Int. J. Nonlinear Anal. Appl. 14 (2023) 6, 273–280
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.23562.2558

Integration as a generalization of the integral operator

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(Communicated by Ali Jabbari)

Abstract

Let $\mathfrak{A}$ be an algebra. A *derivation* on $\mathfrak{A}$ is a linear mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in \mathfrak{A}$. As a dual to this notion, we consider a linear mapping $\Delta : \mathfrak{A} \to \mathfrak{A}$ with the property $\Delta(a)\Delta(b) = \Delta(\Delta(a)b + a\Delta(b))$ for every $a, b \in \mathfrak{A}$ and we call it an *integration*. In this paper, we give some examples, counterexamples and facts concerning integrations on algebras. Furthermore, we state and prove a characterization for integrations on finite dimensional matrix algebras.

Keywords: Derivation, integration, C*-algebra , matrix algebra
2020 MSC: 46L57, 47B47, 46L10, 47C15, 13N15, 16W25, 17A36

1 Introduction

Recall that *the Leibniz rule* for derivatives states that $(fg)' = f'g + fg'$ for each two differentiable functions. This is the main idea for the Leibniz property of a derivation on an algebra. By its definition, a linear mapping δ on an algebra $\mathfrak{A}$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in \mathfrak{A}$. The dual process to the Leibniz rule is *integration by parts* or *partial integration* process which is stated as $\int udv = uv - \int vdu$ or, equivalently, $uv = \int udv + \int vdu$. Substituting u and v into $\int t$ and $\int s$ respectively, we arrive at $\int t \int s = \int((\int t)s + t(\int s))$. This motivates us to consider those linear mappings $\Delta : \mathfrak{A} \to \mathfrak{A}$ with the property $\Delta(a)\Delta(b) = \Delta(\Delta(a)b + a\Delta(b))$ for every $a, b \in \mathfrak{A}$. We use the terminology *integration* for such linear mappings and we are interested to investigate the relation between derivations and integrations on algebras.

The integration operator is a special case of Rota-Baxter operators introduced by G. Baxter in 1960 [1].

It is not so surprising to us that there should be a calculus theory to link these notions to each other. Once we define an integration, we can consider many other notions concerning it as a dual to the notions of inner derivation, approximately inner derivations, local derivations, Jordan derivations and so on (see, for example [2, 4, 6, 7, 8]).

In Section 2, we give some examples, counterexamples and facts concerning integrations on algebras. In Section 3, we state and prove a characterization for integrations on finite dimensional matrix algebras.

Throughout the paper, $\mathfrak{A}$ is an unital algebra with unit ι and for a positive integer n, the algebra of all complex $n \times n$ matrices is denoted by $M_n(\mathbb{C})$. Recall that the matrix algebra $M_n(\mathbb{C})$ has a *system of matrix units* $\{E_{ij}\}_{1 \leqslant i,j \leqslant n}$ with the following properties:

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Received: June 2021 *Accepted:* April 2022

i. $E_{ij}^* = E_{ji}$;

ii. $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$;

ii. $\sum_{i=1}^n E_{ii}$ is the $n \times n$ identity matrix I_n,

where δ_{jk} is the Kronecker delta. By corollary 1.28 [15], a factor of type $\mathbf{I}_n$ is nothing but the $M_n(\mathbb{C})$ and then has such a system of matrix units.

2 Preliminaries

We begin this section with the definition of an integration.

Definition 2.1. A linear mapping $\Delta : \mathfrak{A} \to \mathfrak{A}$ is called an *integration* if $\Delta(a)\Delta(b) = \Delta(\Delta(a)b + a\Delta(b))$ for every $a, b \in \mathfrak{A}$.

Recall that an element ε of an algebra $\mathfrak{A}$ is called *idempotent* if $\varepsilon^2 = \varepsilon$ and an element ν is called a *square nilpotent* if $\nu^2 = 0$.

Example 2.2. Let $\mathcal{A}$ be an associative algebra and let x_0 be a square nilpotent of $\mathcal{A}$, i.e. $x_0^2 = 0$. A linear mapping $\Delta : \mathcal{A} \to \mathcal{A}$ defined by $\Delta(a) = ax_0$ is an integration on $\mathcal{A}$.

In the following proposition we see that the above example is a typical example of an integration. The proof is straightforward and so we omit it.

Proposition 2.3. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. If Δ satisfies one of the following conditions, then Δ is an integration on $\mathfrak{A}$.

i. there is a square nilpotent ν such that $\Delta(a) = \nu a \nu$ for all $a \in \mathfrak{A}$;

ii. there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = \nu$ such that $\Delta(a) = \nu a \varepsilon$ for all $a \in \mathfrak{A}$;

iii. there is a square nilpotent ν and an idempotent ε with $\nu\varepsilon = \nu$ such that $\Delta(a) = \varepsilon a \nu$ for all $a \in \mathfrak{A}$;

iv. there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = \nu$ and $\nu\varepsilon = 0$ such that $\Delta(a) = \varepsilon a \nu - \nu a \varepsilon$ for all $a \in \mathfrak{A}$;

v. there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = 0$ and $\nu\varepsilon = \nu$ such that $\Delta(a) = \varepsilon a \nu - \nu a \varepsilon$ for all $a \in \mathfrak{A}$.

The above proposition provides a collection of non-trivial examples of integrations and gives a good idea for the following definition. Prior to define the following notion, we recall that an *inner derivation* δ_{a_0} *implemented by* an element a_0 of an algebra $\mathfrak{A}$ is the derivation defined by $\delta_{a_0}(a) = a_0 a - a a_0$ for each $a \in \mathfrak{A}$. It is known that each derivation on $M_n(\mathbb{C})$ is inner and the celebrated Kadison-Sakai theorem [6, 11, 14] states that every derivation on a von Neumann algebra is inner. One of our goal in this paper is to find an appropriate definition for an inner integration.

Definition 2.4. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Then

i. Δ is called a *square nilpotent integration* if there is a square nilpotent ν such that $\Delta(a) = \nu a \nu$, for all $a \in \mathfrak{A}$;

ii. Δ is called a *nil-idempotent integration* if there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = \nu$ such that $\Delta(a) = \nu a \varepsilon$, for all $a \in \mathfrak{A}$;

iii. Δ is called an *idem-nilpotent integration* if there is a square nilpotent ν and an idempotent ε with $\nu\varepsilon = \nu$ such that $\Delta(a) = \varepsilon a \nu$, for all $a \in \mathfrak{A}$;

iv. Δ is called a *left nil integration* if there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = \nu$ and $\nu\varepsilon = 0$ such that $\Delta(a) = \varepsilon a \nu - \nu a \varepsilon$, for all $a \in \mathfrak{A}$;

v. Δ is called a *right nil integration* if there is a square nilpotent ν and an idempotent ε with $\varepsilon\nu = 0$ and $\nu\varepsilon = \nu$ such that $\Delta(a) = \varepsilon a\nu - \nu a\varepsilon$, for all $a \in \mathfrak{A}$.

Proposition 2.5. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Then

i. if Δ is a square nilpotent integration, then $\Delta(\Delta(a)b) = \Delta(a\Delta(b)) = 0$ for all $a, b \in \mathfrak{A}$;

ii. if Δ is a nil-idempotent integration, then $\Delta(\Delta(a)b) = \Delta^2(a\Delta(b)) = 0$ for all $a, b \in \mathfrak{A}$;

iii. if Δ is an idem-nilpotent integration, then $\Delta^2(\Delta(a)b) = \Delta(a\Delta(b)) = 0$ for all $a, b \in \mathfrak{A}$;

iv. if Δ is a left nil integration, then $\Delta^2(\Delta(a)b) = \Delta^2(a\Delta(b)) = 0$ for all $a, b \in \mathfrak{A}$;

v. if Δ is a right nil integration, then $\Delta^2(\Delta(a)b) = \Delta^2(a\Delta(b)) = 0$ for all $a, b \in \mathfrak{A}$;

Proof . Straightforward. $\square$

Definition 2.6. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Then Δ is called an *integration of nilpotency* r if there is a positive integer r such that $\Delta^r(\Delta(a)b) = \Delta^r(a\Delta(b)) = 0$ for each $a, b \in \mathfrak{A}$.

Definition 2.7. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Then Δ is called an *inner integration* if there is a positive integer m, there are positive linear functionals $\varphi_k : \mathfrak{A} \to \mathbb{C}$ $(1 \leqslant k \leqslant m)$ and there are $t_1, \ldots, t_m \in \mathfrak{A}$ with the following properties

$$\begin{aligned} \Delta(a) &= \sum_{k=1}^{m} \varphi_k(a)t_k, \quad a \in \mathfrak{A} \\ t_k t_\ell &= \alpha_{k\ell}t_k + \beta_{k\ell}t_\ell \quad 1 \leqslant k, \ell \leqslant m, \end{aligned}$$

for some $\alpha_{k\ell}, \beta_{k\ell} \in \mathbb{C}$, that at least one of the complex numbers $\alpha_{k\ell}$ or $\beta_{k\ell}$ is zero. In this case we say that Δ is an *inner integration implemented by* $\{\varphi_k\}_{1\leqslant k\leqslant m} \cup \{t_k\}_{1\leqslant k\leqslant m}$ with respect to $\{\alpha_{k\ell}\}_{1\leqslant k,\ell\leqslant m}$ or $\{\beta_{k\ell}\}_{1\leqslant k,\ell\leqslant m}$.

Note that if $t_k t_\ell = 0$ for all $1 \leqslant k, \ell \leqslant m$, then Δ is an inner integration of nilpotency 1. In this case, we have $\Delta(a)\Delta(b) = 0$ for each $a, b \in \mathfrak{A}$.

Example 2.8. A linear mapping $\Delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$\Delta(A) = E_{12}A = a_{21}E_{11} + a_{22}E_{12} = a_{21}\Delta(E_{21}) + a_{22}\Delta(E_{22}) = \varphi_1(a)t_1 + \varphi_2(a)t_2$$

is an inner integration implemented by $\{\varphi_1, \varphi_2, \Delta(E_{21}), \Delta(E_{22})\}$, for every $A = (a_{ij}) \in M_n(\mathbb{C})$.

Proposition 2.9. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be an inner integration implemented by $\{\varphi_k\}_{1\leqslant k\leqslant m} \cup \{t_k\}_{1\leqslant k\leqslant m}$ with respect to $\{\alpha_{k\ell}\}_{1\leqslant k,\ell\leqslant m}$ with $\alpha_{k\ell} = \delta_{k\ell}$ and $\beta_{k\ell} = 0$. Then $\jmath = \sum_{k=1}^{m} t_k$ is the identity of the algebra $\Delta(\mathfrak{A})$.

Proof . We have

$$\begin{aligned} \Delta(a)(\sum_{\ell=1}^{m} t_\ell) &= (\sum_{k=1}^{m} \varphi_k(a)t_k)(\sum_{\ell=1}^{m} t_\ell) = \sum_{k=1}^{m}\sum_{\ell=1}^{m} \varphi_k(a)t_k t_\ell \\ &= \sum_{k=1}^{m}\sum_{\ell=1}^{m} \varphi_k(a)\delta_{k\ell}t_k \\ &= \sum_{k=1}^{m} \varphi_k(a)t_k = \Delta(a). \end{aligned}$$

$\square$

In the present section, we give some elementary facts concerning integrations and inner integrations. Note that square nilpotent, nil-idempotent, idem-nilpotent, left and right nil integrations are all of nilpotency at most two.

Proposition 2.10. Let $\mathfrak{A}$ be an algebra, Δ be an integration on $\mathfrak{A}$ and let $n \geqslant 2$ be a positive integer. Then

$$\Pi_{i=1}^{n}\Delta(a_i) = \Delta\left(\Sigma_{i=1}^{n}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i \cdot \Pi_{j=i+1}^{n}\Delta(a_j)\right) \quad (*)$$

for every $a_1, \ldots, a_n \in \mathfrak{A}$.

Proof . We use induction on n. For $n = 2$ the result is true by the definition of an integration. Let us assume that $(*)$ is true for n. For $n+1$ we have

$$\begin{aligned}
\Pi_{i=1}^{n+1}\Delta(a_i) &= \Pi_{i=1}^{n}\Delta(a_i)\cdot\Delta(a_{n+1}) \\
&= \Delta\left(\Sigma_{i=1}^{n}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i\cdot\Pi_{j=i+1}^{n}\Delta(a_j)\right)\cdot\Delta(a_{n+1}) \\
&= \Delta(\Delta(\Sigma_{i=1}^{n}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i\cdot\Pi_{j=i+1}^{n}\Delta(a_j))\cdot a_{n+1} \\
&\quad +\Sigma_{i=1}^{n}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i\cdot\Pi_{j=i+1}^{n}\Delta(a_j)\cdot\Delta(a_{n+1})) \\
&= \Delta(\Pi_{i=1}^{n}\Delta(a_i)\cdot a_{n+1} \\
&\quad +\Sigma_{i=1}^{n}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i\cdot\Pi_{j=i+1}^{n+1}\Delta(a_j)) \\
&= \Delta(\Sigma_{i=1}^{n+1}\Pi_{j=1}^{i-1}\Delta(a_j)\cdot a_i\cdot\Pi_{j=i+1}^{n+1}\Delta(a_j)).
\end{aligned}$$

□

Corollary 2.11. Let $\mathfrak{A}$ be a unital algebra with unit ι. Let Δ be an integration on $\mathfrak{A}$ and let n be a positive integer. If $x = \Delta(\iota)$ then $\Delta(x^{n-1}) = \frac{x^n}{n}$.

Proof . Putting $a_1 = \ldots = a_n = \iota$ in Proposition 2.10, we have

$$x^n = \Delta(\iota)^n = \Delta(\Sigma_{i=1}^{n}x^{i-1}\iota x^{n-i}) = n\Delta(x^{n-1}).$$

□

Though, there are many examples of derivations on algebras whose ranges are not algebras, we can easily see that the range of an integration is obviously an algebra. However, even if the algebra $\mathfrak{A}$ has unit ι we can show that $\iota \notin \Delta(\mathfrak{A})$. In contrary, suppose that $\iota = \Delta(a)$ for some $a \in \mathfrak{A}$ then

$$\iota = \iota^2 = \Delta(a)^2 = \Delta(\Delta(a)a + a\Delta(a)) = 2\Delta(a) = 2\iota,$$

which is a contradiction. Nevertheless, the example $\Delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\Delta(A) = E_{12}AE_{11}$ for each $A \in M_n(\mathbb{C})$ shows that the range of an integration can be unital. Note that the mentioned integration is a nil-idempotent integration and the unit of its range is E_{11}.

It is known for two derivations δ_1 and δ_2 on an algebra $\mathfrak{A}$ and a scalar c, the linear mapping $c\delta_1 + \delta_2$ is again a derivation. A natural question which arises is whether this fact true for integrations or not. The following proposition gives a necessary and sufficient condition for an affirmative answer is some cases.

Proposition 2.12. Let Δ_1 and Δ_2 be two integration on an algebra $\mathfrak{A}$ and c be a scalar. Then $c\Delta_1 + \Delta_2$ is an integration on $\mathfrak{A}$ if and only if

$$\Delta_1(a)\Delta_2(b) + \Delta_2(a)\Delta_1(b) = \Delta_1(\Delta_2(a)b + a\Delta_2(b)) + \Delta_2(\Delta_1(a)b + a\Delta_1(b),$$

In particular, a scalar multiple of an integration is again an integration.

Proof . Straightforward.

□

As the final part of this section we give a transient consideration on a generalization of the Leibniz rule and the notion of a higher derivation.

Considering the Leibniz rule we can inductively prove that $\delta^n(ab) = \Sigma_{i=0}^{n}\binom{n}{i}\delta^i(a)\delta^{n-i}(b)$, for a derivation δ and a positive integer n. This is the starting point of studying the behaviour of the sequence $\{d_n\}_{n=0}^{\infty}$, where $d_n = \frac{\delta^n}{n!}$. The sequence is an example of a higher derivation and there are some characterizations for higher derivations on algebras (see [3, 5, 9, 10, 12, 13]). Using this idea, we have the following theorem.

Theorem 2.13. Let $\mathfrak{A}$ be an algebra, Δ be an integration on $\mathfrak{A}$ and let n be a positive integer. Then

$$\Delta^n(a)\Delta^n(b) = \Delta^n\left(\sum_{i=0}^{n}\binom{n}{i}\Delta^i(a)\Delta^{n-i}(b)\right),$$

for every $a, b \in \mathfrak{A}$.

Proof . We can inductively prove the result. For $n = 1$ the result is obvious by the definition of an integration. Let the result be true for n. For $n + 1$ we have

$$\begin{aligned}
\Delta^{n+1}(a)\Delta^{n+1}(b) &= \Delta^n(\Delta(a))\Delta^n(\Delta(b)) \\
&= \Delta^n\left(\sum_{i=0}^{n}\binom{n}{i}\Delta^i(\Delta(a))\Delta^{n-i}(\Delta(b))\right) \\
&= \Delta^n\left(\sum_{i=0}^{n}\binom{n}{i}\Delta(\Delta^i(a))\Delta(\Delta^{n-i}(b))\right).
\end{aligned}$$

Now using the partial integration process for the integration Δ, we can write

$$\begin{aligned}
&\Delta^{n+1}(a)\Delta^{n+1}(b) \\
= \;& \Delta^n\Delta\left(\sum_{i=0}^{n}\binom{n}{i}\Delta^{i+1}(a)\Delta^{n+1-(i+1)}(b) + \sum_{i=0}^{n}\binom{n}{i}\Delta^i(a)\Delta^{n+1-i}(b)\right) \\
= \;& \Delta^n\Delta\left(\sum_{k=1}^{n+1}\binom{n}{k-1}\Delta^k(a)\Delta^{n+1-k}(b) + \sum_{k=0}^{n}\binom{n}{k}\Delta^k(a)\Delta^{n+1-k}(b)\right) \\
= \;& \Delta^{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}\Delta^k(a)\Delta^{n+1-k}(b)\right).
\end{aligned}$$

□

Prior to the definition of a higher integration, let us give a simple corollary from the above theorem. We use this in the next section. Note that for an integration Δ the linear mapping Δ^2 is not necessarily an integration.

Corollary 2.14. Let $\mathfrak{A}$ be an algebra and $\Delta : \mathfrak{A} \to \mathfrak{A}$ be an integration of nilpotency r. Then $D : \mathfrak{A} \to \mathfrak{A}$ defined by $D = \Delta^r$ is an integration of nilpotency 1.

Proof . We have

$$\begin{aligned}
D(a)D(b) &= \Delta^r(a)\Delta^r(b) \\
&= \Delta^r\left(\sum_{i=0}^{r}\binom{r}{i}\Delta^i(a)\Delta^{r-i}(b)\right) \\
&= 0 \\
&= \Delta^r(\Delta^r(a)b + a\Delta^r(b)) \\
&= D(D(a)b + aD(b)).
\end{aligned}$$

□

Definition 2.15. Let $\mathfrak{A}$ be an algebra. A sequence $\{D_n\}_{n=0}^{\infty}$ is called a *higher integration* if D_0 is the identity mapping on $\mathfrak{A}$ and

$$D_n(a)D_n(b) = D_n(\Sigma_{i=0}^{n} D_k(a)D_{n-k}(b))$$

for every $a, b \in \mathfrak{A}$.

Theorem 2.16. Let Δ be an integration on an algebra $\mathfrak{A}$. Define $D_n : \mathfrak{A} \to \mathfrak{A}$ by $D_n = \frac{\Delta^n}{n!}$ and let D_0 be the identity mapping on $\mathfrak{A}$. Then $\{D_n\}_{n=0}^{\infty}$ is higher integration.

Proof . Divide both sides of the equation mentioned in Theorem 2.13 by $n!^2$. □

3 Integrations on simle finite dimensional C*-algebras

In this section we assume that $\mathscr{A}$ is a simle finite dimensional C*-algebra. By proof of the theorem 11.2 [15], each element $a \in \mathscr{A}$ can be written as $a = \sum_{i,j=1}^{n} a_{ij}(a)e_{ij}$, such that $a_{ij}(a) \in M_n(\mathbb{C})$ and $\{e_{ij}\}_{1\leqslant i,j\leqslant n}$ is a *finite system of matrix units* with the following properties:

i. $e_{ij}^* = e_{ji}$;

ii. $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$;

ii. $\sum_{i=1}^{n} e_{ii}$ is the identity ι of $\mathscr{A}$,

where δ_{jk} is the Kronecker delta. In fact the map $T : \mathscr{A} \to M_n(\mathbb{C})$ defined by $T(a) = (a_{ij}(a))_{ij}$, is a $*-$isomorphism. For two elements $a, b \in \mathscr{A}$ we have

$$ab = (\sum_{i,j=1}^{n} a_{ij}(a)e_{ij})(\sum_{r,s=1}^{n} b_{rs}(b)e_{rs}) = \sum_{i,s=1}^{n}(\sum_{r=1}^{n} a_{ir}(a)b_{rs}(b))e_{is}.$$

The trace of a, denoted by $\mathrm{tr}(a)$, is defined by $\mathrm{tr}(a) = \sum_{i=1}^{n} a_{ii}(a)$. The following fact about the trace functional is useful.

Lemma 3.1. Let $a, b \in \mathscr{A}$. Then $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(ae_{ij}) = a_{ji}(a)$.

Proof . Straightforward. □

As an example, if $\mathfrak{H}$ is a finite dimensional Hilbert space, then $\mathbf{B}(\mathfrak{H})$ has such a matrix unit. We can therefore deduce that results of this section are true for $M_n(\mathbb{C})$. In this section, our ultimate goal is to prove that each integration on such algebras is inner.

Lemma 3.2. Let $\Delta : \mathscr{A} \to \mathscr{A}$ be a linear mapping and $\dim(\Delta(\mathscr{A})) = m$ for a positive integer m. Then there are $f_1, \ldots, f_m, t_1, \ldots, t_m \in \mathscr{A}$ such that $\Delta(a) = \sum_{k=1}^{m} \mathrm{tr}(af_k)t_k$.

Proof . Let $\{t_1, \ldots, t_m\}$ be a basis for $\Delta(\mathscr{A})$. Thus there are linear functionals $\varphi_k : \mathscr{A} \to \mathbb{C}$ $(1 \leqslant k \leqslant m)$ such that $\Delta(a) = \sum_{k=1}^{m} \varphi_k(a)t_k$. Now let $f_k = \sum_{i,j=1}^{n} \varphi_k(e_{ji})e_{ij}$. This implies that $(f_k)_{ij} = \varphi_k(e_{ji})$. We can write

$$\begin{aligned}
\Delta(a) &= \sum_{k=1}^{m} \varphi_k(a)t_k = \sum_{k=1}^{m} \varphi_k(\sum_{i,j=1}^{n} a_{ij}(a)e_{ij})t_k \\
&= \sum_{k=1}^{m}\sum_{i,j=1}^{n} a_{ij}(a)\varphi_k(e_{ij})t_k = \sum_{k=1}^{m}\sum_{i,j=1}^{n} a_{ij}(a)(f_k)_{ji}t_k \\
&= \sum_{k=1}^{m}\sum_{i=1}^{n}(af_k)_{ii}t_k = \sum_{k=1}^{m} \mathrm{tr}(af_k)t_k.
\end{aligned}$$

□

Definition 3.3. Let $\Delta : \mathscr{A} \to \mathscr{A}$ be a linear mapping. We say that Δ *is representable by* $\{f_k\}_{1\leqslant k\leqslant m} \cup \{t_k\}_{1\leqslant k\leqslant m}$ if $\Delta(a) = \sum_{k=1}^{m} \mathrm{tr}(af_k)t_k$.

Proposition 3.4. Let $\Delta : \mathscr{A} \to \mathscr{A}$ be an integration. Then Δ can be represented by $\{f_k\}_{1\leqslant k\leqslant m} \cup \{t_k\}_{1\leqslant k\leqslant m}$ with the following properties

i. there is a subset $N = \{(i_k, j_k) : 1 \leqslant k \leqslant m\}$ of the set $N_n^2 = \{(i, j) : 1 \leqslant i, j \leqslant n\}$ such that $t_k = \Delta(e_{i_k j_k})$ for $1 \leqslant k \leqslant m$ and $\Delta(e_{ij}) = 0$ for $(i, j) \notin N$;

ii. $f_k = e_{j_k i_k}$ for $1 \leqslant k \leqslant m$;

iii. $t_k t_\ell = \alpha_{k\ell}t_k + \beta_{k\ell}t_\ell$ for $1 \leqslant k, \ell \leqslant m$, that at least one of the complex numbers $\alpha_{k\ell}$ or $\beta_{k\ell}$ is zero.

Proof . Let $\dim(\Delta(\mathscr{A})) = m$. We know that $U = \{\Delta(e_{ij}) : (i,j) \in N_n^2\}$ generates $\Delta(\mathscr{A})$. Thus it contains a basis $\{\Delta(e_{i_k j_k}) : 1 \leqslant k \leqslant m\}$. Put $t_k = \Delta(e_{i_k j_k})$.

Applying the notations used in Lemma 3.2 we have $\Delta(a) = \sum_{k=1}^m \mathrm{tr}(af_k)t_k$. Considering $t_\ell = \Delta(e_{i_\ell j_\ell}) = \sum_{k=1}^m \mathrm{tr}(e_{i_\ell j_\ell} f_k)t_k$ and the fact that $\{t_1, \ldots, t_m\}$ is an independence set, we can deduce that $(f_k)_{j_\ell i_\ell} = \mathrm{tr}(e_{i_\ell j_\ell} f_k) = \delta_{k\ell}$. Thus $f_k = e_{j_k i_k}$.

Now we have

$$\begin{aligned}
t_k t_\ell &= \Delta(e_{i_k j_k})\Delta(e_{i_\ell j_\ell}) \\
&= \Delta(\Delta(e_{i_k j_k})e_{i_\ell j_\ell} + e_{i_k j_k}\Delta(e_{i_\ell j_\ell})) \\
&= \Delta(t_k e_{i_\ell j_\ell} + e_{i_k j_k} t_\ell) \\
&= \sum_{r=1}^m \mathrm{tr}(t_k e_{i_\ell j_\ell} f_r)t_r + \sum_{r=1}^m \mathrm{tr}(e_{i_k j_k} t_\ell f_r)t_r \\
&= \sum_{r=1}^m \mathrm{tr}(t_k e_{i_\ell j_\ell} e_{j_r i_r})t_r + \sum_{r=1}^m \mathrm{tr}(e_{i_k j_k} t_\ell e_{j_r i_r})t_r \\
&= \sum_{r=1}^m \delta_{j_\ell j_r}\mathrm{tr}(t_k e_{i_\ell i_r})t_r + \sum_{r=1}^m \mathrm{tr}(e_{j_r i_r} e_{i_k j_k} t_\ell)t_r \\
&= \sum_{r=1}^m \delta_{j_\ell j_r}\mathrm{tr}(t_k e_{i_\ell i_r})t_r + \sum_{r=1}^m \delta_{i_r i_k}\mathrm{tr}(e_{j_r j_k} t_\ell)t_r \\
&= \sum_{r=1}^m \delta_{j_\ell j_r}\delta_{i_\ell i_r}(t_k)_{i_r i_r} t_r + \sum_{r=1}^m \delta_{i_r i_k}\delta_{j_r j_k}(t_\ell)_{j_r j_r} t_r \\
&= (t_k)_{i_\ell i_\ell} t_\ell + (t_\ell)_{j_k j_k} t_k.
\end{aligned}$$

We can therefore deduce that

$$t_k^2 = \alpha_k t_k, \quad t_k t_\ell = \alpha_{k\ell} t_k + \beta_{k\ell} t_\ell \quad (*)$$

for some $\alpha_k, \alpha_{k\ell}, \beta_{k\ell} \in \mathbb{C}$. We have $\alpha_k t_k t_\ell = t_k^2 t_\ell = \alpha_{k\ell} t_k^2 + \beta_{k\ell} t_k t_\ell = \alpha_k \alpha_{k\ell} t_k + \beta_{k\ell} t_k t_\ell$.

Now there are two cases:

If $\alpha_k = 0$, then $\beta_{k\ell} = 0$ or $t_k t_\ell = 0$. Regarding to $(*)$, in any case we have $\beta_{k\ell} = 0$.

And if $\alpha_k \neq 0$ and $\beta_{k\ell} \neq 0$, then

$$(\alpha_k \beta_{k\ell} - \beta_{k\ell}^2)t_\ell = \alpha_{k\ell}\beta_{k\ell} t_k,$$

and since $\{t_k, t_\ell\}$ is an independence set we have $\alpha_{k\ell} = 0$. Anyhow, at least one of the complex numbers $\beta_{k\ell}$ or $\alpha_{k\ell}$ is zero, and thus we have the result. □

Theorem 3.5. Let $\mathscr{A}$ be a simple finite dimensional C*-algebra, and let $\Delta : \mathscr{A} \to \mathscr{A}$ be a linear mapping. Then Δ is an integration if and only if it is an inner integration implemented by the trace functionals $\varphi_k : \mathscr{A} \to \mathbb{C}$ defined by $\varphi_k(a) = \mathrm{tr}(a e_{j_k i_k})$ and the set $\{t_k = \Delta(e_{i_k j_k}) : 1 \leqslant k \leqslant m\}$ with respect to $\alpha_{k\ell} = \Delta(e_{i_\ell j_\ell})_{j_k j_k}$ or $\beta_{k\ell} = \Delta(e_{i_k j_k})_{i_\ell i_\ell}$, that at least one of the complex numbers $\alpha_{k\ell}$ or $\beta_{k\ell}$ is zero. i.e.,

$$\begin{aligned}
\Delta(a) &= \sum_{k=1}^n \mathrm{tr}(a e_{j_k i_k})\Delta(e_{i_k j_k}), \quad a \in \mathscr{A} \\
\Delta(e_{i_k j_k})\Delta(e_{i_\ell j_\ell}) &= \Delta(e_{i_\ell j_\ell})_{j_k j_k}\Delta(e_{i_k j_k}) + \Delta(e_{i_k j_k})_{i_\ell i_\ell}\Delta(e_{i_\ell j_\ell}) \quad 1 \leqslant k, \ell \leqslant m.
\end{aligned}$$

Proof . Straightforward. □

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