire Category theorem, there is a  $k \geq 1$  such that  $k\overline{[T(N_\theta(\frac{r}{2}))]}$  has nonempty interior thus  $M = \text{int}\overline{[T(N_\theta(\frac{r}{2}))]} \neq \emptyset$ . If  $q_0 \in M$ , let  $s > 0$  such that  $N_{q_0}(S) \subseteq \overline{[T(N_\theta(\frac{r}{2}))]}$ . Let  $q \in W$ ,  $q \in N'_{\theta'}(S)$  where  $\theta'$  is the null vector in  $W$ . Since  $q_0 \in \overline{[T(N_\theta(\frac{r}{2}))]}$ , there exist a sequence  $\{p_n\}$  in  $N_\theta(\frac{r}{2})$  such that  $T(p_n) \rightarrow q_0$ . There is also a sequence  $\{p'_n\}$  in  $N_{\theta'}(\frac{r}{2})$  such that  $T(p'_n) \rightarrow q_0 + q$ . Thus  $T(p'_n - p_n) \rightarrow q$  and  $\{p'_n - p_n\} \subseteq N_\theta(r)$ ; that is  $N_\theta(r) \subseteq \overline{[T(N_\theta(r))]}$ . This establish claim (2.2). Now it will be shown that

$$\overline{[T(N_\theta(\frac{r}{2}))]} \subseteq \overline{[T(N_\theta(r))]} \quad (2.3)$$

Note that if (2.3) is proved, then claim (2.2) implies that  $o \in \text{int}[T(N_\theta(r))]$ , for any  $r > 0$  and the theorem is proved. Indeed, if  $G$  is an open subset of  $V$ , then for every  $p \in G$  let  $r_p > 0$  such that  $N_p(r_p) \subseteq G$ . But  $o \in \text{int}[T(N_\theta(r_p))]$  and so  $T(p) \in \text{int}T(N_p(r_p))$ .

Thus there is an  $s_p > 0$  such that  $U_p \equiv N_{T(p)}(s_p) \subseteq T(N_p(r_p))$ . Therefore  $\cup_{p \in G} \{U_p\} \subseteq T(G)$ . But  $T(p) \in U_p$  so  $\cup_{p \in G} \{U_p\} = T(G)$  and hence  $T(G)$  is open.

To prove (2.3) fix  $q_1 \in \overline{[T(N_\theta(\frac{r}{2}))]}$ , by (2.2)  $0 \in \text{int}\overline{[T(N_\theta(\frac{r}{4}))]}$ . Hence

$$[q_1 - \overline{[T(N_\theta(\frac{r}{2}))]}] \cap T(N_\theta(\frac{r}{2})) \neq \emptyset.$$

Let  $p_1 \in N_\theta(\frac{r}{2})$  such that  $T(p_1) \in [q_1 - \overline{[T(N_\theta(\frac{r}{4}))]}]$ ; Now  $T(p_1) = q_1 - q_2$  where  $q_2 \in \overline{[T(N_\theta(\frac{r}{4}))]}$ . Using induction, we obtain a sequence  $\{p_n\}$  in  $V$  and a sequence  $\{q_n\}$  in  $W$  such that,

- (i)  $p_n \in N_\theta(\frac{r}{2^n})$ .
- (ii)  $q_n \in \overline{[T(N_\theta(\frac{r}{2^n}))]}$ .
- (iii)  $q_{n+1} = q_n - T(q_n)$ .

But  $d_s(\nu_{p_n}, \varepsilon_0) < \frac{r}{2^n}$ , so  $\sum_1^\infty d_s(\nu_{p_n}, \varepsilon_0) < \infty$ . Hence  $p = \sum_{n=1}^\infty p_n$  exist in  $V$  and  $d_s(\nu_p, \varepsilon_0) < r$ . Also

$$\sum_{k=1}^n T(p_k) = \sum_{k=1}^n (q_k - q_{k+1}) = q_1 - q_{n+1}.$$

The relation (iii) implies that  $q_n \rightarrow 0$ , therefore

$$q_1 = \sum_1^{\infty} T(p_k) = T(p) \in T(N_{\theta}(r)).$$

This proving (2.3) and completing the proof of the theorem.  $\square$

**Theorem 2.9.** (Prochaska)[7]. *Šerstnev space  $(V, \nu, \tau)$  with  $\nu(V) \subseteq D^+$  and  $\tau = \tau_M$  is a locally convex space.*

**Corollary 2.10.** *If  $(V, \nu_1, \tau_M)$  and  $(W, \nu_2, \tau_M)$  be two Šerstnev spaces with  $\nu_1(V) \subseteq D^+$  and  $\nu_2(W) \subseteq D^+$ . Let  $T : V \rightarrow W$  is a continuous surjection, and  $G$  is an open subset of  $V$ , then  $T(G)$  is open.*

*Proof.* Theorems 2.7 and 2.8.  $\square$

**Definition 2.11.** [8]. A linear map  $T : (V, \nu_1, \tau_1) \rightarrow (W, \nu_2, \tau_2)$  between two PN spaces is said to be strongly bounded if there exist a constant  $k > 0$  such that for every  $p \in V$  and for every  $x > 0$ ,

$$\nu_{2_{T(p)}}(x) \geq \nu_{1_p}\left(\frac{x}{k}\right).$$

**Lemma 2.12.** [8]. *Every strongly bounded linear operator  $T : (V, \nu_1, \tau_1) \rightarrow (W, \nu_2, \tau_2)$  between two PN spaces is continuous with respect to the strong topology in  $(V, \nu_1, \tau_1)$  and  $(W, \nu_2, \tau_2)$  respectively.*

**Theorem 2.13.** *Let  $(V, \nu_1, \tau_1)$  and  $(W, \nu_2, \tau_2)$  be two P tp-N spaces and  $T : V \rightarrow W$  is strongly bounded linear operator that is bijective, then  $T^{-1}$  is strongly bounded.*

*Proof.* The operator  $T$  is continuous by Lemma 2. 11. Now the result follows from theorem 2.7.  $\square$

**Theorem 2.14.** *Let  $(V, \nu_1, \tau)$  and  $(W, \nu_2, \tau)$  be two P tp-N spaces under the same triangular function  $\tau$ , and  $T : V \rightarrow W$  is a linear operator such that the graph of  $T$  is closed, where*

$$\text{graph } T \equiv \{(p, T(p)), p \in V\}.$$

*Then  $T$  is continuous.*

*Proof.* Let  $G = \text{graph}T$ , since  $(V \times W, \mu^{\tau}, \tau)$  where  $\mu^{\tau} : V \times W \rightarrow \Delta^+$  is defined by  $\mu^{\tau}(p, q) = \tau(\nu_p, \nu'_q)$ , is a P tp-N space and  $G$  is closed,  $G$  is complete. Define  $\rho : G \rightarrow V$  by  $\rho(p, T(p)) = p$ . It is easy to check that  $\rho$  is bounded and bijective. By the preceding theorem,  $\rho^{-1} : V \rightarrow G$  is continuous.  $T : V \rightarrow W$  is the composition of the continuous map  $\rho^{-1} : V \rightarrow G$  and the continuous map of  $G \rightarrow W$  defined by  $(p, T(p)) \rightarrow T(p)$ . Therefore  $T$  is continuous.  $\square$

**Corollary 2.15.** *Let  $(V, \nu_1, \tau_M)$  and  $(W, \nu_2, \tau_M)$  be two Šerstnev spaces with  $\nu_1(V) \subseteq D^+$  and  $\nu_2(W) \subseteq D^+$  and  $T : V \rightarrow W$  be a linear operator such that graph of  $T$  is closed. Then  $T$  is continuous.*

*Proof.* It is obviously obtained from Theorems 2.8 and Theorem 2.13.  $\square$

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