Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 603–612
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.28244.3843

Notes on f-hom-ders associated with a system of additive functional equations in unital algebras

Choonkil Park[a], Jedsada Senasukh[b,*]

[a]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
[b]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

(Communicated by Abbas Najati)

Abstract

The following system of additive functional equations $2f(x+y) = g(x) + g(y)$, $g(x+y) = 2f(y-x) + 4f(x)$ in a Banach algebra was investigated by Paokanta et al. [S. Paokanta, M. Dehghanian, C. Park and Y. Sayyari, System of additive functional equations in Banach algebras (preprint)]. By using the system (A), Paokanta et al. [S. Paokanta, M. Dehghanian, C. Park and Y. Sayyari, System of additive functional equations in Banach algebras (preprint)] also investigated the notion of an f-hom-der in Banach algebras. In this note, we first show that the system of additive functional equations (A) is equivalent to an additive functional equation and then some aspects related to f-hom-ders are discussed. Finally, we present stability results of f-hom ders associated with the system of additive functional equations (A).

Keywords: f-hom-der, unital algebra, Banach algebra, system of additive functional equations, stability result
2020 MSC: 17B40, 39B52, 39B72, 39B82

1 Introduction and preliminaries

By an algebra $\mathcal{A}$, we mean that $\mathcal{A}$ is a complex linear vector space together with a bilinear mapping $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathcal{A}$. In this case, such a bilinear mapping is usually called an *associative multiplication* in $\mathcal{A}$. If $\mathcal{A}$ contains a special element e, called an *identity*, satisfying $e \cdot x = x \cdot e = x$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ is named as a *unital algebra*. One of the important associative algebras is the algebra of complex n-matrices $M_n(\mathbb{C})$ under the matrix multiplication. We note that $M_n(\mathbb{C})$ is a unital algebra and it is not commutative (for $n > 1$). In this case, $M_n(\mathbb{C})$ is considered as a non-commutative algebra. For the case $n = 1$, we consider $M_n(\mathbb{C})$ as the algebra $\mathbb{C}$. It is clear that $\mathbb{C}$ is unital and commutative. Moreover, every non-zero element of $\mathbb{C}$ has a multiplicative inverse. Hence, $\mathbb{C}$ is an example of *invertible algebras*.

An interesting topic studied in an algebra is the notion of a (linear) *derivation*. A $\mathbb{C}$-linear mapping $g : \mathcal{A} \to \mathcal{A}$ is said to be a *derivation* if it satisfies the identity (related to *Leibniz's rule*)

$$g(xy) = xg(y) + g(x)y \quad \text{for all } x, y \in \mathcal{A}.$$

*Corresponding author
Email addresses: baak@hanyang.ac.kr (Choonkil Park), senasukh@kkumail.com (Jedsada Senasukh)

Received: July 2022 *Accepted:* September 2022

In the complex algebra $C^\infty(0,1)$ provided the pointwise product, the algebra of all infinitely differentiable complex functions on $(0,1)$, we see that the mapping $C^\infty(0,1) \ni f \mapsto f'$ induces a *derivation* in $C^\infty(0,1)$. To generalize this concept, Mirzavaziri and Moslehian [13, 14] investigated a notion of an *f-derivation.* For $\mathbb{C}$-linear mappings $f, g : \mathcal{A} \to \mathcal{A}$, the mapping g is an f-derivation if

$$g(xy) = f(x)g(y) + g(x)f(y) \quad \text{for all } x, y \in \mathcal{A}.$$

Obviously that every derivation is an I-derivation where I denotes the identity mapping. Moreover, the notion of f-derivation also covers the algebra homomorphisms. It it easy to see that if $g : \mathcal{A} \to \mathcal{A}$ is an algebra homomorphism (that is, g is $\mathbb{C}$-linear and multiplicative), then g is a $\frac{1}{2}g$-derivation. In fact, for any $x, y \in \mathcal{A}$ we see that

$$g(xy) = g(x)g(y) = \frac{1}{2}g(x)g(y) + \frac{1}{2}g(x)g(y) = \left(\frac{1}{2}g(x)\right) g(y) + g(x)\left(\frac{1}{2}g(y)\right).$$

Some results related to this notion can be seen for examples in [7, 10, 17, 19].

Combining the concepts of both derivations and algebra homomorphisms, Kheawborisut et al. [12] introduced a hom-der in a fuzzy Banach algebra by using the formula

$$g(xy) = xg(y) + g(x)y.$$

If g forms a hom-der, then g is a derivation if and only if it is an algebra homomorphism. That is, the derivation and the homomorphism are identical in the class of hom-ders. An example supporting the definition is the mapping $x \mapsto 2x$. As a generalization of hom-ders, Paokanta et al. [15] defined an f-hom-der as in the following definition:

Definition 1.1. [15, Definition 1.1] Suppose that $\mathcal{A}$ is an algebra and $f, g : \mathcal{A} \to \mathcal{A}$ are $\mathbb{C}$-linear mappings. We say that g is an *f-hom-der* if f is an algebra homomorphism and

$$g(x)g(y) = f(x)g(y) + g(x)f(y) \quad \text{for all } x, y \in \mathcal{A}. \tag{1.1}$$

For a given algebra homomorphism f, we see that the mapping $\mathcal{A} \ni x \mapsto 2f(x)$ induces an f-hom-der. Hence, the mapping $2f$ is an f-hom-der.

Note that the formula (1.1) is reasonable for certain mappings $f, g : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are algebras.

We observe that all derivations we are discussing concerns linear mappings (in particular, they are additive mappings) in certain complex linear vector spaces. With this reason, it is natural to consider its approximate versions based on the idea of *Ulam.* In 1940, Ulam [22], proposed an interesting question related to approximate group homomorphisms in a metric commutative group. The question is known as *Ulam's stability problem.* More precisely, he proposed the following problem: *For given a commutative group $(G, +)$, a metric commutative group $(H, +, d)$ and a positive real number ε, does there exists a group homomorphism $F : G \to H$ and a real number $\delta > 0$ such that $d(F(x), f(y)) \le \varepsilon$ for all approximate group homomorphisms $f : G \to H$ satisfying $d(f(x + y), f(x) + f(y)) \le \delta$ for all $x, y \in G$?* Hyers [5] confirmed that the answer of Ulam's stability problem is *affirmative* in Banach spaces with $\delta = \varepsilon$. A natural generalization of Hyers' stability result was presented by Aoki [1]. Rassias [21] independently presented a generalization of Hyers' result similar to that of Aoki [1]. Moreover, Rassias [21] also investigated the linearity condition in his result. For some pioneered stability results regarding the stability of additive and related functional equations can be seen for instances in [3, 4, 8, 11, 20] (and references therein). As a consequence of a fixed point theorem proposed by Diaz and Margolis [6] and the direct method, many stability results concerning various kinds of derivations in different spaces have been extensively studied by many authors and some examples are in [2, 9, 12, 16, 18, 23].

Paokanta et al. [15] applied the fixed point result in proving some stability results of f-hom-ders in complex Banach algebras associated with the *system of additive functional equations*:

$$\begin{cases} 2f(x + y) = g(x) + g(y), \\ g(x + y) = 2f(y - x) + 4f(x). \end{cases} \tag{1.2}$$

By a direct computation, if f is additive and $g := 2f$ then f and g are solutions of the system of additive functional equations (1.2).

In this paper, we manage our study as in the following two sections:

- In Section 2, we first solve the system of additive functional equations (1.2) in commutative groups and we also give some observations on the f-hom-ders in a unital algebra. Finally, we present a relation between the system (1.2) and f-hom-ders;
- In Section 3, we verify some stability results of f-hom-ders associated with the system of additive functional equations (1.2) by using the direct method.

2 System of additive functional equations (1.2) and f-hom-ders

In this section, we solve the system of additive functional equations (1.2) in commutative groups. Moreover, under some suitable conditions we give an explicit form of an f-hom-der in a unital algebra.

Now, we begin this section with solving the system of additive functional equations in certain commutative groups. In addition, the following theorem covers [15, Lemma 2.1]

Theorem 2.1. *Suppose that G is a group and H is a commutative group in which each of its element is of order different from 2 and 5. Then mappings $f, g : G \to H$ satisfy the system of additive functional equations*

$$\begin{cases} 2f(x+y) = g(x) + g(y), \\ g(x+y) = 2f(y-x) + 4f(x) \end{cases} \quad \text{for all } x, y \in G \tag{2.1}$$

if and only if f, g are additive and $g = 2f$.

Proof . ($\Longrightarrow$) We first note that if $u, v \in H$ and $2u = 2v$ then the commutativity of H implies that $2(u-v) = 2u - 2v = 0$. Since $o(u-v) \neq 2$ (where $o(u-v)$ denotes the order of $u-v$ in H), one obtains that $u - v = 0$.

By letting $x = y = 0$ in (2.1) we have that

$$2f(0) = 2g(0) \quad \text{and} \quad g(0) = 6f(0).$$

It follows that $f(0) = g(0)$ and hence $f(0) = g(0) = 6f(0)$. Using the cancellation law of the group H, we have that $5f(0) = 0$. Note that $o(f(0)) \neq 2$ and $o(f(0)) < 5$. In fact, $o(f(0)) \in \{1, 3, 4\}$. We consider the following.

- If $o(f(0)) = 4$, then $0 = 5f(0) = 4f(0) + f(0) = 0 + f(0) = f(0)$ which is impossible. Hence, $o(f(0)) \neq 4$.
- If $f(0) \neq 0$ and $o(f(0)) = 3$, then $0 = 5f(0) = 3f(0) + 2f(0) = 2f(0)$ which is impossible. Thus, $f(0) = 0$ or $o(f(0)) \in \{1, 4\}$. By above consideration, we can see that $o(f(0)) = 1$, that is, $f(0) = 0$ and then $g(0) = f(0) = 0$.

Letting $y = 0$ in (2.1) shows that $g = 2f$. This implies the equality

$$2f(x+y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in G. \tag{2.2}$$

Hence, f is additive. Since $g = 2f$, (2.2) shows that g is additive.

($\Longleftarrow$) Let $x, y \in G$ be given. Since $g = 2f$, we have that

$$2f(x+y) = 2f(x) + 2f(y) = g(x) + g(y)$$

and

$$g(x+y) = 2f(x+y) = 2f(y) + 2f(x) = (2f(y) - 2f(x)) + 4f(x) = 2f(y-x) + 4f(x).$$

This completes the proof. $\square$

It is known that $(\mathbb{Z}_n, +)$ is a commutative group. If $n > 1$ is odd and $5 \nmid n$, then every element of $(\mathbb{Z}_n, +)$ is of order different from $2, 5$.

Corollary 2.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are algebras. If $f, g : \mathcal{A} \to \mathcal{B}$ are $\mathbb{C}$-linear solutions of the system* (1.2) *and f is an algebra homomorphism, then g is an f-hom-der. In particular, g is a $\frac{1}{2}f$-derivation* if $\mathcal{A} = \mathcal{B}$.

Proof . Theorem 2.1 shows that $g = 2f$. For any $x, y \in \mathcal{A}$, it follows that

$$\begin{aligned} g(x)g(y) &= (2f(x))(2f(y)) \\ &= f(x)(2f(y)) + (2f(x))f(y) \\ &= f(x)g(y) + g(x)f(y). \end{aligned}$$

This shows that g is an f-hom-der. We see that

$$\begin{aligned} g(xy) &= f(xy) + f(xy) \\ &= f(x)f(y) + f(x)f(y) \\ &= \left(\frac{1}{2}f(x)\right)(2f(y)) + (2f(x))\left(\frac{1}{2}f(y)\right) \\ &= \left(\frac{1}{2}f(x)\right)g(y) + g(x)\left(\frac{1}{2}f(y)\right). \end{aligned}$$

This finishes our proof. □

It is natural to ask that: *Does the converse of Corollary 2.2 hold true?*. The following theorem shows that the answer is *affirmative* in a paticular situation.

For algebras $\mathcal{A}, \mathcal{B}$ and a mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$Z^R_{f(\mathcal{A})} := \{b \in \mathcal{B} : f(x) = f(x)b \quad \text{for all } x \in \mathcal{A}\}.$$

It is clear that if $\mathcal{B}$ is unital then its indentity belongs to $Z^R_{f(\mathcal{A})}$.

Theorem 2.3. *Suppose that $\mathcal{A}$ is an algebra and $\mathcal{B}$ is a unital algebra. Suppose that $f, g : \mathcal{A} \to \mathcal{B}$ are given $\mathbb{C}$-linear mappings such that there exists $u \in \mathcal{A}$ such that $g(u) - f(u)$ is invertible and $g(u)(g(u) - f(u))^{-1} \in 2Z^R_{f(\mathcal{A})}$. If g is an f-hom-der, then f and g are sulutions of the system* (1.2)*, that is, $g = 2f$.*

Proof . Since $g(u)(g(u) - f(u))^{-1} \in 2Z^R_{f(\mathcal{A})}$, there exists $u' \in Z^R_{f(\mathcal{A})}$ such that $g(u)(g(u) - f(u))^{-1} = 2u'$. It follows that

$$f(x)g(u)(g(u) - f(u))^{-1} = f(x)(2u') = 2f(x)u' = 2f(x) \quad \text{for all } x \in \mathcal{A}.$$

By letting $y = u$ in (1.1), we see that

$$\begin{aligned} & g(x)g(u) = f(x)g(u) + g(x)f(u) \\ \Longrightarrow\ & g(x)g(u) - g(x)f(u) = f(x)g(u) \\ \Longrightarrow\ & g(x)(g(u) - f(u)) = f(x)g(u) \\ \Longrightarrow\ & g(x) = f(x)g(u)(g(u) - f(u))^{-1} = 2f(x) \end{aligned}$$

for all $x \in \mathcal{A}$. By Theorem 2.1, our proof is finished. □

Proposition 2.4. *Suppose that $\mathcal{A}, \mathcal{B}$ are algebras in which $\mathcal{B}$ is unital, $u \in \mathcal{A}$, and $f, g : \mathcal{A} \to \mathcal{B}$ are given $\mathbb{C}$-linear mappings. Then the following assertions are equivalent.*

1) *g is an f-hom-der and there exists $u \in \mathcal{A}$ such that $g(u) - f(u)$ is invertible and $g(u)(g(u) - f(u))^{-1} \in 2Z^R_{f(\mathcal{A})}$.*
2) *$g = 2f$ and f is an algebra homomorphism such that $f(u)$ is invertible.*

Proof . ($\Longrightarrow$) By Theorem 2.3, $g = 2f$. We also see that

$$g(u) - f(u) = 2f(u) - f(u) = f(u)$$

and hence $f(u)$ is invertible. This implies that 2) holds.

($\Longleftarrow$) Note that $g(u) - f(u) = f(u)$. So, $g(u) - f(u)$ is invertible. We see that

$$g(u)(g(u) - f(u))^{-1} = g(u)f(u)^{-1} = 2f(u)f(u)^{-1} = 2e \in Z_{f(\mathcal{A})}.$$

This completes our proof. □

Corollary 2.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras. Suppose that $f, g : \mathcal{A} \to \mathcal{B}$ are given $\mathbb{C}$-linear mappings such that $g(e) = 2e$ and f is an algebra homomorphism with $f(e) = e$. Then g is an f-hom-der if and only if $g = 2f$.*

Proof . It is obvious since e is invertible. □

Remark 2.6. If $g = 2f$ and $\mathcal{A} = \mathcal{B}$, then g is an f*-Lie-hom-der* in a Lie algebra $(\mathcal{A}, [\cdot,\cdot])$, that is,

$$[g(x), g(y)] = [f(x), g(y)] + [g(x), f(y)] \quad \text{for all } x, y \in \mathcal{A}.$$

We end this subsection with the following example.

Example 2.7. Let $\mathcal{A} = \mathcal{B} := M_2(\mathbb{C})$. Note that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras which are not invertible. Define two mappings $f, g : \mathcal{A} \to \mathcal{B}$ by

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$$

for all $a, b, c, d \in \mathbb{C}$. It is clear that both f and g are $\mathbb{C}$-linear mappings. Moreover, they are not invertible. We note that $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is invertible and

$$f\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} = -f\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

So, we have

$$g\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\left(g\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - f\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\right)^{-1} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix}^{-1} = 2I_2$$

By letting $u := \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, it follows that $g(u) - f(u)$ is invertible and $g(u)(g(u) - f(u))^{-1}$ commutes to all elements of $\mathcal{A}$. Moreover, $g \neq 2f$. If we define $g' := 2f$, then we easily see that

$$g(u) - f(u) = f(u) \quad \text{and} \quad g(u)(g(u) - f(u))^{-1} = 2I_2.$$

3 Stability results

In this section, we verify some stability results of f-hom-ders associated with the system of additive functional equations (1.2) by using the direct method.

Firstly, we consider the approximate version of the system of additive functional equations (1.2) in a normed linear space as in the following lemma.

Lemma 3.1. *Suppose that X is a linear vector spaces, $(Y, \|\cdot\|)$ is a normed linear space, and $\varphi_1, \varphi_2 : X \times X \to \mathbb{R}_+ := [0, \infty)$ are given functions. If $f, g : X \to Y$ are mappings such that*

$$\|2f(x+y) - g(x) - g(y)\| \le \varphi_1(x, y); \tag{3.1}$$

$$\|g(x+y) - 2f(y-x) - 4f(x)\| \le \varphi_2(x, y) \tag{3.2}$$

for all $x, y \in X$, then f and g satisfy

$$\|2f(x) - g(x)\| \le \phi(x); \tag{3.3}$$

$$\|f(2x) - 2f(x)\| \le \psi(x) \tag{3.4}$$

for all $x \in X$ where $\phi : X \to Y, \psi : X \to Y, K_1$ and K_2 are defined by

$$K_1 := \varphi_1(0,0) + 2\varphi_2(0,0) \quad \textit{and} \quad K_2 := \frac{1}{5}\left(3\varphi_2(0,0) + \varphi_2(0,0)\right);$$

$$\phi(x) := \min\{\varphi_1(x,0) + K_2, \varphi_2(0,x) + K_1\} \quad \textit{and} \quad \psi(x) := \frac{1}{2}\varphi_1(x,x) + \phi(x).$$

Proof . By letting $x = y = 0$ in (3.1) and (3.2), we obtain that

$$\|2f(0) - 2g(0)\| \leq \varphi_1(0,0) \quad \text{and} \quad \|g(0) - 6f(0)\| \leq \varphi_2(0,0). \tag{3.5}$$

It follows that

$$\begin{aligned}\|10f(0)\| &= \|2f(0) - 12f(0)\| \\ &\leq \|2f(0) - 2g(0)\| + \|2g(0) - 12f(0)\| \\ &\leq \varphi_1(0,0) + 2\varphi_2(0,0).\end{aligned}$$

We immidiately obtain that

$$\|f(0)\| \leq \frac{1}{10}\varphi_1(0,0) + \frac{1}{5}\varphi_2(0,0).$$

We note from (3.5) that

$$\begin{aligned}\|g(0)\| &\leq \min\left\{\frac{1}{2}\varphi_1(0,0) + \|f(0)\|, \varphi_2(0,0) + 6\|f(0)\|\right\} \\ &\leq \min\left\{\frac{1}{5}\left(3\varphi_1(0,0) + \varphi_2(0,0)\right), \frac{1}{2}\varphi_1(0,0) + \frac{1}{5}\left(3\varphi_1(0,0) + 11\varphi_2(0,0)\right)\right\} \\ &= \frac{1}{5}\left(3\varphi_1(0,0) + \varphi_2(0,0)\right).\end{aligned}$$

So, we now let

$$K_1 := \frac{1}{5}\left(\frac{1}{2}\varphi_1(0,0) + \varphi_2(0,0)\right) \quad \text{and} \quad K_2 := \frac{1}{5}\left(3\varphi_2(0,0) + \varphi_2(0,0)\right).$$

And we have that $\|f(0)\| \leq K_1$ and $\|g(0)\| \leq K_2$.

Next, we show that (3.3) holds for all $x \in X$. We see from (3.1) by letting $y = 0$ that

$$\|2f(x) - g(x)\| \leq \|2f(x) - g(x) - g(0)\| + \|g(0)\| \leq \varphi_1(x,0) + K_2 \quad \text{for all } x \in X.$$

By letting $x = 0$ in (3.2), we have

$$\|g(y) - 2f(y)\| \leq \varphi_2(0,y) + 4\|f(0)\| \leq \varphi_2(0,y) + 4K_1 \quad \text{for all } y \in X.$$

These two inequalities imply that

$$\|2f(x) - g(x)\| \leq \min\left\{\varphi_1(x,0) + K_2, \varphi_2(0,x) + 4K_1\right\}$$

for all $x \in X$. This means that (3.3) holds by choosing

$$\phi(x) := \min\left\{\varphi_1(x,0) + K_2, \varphi_2(0,x) + K_1\right\}.$$

Finally, we show that (3.4) holds. By letting $x = y$ in (3.1), we see that

$$\|2f(2x) - 2g(x)\| \leq \varphi_1(x,x) \quad \text{for all } x \in X.$$

So, we have

$$\|2f(2x) - 4f(x)\| \leq \|2f(2x) - 2g(x)\| + \|2g(x) - 4f(x)\| \leq \varphi_1(x,x) + 2\phi(x).$$

We have that

$$\|f(2x) - 2f(x)\| \leq \frac{1}{2}\varphi_1(x,x) + \phi(x) \quad \text{for all } x \in X.$$

This completes the proof. □

To make the simplicity for our further consideration, we give the following remark.

Remark 3.2. Suppose that $f(0)=g(0)=0$. We define $\varphi_1^*, \varphi_2^* : X\times X \to \mathbb{R}_+$ by

$$\varphi_i^*(x,y) := \begin{cases} 0 & \text{if } x=y=0; \\ \varphi_i(x,y) & \text{if } x\neq 0 \text{ or } y\neq 0. \end{cases}$$

for $i=1,2$. If f and g satisfy (3.1) and (3.2), then

$$\|2f(x+y)-g(x)-g(y)\| \le \varphi_1^*(x,y);$$
$$\|g(x+y)-2f(y-x)-4f(x)\| \le \varphi_2^*(x,y)$$

for all $x,y\in X$. Moreover, we also have

- $\varphi_1^* \le \varphi_1$ and $\varphi_2^* \le \varphi_2$;
- $K_1 = K_2 = 0$ and $\phi(0)=\psi(0)=0$;
- $\phi(x) = \min\{\varphi_1^*(x,0), \varphi_2^*(0,x)\}$ for all $x\neq 0$;
- $\psi(x) = \frac{1}{2}\varphi_1^*(x,x) + \min\{\varphi_1^*(x,0), \varphi_2^*(0,x)\}$ for all $x \neq 0$.

To support Lemma 3.1, we consider the following example.

Example 3.3. Let $f,g:\mathbb{C}\to\mathbb{C}$ be mappings defined by

$$f(x) := 2\sqrt{|x|} \quad \text{and} \quad g(x) := 2\sqrt{2}\sqrt{|x|} \quad \text{for all } x\in\mathbb{C}.$$

For any $x,y\in\mathbb{C}$, we see that

$$|2f(x+y)-g(x)-g(y)| = \left|4\sqrt{|x+y|} - 2\sqrt{2}\sqrt{|x|} - 2\sqrt{2}\sqrt{|y|}\right| =: \varphi_1(x,y);$$
$$|g(x+y)-2f(y-x)-4f(x)| = \left|2\sqrt{2}\sqrt{|x+y|} - 4\sqrt{|y-x|} - 8\sqrt{|x|}\right| =: \varphi_2(x,y).$$

We also note that

$$\varphi_1(x,0) = 2(2-\sqrt{2})\sqrt{|x|} \quad \text{and} \quad \varphi_2(0,x) = 2(2-\sqrt{2})\sqrt{|x|};$$
$$\varphi_1(x,x) = (4\sqrt{2}-4\sqrt{2})\sqrt{|x|} = 0.$$

It follows that

$$\phi(x) := \min\{\varphi_1(x,0), \varphi_2(0,x)\} = 2(2-\sqrt{2})\sqrt{|x|};$$
$$\psi(x) := \frac{1}{2}\varphi_1(x,x) + \phi(x) = \phi(x) = 2(2-\sqrt{2})\sqrt{|x|}.$$

Lemma 3.1 asserts that

$$\left|4-2\sqrt{2}\right|\sqrt{|x|} = |2f(x)-g(x)| \le \phi(x) = 2(2-\sqrt{2})\sqrt{|x|};$$
$$\left|2\sqrt{2}-4\right|\sqrt{|x|} = |f(2x)-2f(x)| \le \psi(x) = 2(2-\sqrt{2})\sqrt{|x|}.$$

So, $|2f(x)-g(x)| = \phi(x)$ and $|f(2x)-2f(x)| = \psi(x)$ for all $x\in\mathbb{C}$. This shows that our estimations (3.3) and (3.2) are *sharp* for a particular case.

The following lemma is obvious.

Lemma 3.4. *Suppose that X,Y are linear spaces, $\xi : X\to\mathbb{R}_+$ is a given function, and $f,g : X\to Y$ are mappings such that*

$$\|2f(x)-g(x)\| \le \xi(x) \quad \textit{for all } x\in X. \tag{3.6}$$

For $i\in\{-1,1\}$, if $\lim_{n\to\infty}\frac{1}{2^{in}}\xi(2^{in}x) = 0$ for all $x\in X$ then $\left(\frac{1}{2^{in}}f(2^{in}x)\right)$ converges if and only if $\left(\frac{1}{2^{in}}g(2^{in}x)\right)$ converges. In this case, we have

$$\lim_{n\to\infty}\frac{1}{2^{in}}g(2^{in}x) = 2\lim_{n\to\infty}\frac{1}{2^{in}}f(2^{in}x).$$

Proof . The proof is straightforward. □

By a *Banach algebra* $\mathcal{B}$, we mean an algebra $\mathcal{B}$ together with a complete norm $\|\cdot\|$ on $\mathcal{B}$ which is *sub-multiplicative*, that is,

$$\|xy\| \leq \|x\|\|y\| \quad \text{for all } x, y \in \mathcal{B}.$$

By using Hyers' direct method, we are in a position to verify some stability results of f-hom-ders associated with the system of additive functions (1.2) in Banach algebras.

Theorem 3.5. *Suppose that $\mathcal{A}$ is an algebra and $(\mathcal{B}, \|\cdot\|)$ is a Banach algebra. Suppose that $\varphi_1, \varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$ are functions with $\varphi_1(0,0) = 0$ and $f, g : \mathcal{A} \to \mathcal{B}$ are mappings that satisfy*

$$\|2f(\lambda x + y) - g(\lambda x) - \lambda g(y)\| \leq \varphi_1(x, y);$$
$$\|g(\lambda x + \lambda y) - 2f(\lambda y - \lambda x) - 4\lambda f(x)\| \leq \varphi_1(x, y);$$
$$\|f(xy) - f(x)f(y)\| \leq \varphi_2(x, y)$$

for all $x, y \in \mathcal{A}$ and all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If φ_1 and φ_2 satisfy

$$\Phi(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi_1(2^n x, 2^n y) < \infty \quad \text{for all } x, y \in \mathcal{A}; \tag{3.7}$$
$$\lim_{n\to\infty} \frac{1}{4^n} \varphi_2(2^n x, 2^n y) = 0 \quad \text{for all } x, y \in \mathcal{A},$$

then there exist uniquely $\mathbb{C}$-linear mappings $F, G : \mathcal{A} \to \mathcal{B}$ such that G is an F-hom-der satisfying the inequalities

$$\begin{cases} \|F(x) - f(x)\| \leq \Lambda(x) & \text{for all } x \in \mathcal{A}; \\ \|G(x) - g(x)\| \leq 2\Lambda(x) + \min\{\varphi_1(x, 0), \varphi_1(0, x)\} & \text{for all } x \in \mathcal{A}, \end{cases}$$

where $\Lambda(x) := \frac{1}{2}\left(\frac{1}{2}\Phi(x, x) + \min\{\Phi(x, 0), \Phi(0, x)\}\right)$. Moreover, $G = 2F$.

Proof . Obviously $f(0) = g(0) = 0$. Lemma 3.1 states that

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \leq \frac{1}{4}\varphi_1(x, x) + \frac{1}{2}\min\{\varphi_1(x, 0), \varphi_1(0, x)\} \quad \text{for all } x \in \mathcal{A}.$$

For any $m > n$, it can be seen that

$$\begin{aligned} \left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\right\| &= \sum_{k=n}^{m-1} \left\|\frac{1}{2^k}f(2^k x) - \frac{1}{2^{k+1}}f(2^{k+1}x)\right\| \\ &\leq \frac{1}{4}\sum_{k=n}^{m-1} \frac{1}{2^k}\varphi_1(2^k x, 2^k x) + \frac{1}{2}\min\left\{\sum_{k=n}^{m-1} \frac{1}{2^k}\varphi_1(2^k x, 0), \sum_{k=n}^{m-1} \frac{1}{2^k}\varphi_1(0, 2^k x)\right\} \end{aligned}$$

for all $x \in \mathcal{A}$. It follows from (3.7) that the sequence $(\frac{1}{2^n}f(2^n x))_{n=0}^{\infty}$ is Cauchy for all $x \in \mathcal{A}$. So, we can define $F : \mathcal{A} \to \mathcal{B}$ by $F(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$ for all $x \in \mathcal{A}$. We see from Lemma 3.1 that

$$\|2f(x) - g(x)\| \leq \min\{\varphi_1(x, 0) + \varphi_1(0, x)\} \quad \text{for all } x \in \mathcal{A}.$$

Lemma 3.4 and (3.7) assert that $G(x) := \lim_{n\to\infty} \frac{1}{2^n}g(2^n x) = 2F(x)$ for all $x \in \mathcal{A}$. We also see that

$$\|G(x) - g(x)\| \leq 2\Lambda(x) + \min\{\varphi_1(x, 0) + \varphi_1(0, x)\} \quad \text{for all } x \in \mathcal{A}.$$

As in the proof of [15, Theorem 2.2] and Theorem 2.1, we have that F and G are $\mathbb{C}$-linear mappings. Moreover, F is an algebra homomorphism. Hence, G is an F-hom-der by Corollary 2.2.

This completes the proof. □

Remark 3.6. According to Theorem 3.5, if there exists another mapping $H : \mathcal{A} \to \mathcal{B}$ such that H is an F-hom-der satisfying

$$\begin{cases} 2F(x + y) = H(x) + H(y); \\ H(x + y) = 2F(y - x) + 4F(x) \end{cases} \quad \text{for all } x, y \in \mathcal{A},$$

then $H = G$. This shows us that $2F$ is the unique F-hom-der in the class of solutions of the system of functional equations (1.2).

Theorem 3.7. *Suppose that $\mathcal{A}$ is an algebra and $(\mathcal{B}, \|\cdot\|)$ is a Banach algebra. Suppose that $\varphi_1, \varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$ are functions and $f, g : \mathcal{A} \to \mathcal{A}$ are mappings that satisfy*

$$\|2f(\lambda x + y) - g(\lambda x) - \lambda g(y)\| \le \varphi_1(x, y);$$
$$\|g(\lambda x + \lambda y) - 2f(\lambda y - \lambda x) - 4\lambda f(x)\| \le \varphi_1(x, y);$$
$$\|f(xy) - f(x)f(y)\| \le \varphi_2(x, y)$$

for all $x, y \in \mathcal{A}$ and all $\lambda \in \mathbb{T}^1$. If φ_1 and φ_2 satisfy

$$\Phi(x, y) := \sum_{n=1}^{\infty} 2^n \varphi_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \quad \text{for all } x, y \in \mathcal{A};$$
$$\lim_{n\to\infty} 4^n \varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad \text{for all } x, y \in \mathcal{A},$$

then there exist uniquely $\mathbb{C}$-linear mappings $F, G : \mathcal{A} \to \mathcal{B}$ such that G is an F-hom-der satisfying the inequalities

$$\begin{cases} \|F(x) - f(x)\| \le \Lambda(x) & \text{for all } x \in \mathcal{A}; \\ \|G(x) - g(x)\| \le 2\Lambda(x) + \min\{\varphi_1(x, 0), \varphi_1(0, x)\} & \text{for all } x \in \mathcal{A}, \end{cases}$$

where $\Lambda(x) := \frac{1}{2}\left(\frac{1}{2}\Phi(x, x) + \min\{\Phi(x, 0), \Phi(0, x)\}\right)$. Moreover, $G = 2F$.

Proof . The proof is quite similar to that of Theorem 3.5. □

To see the variety of the control functions $\varphi_1, \varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$, the following example is presented.

Example 3.8. Suppose that $\varphi_1, \varphi_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$ are functions and L_1, L_2 are positive real numbers such that $L_1, L_2 < 1$. Then the following assertions are true.

1) If $\varphi_i(2x, 2y) \le 2^i L_i \varphi_i(x, y)$ for all $x, y \in \mathcal{A}$ and all $i = 1, 2$, then

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \varphi_1(2^n x, 2^n y) \le \frac{1}{1 - L_1} \varphi_1(x, y) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{4^n} \varphi_2(2^n x, 2^n y) = 0$$

for all $x, y \in \mathcal{A}$.

2) If $\varphi_i\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L_i}{2^i} \varphi_i(x, y)$ for all $x, y \in \mathcal{A}$ and all $i = 1, 2$, then

$$\sum_{n=1}^{\infty} 2^n \varphi_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \frac{L}{1 - L_1} \varphi_1(x, y) \quad \text{and} \quad \lim_{n\to\infty} 4^n \varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in \mathcal{A}$.

Acknowledgments

The second author would like to thank the Development and Promotion of Science and Technology talents project (DPST).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] Y. Cho, C. Park and Y. Yang, *Stability of derivations in fuzzy normed algebras*, J. Nonlinear Sci. Appl. **8** (2015), 1–7.

[3] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.

[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[5] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[6] J.B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.

[7] I. Hwang and C. Park, *Bihom derivations in Banach algebras*, J. Fixed Point Theory Appl. **21** (2019), Paper No. 81.

[8] G. Isac and Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Int. J. Math. Math. Sci. **19** (1996), 219–228.

[9] M. Isar, G. Lu, Y. Jin and C. Park, *A general additive functional inequality and derivation in Banach algebras*, J. Math. Inequal. **15** (2021), 305–321.

[10] Y. Jin, C. Park and M.T. Rassias, *Hom-derivations in C^*-ternary algebras*, Acta. Math. Sin.-English Ser. **36** (2020), 1025–1038.

[11] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[12] A. Kheawborisut, S. Paokanta, J. Senasukh and C. Park, *Ulam stability of hom-ders in fuzzy Banach algebars*, AIMS Math. **7** (2022), 16556–16568.

[13] M. Mirzavaziri and M.S. Moslehian, *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Am. Math. Soc. **134** (2006), 3319–3327.

[14] M. Mirzavaziri and M.S. Moslehian, *σ-Derivations in Banach algebras*, arXiv:math.FA/0505319

[15] S. Paokanta, M. Dehghanian, C. Park and Y. Sayyari, *System of additive functional equations in Banach algebras*, preprint.

[16] S. Paokanta, C. Park and J.R. Lee, *An additive (u, β)-functional equation and linear mappings in Banach modules*, Int. J. Nonlinear Anal. Appl. **13** (2022), 1747–1755.

[17] C. Park, *Derivation-homomorphism functional inequalities*, J. Math. Inequal. **15** (2021), 95–105.

[18] C. Park and A. Bodaghi, *On the stability of $*$-derivations on Banach $*$-algebras*, Adv. Difference Equ. **2012** (2012), Paper No. 138.

[19] C. Park and X. Wu, *Homomorphism-derivation functional inequalities in C^*-algebras*, AIMS Math. **5** (2020), 4482–4493.

[20] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[21] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.

[22] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.

[23] S.Y. Yang, A. Bodaghi and K. Atan, *Approximate cubic $*$-derivations on Banach $*$-algebras*, Abstr. Appl. Anal. **2012** (2012), Article ID 684179.