Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1237–1247
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25299.2983

Best proximant for set-valued maps via proximal relations

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(Communicated by Ali Farajzadeh)

Abstract

We familiarize $\mathcal{AM}$ and $\mathcal{SAM}-$contractions involving rational terms to prove the best proximant for discontinuous set-valued maps in partially ordered metric spaces. In the sequel, we demonstrate that completeness of space or subspace is not mandatory for the survival of the best proximant of set-valued maps. Obtained outcomes are unifications, extensions, improvements, and generalizations of some of the widely known results. We provide non-trivial illustrations to exhibit the importance of our explorations.

Keywords: Weak $P-$ property, α-proximal admissible, partially ordered
2020 MSC: 41A65, 47H10, 54H25

1 Introduction and Preliminaries

Innumerable real-world problems may be re-framed as a problem of finding a fixed point. There are situations when it is not possible to get a fixed point for a certain map. In such cases, it is normal to attain an approximate fixed point in place of a fixed point. A non-self map $\mathcal{A}:\mathcal{V}\to\mathcal{W}$ of a metric space $(\mathcal{Z},d)$ have a fixed point in $\mathcal{V}$ if $\mathcal{A}\mathfrak{v}=\mathfrak{v}$ has an exact solution, i.e., $d(\mathfrak{v},\mathcal{A}\mathfrak{v})=0$. If exact solution of $\mathcal{A}\mathfrak{v}=\mathfrak{v}$ does not exist, then $d(\mathfrak{v},\mathcal{A}\mathfrak{v})>0$. In that condition, we wish to obtain $\mathfrak{v}\in\mathcal{V}$ (a best proximant of $\mathcal{A}$ in $\mathcal{V}$), so that $d(\mathfrak{v},\mathcal{A}\mathfrak{v})$ is minimum. It is interesting to see that an optimization problem may be converted to investigating a best proximity point. The best proximant is the same as a fixed point if $d(\mathcal{V},\mathcal{W})=0$, i.e., if the involved map is presumed to be a self map. It is worth mentioning here that Ky Fan [6] was the first to answer the question that what happens if a map under consideration does not have a fixed point.

In the present work, we introduce $\mathcal{SAM}$ and $\mathcal{AM}-$contractions involving rational terms to establish a best proximant for set-valued maps in a partially ordered metric space. In the sequel, we demonstrate that completeness of underlying space or subspace is not a necessary assertion for the survival of a best proximant of a single or a pair of set-valued maps.

Let $\mathcal{V}$ and $\mathcal{W}$ be non-void subsets of a metric space $(\mathcal{Z},d)$. We symbolize a partially ordered metric space by the

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Received: November 2021 *Accepted:* October 2022

triplet $(\mathcal{Z}, d, \preceq)$. Let

$$\begin{aligned}
&\delta(\mathcal{V},\mathcal{W}) := \sup\{d(\mathfrak{v},\mathfrak{w}) : \mathfrak{v}\in\mathcal{V} \text{ and } \mathfrak{w}\in\mathcal{W}\};\\
&\mathcal{D}(\mathcal{V},\mathcal{W}) := \inf\{d(\mathfrak{v},\mathfrak{w}) : \mathfrak{v}\in\mathcal{V} \text{ and } \mathfrak{w}\in\mathcal{W}\};\\
&\mathcal{V}_0 := \{\mathfrak{v}\in\mathcal{V} : d(\mathfrak{v},\mathfrak{w}) = \mathcal{D}(\mathcal{V},\mathcal{W}), \text{ for some } \mathfrak{w}\in\mathcal{W}\};\\
&\mathcal{W}_0 := \{\mathfrak{v}\in\mathcal{V} : d(\mathfrak{v},\mathfrak{w}) = \mathcal{D}(\mathcal{V},\mathcal{W}), \text{ for some } \mathfrak{v}\in\mathcal{V}\}.
\end{aligned}$$

Definition 1.1. [1] An element $\mathfrak{v}\in\mathcal{V}$ of a set-valued map $\mathcal{A} : \mathcal{V} \longrightarrow 2^{\mathcal{W}}$ is a best proximity point of $\mathcal{A}$ if $\mathcal{D}(\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \mathcal{D}(\mathcal{V},\mathcal{W})$.

Definition 1.2. [12]An element $\mathfrak{v}\in\mathcal{V}$ of set-valued maps $\mathcal{S}, \mathcal{A} : \mathcal{V} \longrightarrow 2^{\mathcal{W}}$ is a common best proximity point of $\mathcal{S}$ and $\mathcal{A}$ if $\mathcal{D}(\mathfrak{v}, \mathcal{S}\mathfrak{v}) = \mathcal{D}(\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \mathcal{D}(\mathcal{V},\mathcal{W})$.

Remark 1.3. It is interesting to mention here that proximal points were first defined by Pai [10] as: $\mathfrak{v}\in\mathcal{V}$ and $\mathfrak{w}\in\mathcal{W}$ are proximal points if $\|\mathfrak{v}-\mathfrak{w}\| = \mathcal{D}(\mathcal{V},\mathcal{W})$. A pair $(\mathfrak{v},\mathfrak{w})$ is named a best proximity pair of $\mathcal{V}$ and $\mathcal{W}$ by Xu [15]. Eldred and Veeramani [4] (Eldred et al. [5]) appears first to name a point $\mathfrak{v}$ to be a best proximity point of $\mathcal{A} : \mathcal{V}\cup\mathcal{W} \longrightarrow \mathcal{V}\cup\mathcal{W}$ if $\mathcal{A}(\mathcal{V}) \subseteq \mathcal{W}$, $\mathcal{A}(\mathcal{W}) \subseteq \mathcal{V}$ and $d(\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \mathcal{D}(\mathcal{V},\mathcal{W})(\|\mathfrak{v}-\mathcal{A}\mathfrak{v}\| = \mathcal{D}(\mathcal{V},\mathcal{W}))$, $\mathfrak{v}\in\mathcal{V}\cup\mathcal{W}$.

Definition 1.4 ([7] and [16]). A pair $(\mathcal{V},\mathcal{W})$ have the weak $P-$property iff

$$\begin{cases} d(\mathfrak{v}_1,\mathfrak{w}_1) = \mathcal{D}(\mathcal{V},\mathcal{W}) \\ d(\mathfrak{v}_2,\mathfrak{w}_2) = \mathcal{D}(\mathcal{V},\mathcal{W}) \end{cases} \Longrightarrow d(\mathfrak{v}_1,\mathfrak{v}_2) \le d(\mathfrak{w}_1,\mathfrak{w}_2),$$

$\mathcal{V}_0 \neq \emptyset$, $\mathfrak{v}_1$, $\mathfrak{v}_2\in\mathcal{V}_0$ and $\mathfrak{w}_1$, $\mathfrak{w}_2\in\mathcal{W}_0$.

Definition 1.5. [11] The proximal relations in a partially ordered metric space $(\mathcal{Z}, d, \preceq)$ between two non-empty subsets $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{W}_0$, for $\mathfrak{v}_1, \mathfrak{v}_2\in\mathcal{V}_0$ are

- **(i)** $\mathcal{W}_1 \prec_{(1)} \mathcal{W}_2$ if $\mathfrak{w}_1\in\mathcal{W}_1$ with $d(\mathfrak{v}_1,\mathfrak{w}_1) = \mathcal{D}(\mathcal{V},\mathcal{W})$ there exists $\mathfrak{w}_2\in\mathcal{W}_2$ with $d(\mathfrak{v}_2,\mathfrak{w}_2) = \mathcal{D}(\mathcal{V},\mathcal{W})$ such that $\mathfrak{v}_1 \preceq \mathfrak{v}_2$;
- **(ii)** $\mathcal{W}_1 \prec_{(2)} \mathcal{W}_2$ if $\mathfrak{w}_2\in\mathcal{W}_2$ with $d(\mathfrak{v}_2,\mathfrak{w}_2) = \mathcal{D}(\mathcal{V},\mathcal{W})$ there exists $\mathfrak{w}_1\in\mathcal{W}_1$ with $d(\mathfrak{v}_1,\mathfrak{w}_1) = \mathcal{D}(\mathcal{V},\mathcal{W})$ such that $\mathfrak{v}_1 \preceq \mathfrak{v}_2$;
- **(iii)** $\mathcal{W}_1 \prec_{(3)} \mathcal{W}_2$ if $\mathcal{W}_1 \prec_{(1)} \mathcal{W}_2$ and $\mathcal{W}_1 \prec_{(2)} \mathcal{W}_2$.

2 Main Results

The aim is to define $\mathcal{AM}$ and $\mathcal{SAM}-$contractions in $(\mathcal{Z}, d, \preceq)$ to establish the best proximant theorems by giving a short and simple proof exploiting weak $P-$property.

Definition 2.1. Let $\mathcal{V}$ and $\mathcal{W}$ be non-void closed subsets of $(\mathcal{Z}, d, \preceq)$, $\mathcal{V}_0 \neq \emptyset$. Let $\phi : [0,\infty) \to [0,\infty)$ be a control function so that $\phi(t) < t$ and $\phi(t) = 0$ iff $t = 0$. Then a set-valued map $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ is said to be $\mathcal{AM}-$contraction if

$$\begin{aligned}
\delta(\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{w}) \le \mathfrak{m}_1\, d(\mathfrak{v},\mathfrak{w}) + \mathfrak{m}_2\,\phi\big(\max\{d(\mathfrak{v},\mathfrak{w}), \mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W})\}\big)\\
+\mathfrak{m}_3\,\phi\left(\frac{d(\mathfrak{v},\mathfrak{w})[1+\sqrt{d(\mathfrak{v},\mathfrak{w})(\mathcal{D}(\mathfrak{v},A\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}]^2}{(1+d(\mathfrak{v},\mathfrak{w}))^2}\right)
\end{aligned} \tag{2.1}$$

for comparable $\mathfrak{v}$, $\mathfrak{w}\in\mathcal{V}$ and $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3 \ge 0$, $\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3 < 1$.

Definition 2.2. Let $\mathcal{V}$ and $\mathcal{W}$ be non-void closed subsets of $(\mathcal{Z}, d, \preceq)$, $\mathcal{V}_0 \neq \emptyset$. Let $\phi : [0,\infty) \to [0,\infty)$ be a control function so that $\phi(t) < t$ and $\phi(t) = 0$ iff $t = 0$. Then a set-valued map $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ is said to be generalized

$\mathcal{AM}-$contraction if

$$\delta(\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{w}) \le \mathfrak{m}_1\, d(\mathfrak{v},\mathfrak{w}) + \mathfrak{m}_2\, \phi\left(d(\mathfrak{v},\mathfrak{w}) + \max\left\{ \begin{cases} d(\mathfrak{v},\mathfrak{w}), \\ \mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \frac{\mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{w})+\mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v})}{2} - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \frac{\mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{v})+\mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{w})}{2} - \mathcal{D}(\mathcal{V},\mathcal{W}) \end{cases} \right. \right)$$
$$+\mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v},\mathfrak{w})\left[1+\sqrt{d(\mathfrak{v},\mathfrak{w})(\mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}\right]^2}{(1+d(\mathfrak{v},\mathfrak{w}))^2} \right), \tag{2.2}$$

for comparable $\mathfrak{v}$, $\mathfrak{w} \in \mathcal{V}$ and $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3 \ge 0$, and $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 < 1$.

Definition 2.3. Let $\mathcal{V}$ and $\mathcal{W}$ be non-void closed subsets of $(\mathcal{Z}, d, \preceq)$ and $\mathcal{V}_0 \neq \emptyset$. Let $\phi : [0,\infty) \to [0,\infty)$ be a control function so that $\phi(t) < t$ and $\phi(t) = 0$ iff $t = 0$ and $\alpha : \mathcal{V}^2 \longrightarrow [0,\infty)$. Then a pair $(\mathcal{S},\mathcal{A})$ of set-valued maps $\mathcal{S}$, $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ is said to be $\mathcal{SAM}-$contraction if

$$\alpha(\mathfrak{v},\mathfrak{w})\ \delta(\mathcal{A}\mathfrak{v},\mathcal{S}\mathfrak{w}) \le \mathfrak{m}_1\, d(\mathfrak{v},\mathfrak{w}) + \mathfrak{m}_2\, \phi\left(\max\left\{ \begin{cases} d(\mathfrak{v},\mathfrak{w}), \\ \mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \mathcal{D}(\mathfrak{w},\mathcal{S}\mathfrak{w}) - \mathcal{D}(\mathcal{A}\mathfrak{v},\mathcal{S}\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}) \end{cases} \right. \right)$$
$$+\mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v},\mathfrak{w})\left[1+\sqrt{d(\mathfrak{v},\mathfrak{w})(\mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}\right]^2}{(1+d(\mathfrak{v},\mathfrak{w}))^2} \right) \tag{2.3}$$

for comparable $\mathfrak{v}$, $\mathfrak{w} \in V$, $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \ge 0$ and $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 < 1$.

Definition 2.4. Let $\mathcal{V}$ and $\mathcal{W}$ be non-void closed subsets of $(\mathcal{Z}, d, \preceq)$ and $\mathcal{V}_0 \neq \emptyset$. Let $\phi : [0,\infty) \to [0,\infty)$ be a control function so that $\phi(t) < t$ and $\phi(t) = 0$ iff $t = 0$ and $\alpha : \mathcal{V}^2 \longrightarrow [0,\infty)$. Then a pair $(\mathcal{S},\mathcal{A})$ of set-valued maps $\mathcal{S}$, $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ is said to be generalized $\mathcal{SAM}-$contraction if

$$\alpha(\mathfrak{v},\mathfrak{w})\ \delta(\mathcal{A}\mathfrak{v},\mathcal{S}\mathfrak{w}) \le \mathfrak{m}_1\, d(\mathfrak{v},\mathfrak{w}) + \mathfrak{m}_2\, \phi\left(d(\mathfrak{v},\mathfrak{w}) + \max\left\{ \begin{cases} d(\mathfrak{v},\mathfrak{w}), \\ \mathcal{D}(\mathfrak{w},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \mathcal{D}(\mathfrak{w},\mathcal{S}\mathfrak{w}) - \mathcal{D}(\mathcal{A}\mathfrak{v},\mathcal{S}\mathfrak{w}) - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \frac{\mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{v})+D(\mathfrak{w},\mathcal{S}\mathfrak{w})}{2} - \mathcal{D}(\mathcal{V},\mathcal{W}), \\ \frac{D(\mathfrak{v},\mathcal{A}\mathfrak{w})+\mathcal{D}(\mathfrak{w},\mathcal{S}\mathfrak{v})}{2} - \mathcal{D}(\mathcal{V},\mathcal{W}) \end{cases} \right. \right)$$
$$+\mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v},\mathfrak{w})\left[1+\sqrt{d(\mathfrak{v},\mathfrak{w})(\mathcal{D}(\mathfrak{v},\mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}\right]^2}{(1+d(\mathfrak{v},\mathfrak{w}))^2} \right), \tag{2.4}$$

for comparable $\mathfrak{v}$, $\mathfrak{w} \in V$, $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \ge 0$ and $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 < 1$.

Next, we present the first main conclusion for $\mathcal{AM}-$contraction:

Theorem 2.5. Let $\mathcal{A} : \mathcal{V} \longrightarrow 2^{\mathcal{W}}$ be an $\mathcal{AM}-$ contraction (2.1) in a partially ordered metric space $(\mathcal{Z}, d, \preceq)$ satisfying

- **(i)** $\mathcal{A}\mathfrak{v}_0 \subseteq \mathcal{W}_0$, $\mathfrak{v}_0 \in \mathcal{V}_0$,
- **(ii)** $\mathfrak{v}_0$, $\mathfrak{v}_1 \in \mathcal{V}_0$ and $\mathfrak{w}_0 \in \mathcal{A}\mathfrak{v}_0$ satisfying $d(\mathfrak{v}_1,\mathfrak{w}_0) = \mathcal{D}(\mathcal{V},\mathcal{W})$ and $\mathfrak{v}_0 \preceq \mathfrak{v}_1$,
- **(iii)** a pair $(\mathcal{V},\mathcal{W})$ satisfies weak P-property,
- **(iv)** for all $\mathfrak{v}$, $\mathfrak{w} \in \mathcal{V}_0$, $\mathfrak{v} \preceq \mathfrak{w}$ implies that $\mathcal{A}\mathfrak{v} \prec_{(1)} \mathcal{A}\mathfrak{w}$,
- **(v)** if $\{\mathfrak{v}_n\}$ is an increasing sequence in $\mathcal{V}$ so that $\mathfrak{v}_n \to \mathfrak{v}$ then $\mathfrak{v}_n \preceq \mathfrak{v}$, $n \in \mathbb{N}$.

Then $\mathfrak{v} \in \mathcal{V}$ is a best proximity point of $\mathcal{A}$.

Proof . From (ii), there exists $\mathfrak{v}_0$, $\mathfrak{v}_1 \in \mathcal{V}_0$ and $\mathfrak{w}_0 \in \mathcal{A}\mathfrak{w}_0 \subset \mathcal{W}_0$ so that $d(\mathfrak{v}_1,\mathfrak{w}_0) = \mathcal{D}(\mathcal{V},\mathcal{W})$ and $\mathfrak{v}_0 \preceq \mathfrak{v}_1$. By (iv), $\mathcal{A}\mathfrak{v}_0 \prec_{(1)} \mathcal{A}\mathfrak{v}_1$ and there exists $\mathfrak{w}_1 \in \mathcal{A}\mathfrak{v}_1$ and $d(\mathfrak{v}_2,\mathfrak{w}_1) = \mathcal{D}(\mathcal{V},\mathcal{W})$ so that $\mathfrak{v}_1 \preceq \mathfrak{v}_2$ and so on. This implies that,

$$d(\mathfrak{v}_{n+1},\mathfrak{w}_n) = \mathcal{D}(\mathfrak{v}_{n+1},T\mathfrak{v}_n) = \mathcal{D}(V,\mathcal{W}),\ \ \mathfrak{v}_{n+1} \in \mathcal{V}_0 \ \text{ and } \ \mathfrak{w}_n \in \mathcal{A}\mathfrak{v}_n, \tag{2.5}$$

$n \in \mathbb{N} \cup \{0\}$ and $\mathfrak{v}_0 \preceq \mathfrak{v}_1 \preceq \mathfrak{v}_2 \preceq \cdots \mathfrak{v}_n \preceq \mathfrak{v}_{n+1} \preceq \cdots$. If $n_0 \in \mathbb{N} \cup \{0\}$, so that $\mathfrak{v}_{n_0} = \mathfrak{v}_{n_0+1}$, then $\mathcal{D}(\mathfrak{v}_{n_0+1}, \mathfrak{w}_{n_0}) = \mathcal{D}(\mathfrak{v}_{n_0+1}, \mathcal{A}\mathfrak{v}_{n_0}) = \mathcal{D}(\mathfrak{v}_{n_0}, \mathcal{A}\mathfrak{v}_{n_0}) = (\mathcal{D}(\mathcal{V}, \mathcal{W}))$, i.e., $\mathfrak{v}_{n_0}$ is a best proximity point of $\mathcal{A}$. Therefore, the proof is finished. Now, let $\mathfrak{v}_n \neq \mathfrak{v}_{n+1}$, $n \in \mathbb{N} \cup \{0\}$. As $d(\mathfrak{v}_{n+1}, \mathfrak{w}_n) = \mathcal{D}(\mathcal{V}, \mathcal{W})$ and $d(\mathfrak{v}_n, w_{n-1}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$, we obtain

$$d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq d(\mathfrak{w}_{n-1}, \mathfrak{w}_n), \quad \mathfrak{n} \in \mathbb{N}, \quad (\text{using } (iii)). \tag{2.6}$$

Since, $\mathfrak{v}_{n-1} \prec \mathfrak{v}_n$, using inequalities (2.1), (2.5), (2.6), and triangle inequality of d, we obtain

$$\begin{aligned}
d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) &\leq d(\mathfrak{w}_{n-1}, \mathfrak{w}_n) \leq \delta(\mathcal{A}\mathfrak{v}_{n-1}, \mathcal{A}\mathfrak{v}_n) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}_{n-1}, \mathfrak{v}_n), \\ \mathcal{D}(\mathfrak{v}_{n-1}, \mathcal{A}\mathfrak{v}_n) - (\mathcal{D}(\mathcal{V}, \mathcal{W})), \\ d(\mathfrak{v}_n, \mathcal{A}\mathfrak{v}_{n-1}) - (\mathcal{D}(\mathcal{V}, \mathcal{W})) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)[1 + \sqrt{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)(\mathcal{D}(\mathfrak{v}_{n-1}, \mathcal{A}\mathfrak{v}_n) - (\mathcal{D}(\mathcal{V}, \mathcal{W})))}]^2}{(1 + d(\mathfrak{v}_{n-1}, \mathfrak{v}_n))^2} \right) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}_{n-1}, \mathfrak{v}_n), \\ d(\mathfrak{v}_{n-1}, \mathfrak{w}_n) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ d(\mathfrak{v}_n, \mathfrak{w}_{n-1}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3 \phi\left(\frac{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)[1 + \sqrt{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)(d(\mathfrak{v}_{n-1}, \mathfrak{w}_n) - (\mathcal{D}(\mathcal{V}, \mathcal{W})))}]^2}{(1 + d(\mathfrak{v}_{n-1}, \mathfrak{v}_n))^2} \right) \\
&= \mathfrak{m}_1\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}_{n-1}, \mathfrak{v}_n), \\ \mathcal{D}(\mathcal{V}, \mathcal{W}) - (\mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathcal{V}, \mathcal{W}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3 \phi\left(\frac{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)[1 + \sqrt{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)((\mathcal{D}(\mathcal{V}, \mathcal{W}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{(1 + d(\mathfrak{v}_{n-1}, \mathfrak{v}_n))^2} \right) \\
&= \mathfrak{m}_1\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathfrak{m}_2\, \phi\left(\max \left\{ d(\mathfrak{v}_{n-1}, \mathfrak{v}_n), 0, 0 \right. \right) + \mathfrak{m}_3 \phi\left(\frac{d(\mathfrak{v}_{n-1}, \mathfrak{v}_n)}{(1 + d(\mathfrak{v}_{n-1}, \mathfrak{v}_n))^2} \right) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) + \mathfrak{m}_2\, \phi\Big(d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) \Big) + \mathfrak{m}_3\, \phi\Big(d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) \Big).
\end{aligned}$$

Since, $\phi(t) < t$, so

$$d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq (\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3)\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n).$$

Let $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 = \eta$. Then

$$d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \eta\, d(\mathfrak{v}_{n-1}, \mathfrak{v}_n).$$

Similarly,

$$d(\mathfrak{v}_{n-1}, \mathfrak{v}_n) \leq \eta\, d(\mathfrak{v}_{n-2}, \mathfrak{v}_{n-1}).$$

Following this pattern, we attain

$$d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq \eta^n\, d(\mathfrak{v}_0, \mathfrak{v}_1).$$

Since, $\eta \in (0, 1)$, $\eta^n \to 0$, as $n \to \infty$. So

$$\lim_{n \to \infty} d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) = 0. \tag{2.7}$$

Now, we assert that $\{\mathfrak{v}_n\}$ is a Cauchy sequence. Let $m > n$, we have

$$\begin{aligned}
d(\mathfrak{v}_m, \mathfrak{v}_n) &\leq d(\mathfrak{v}_m, \mathfrak{v}_{m-1}) + d(\mathfrak{v}_{m-1}, \mathfrak{v}_{m-2}) + \cdots + d(\mathfrak{v}_{n+1}, \mathfrak{v}_n) \\
&\leq [\eta^{m-1} + \eta^{m-2} + \cdots + \eta^n]\, d(\mathfrak{v}_0, \mathfrak{v}_1) \\
&\leq \left(\frac{\eta^n (1 - \eta^{m-n})}{1 - \eta} \right) d(\mathfrak{v}_0, \mathfrak{v}_1) \\
&\leq \left(\frac{\eta^n}{1 - \eta} \right) d(\mathfrak{v}_0, \mathfrak{v}_1) \to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

i.e., $\{\mathfrak{v}_n\}$ is a Cauchy sequence in a closed set $\mathcal{V}$ and so the limit of each Cauchy sequence contained in $\mathcal{V}$ is also an element of $\mathcal{V}$. Therefore, it converges to $\mathfrak{v} \in \mathcal{V}$. Since, $d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq d(\mathfrak{w}_{n-1}, \mathfrak{w}_n)$, sequence $\{\mathfrak{w}_n\}$ is a Cauchy in a closed set $\mathcal{W}$, converging to $\mathfrak{w} \in \mathcal{W}$. Since, $d(\mathfrak{v}_{n+1}, \mathfrak{w}_n) = \mathcal{D}(\mathcal{V}, \mathcal{W})$, $d(\mathfrak{v}, \mathfrak{w}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$ as $n \longrightarrow \infty$.Next we assert that $\mathfrak{w} \in \mathcal{A}\mathfrak{v}$.

Now,

$$\begin{aligned}
\mathcal{D}(\mathfrak{w}_n, A\mathfrak{v}) &\leq \delta(\mathcal{A}\mathfrak{v}_n, \mathcal{A}\mathfrak{v}) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}_n, \mathfrak{v}) + \mathfrak{m}_2\, \phi\left(\max\left\{ \begin{cases} d(\mathfrak{v}_n, \mathfrak{v}), \\ \mathcal{D}(\mathfrak{v}_n, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathfrak{v}, \mathcal{A}\mathfrak{v}_n) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}_n, \mathfrak{v})[1 + \sqrt{d(\mathfrak{v}_n, \mathfrak{v})(\mathcal{D}(\mathfrak{v}_n, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{\big(1 + d(\mathfrak{v}_n, \mathfrak{v})\big)^2} \right) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}_n, \mathfrak{v}) + \mathfrak{m}_2\, \phi\left(\max\left\{ \begin{cases} d(\mathfrak{v}_n, \mathfrak{v}), \\ d(\mathfrak{v}_n, \mathfrak{w}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ d(\mathfrak{v}_n, \mathfrak{w}_n) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right)
\end{aligned}$$

Taking $n \to \infty$, using $\mathfrak{v}_n \to \mathfrak{v}$, $\mathfrak{w}_n \to w$, and $d(\mathfrak{v}, \mathfrak{w}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$, we have

$$\begin{aligned}
\mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}) \leq \mathfrak{m}_1\, d(\mathfrak{v}, \mathfrak{v}) + \mathfrak{m}_2\, \phi\left(\max\left\{ \begin{cases} d(\mathfrak{v}, \mathfrak{v}), \\ d(\mathfrak{v}, \mathfrak{w}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ d(\mathfrak{v}, \mathfrak{w}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
+ \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}, \mathfrak{v})\big[1 + \sqrt{d(\mathfrak{v}, \mathfrak{v})(d(\mathfrak{v}, \mathfrak{w}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}\big]^2}{\big(1 + d(\mathfrak{v}, \mathfrak{v})\big)^2} \right).
\end{aligned}$$

Using the definition of ϕ,

$$\mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}) \leq 0, \text{ i.e., } \mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}) = 0,$$

i.e., $\mathfrak{w} \in \mathcal{A}\mathfrak{v}$. Thus $D(\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$. □

The following example appreciates that Theorems 2.5 does not give assurance of the uniqueness of the set-valued best proximity point.

Example 2.6. Let partially ordered metric space $(\mathcal{Z}, d, \preceq)$ be defined as:
$d\big((\mathfrak{v}_1, \mathfrak{w}_1), (\mathfrak{v}_2, \mathfrak{w}_2)\big) = |\mathfrak{v}_1 - \mathfrak{v}_2| + |\mathfrak{w}_1 - \mathfrak{w}_2|$, $\mathcal{Z} = (-\infty, 15) \times [0, 15) \subseteq \mathbb{R}^2$ equipped with partial order $\preceq$ so that, $(\mathfrak{v}_1, \mathfrak{w}_1) \preceq (\mathfrak{v}_2, \mathfrak{w}_2)$ if $\mathfrak{v}_1 \leq \mathfrak{v}_2$;and $\mathfrak{w}_1 \leq \mathfrak{w}_2$.
Let $\mathcal{V} = \big\{(0,0), (0,4), (0,7), (0,9), (0,10)\big\}$, and $\mathcal{W} = \big\{(-1,1), (1,3), (1,8), (1,9)\big\}$. Then
$\mathcal{D}(\mathcal{V}, \mathcal{W}) = 2$. Also $d\big((0,0), (-1,1)\big) = d\big((0,4), (1,3)\big) = d\big((0,7), (1,8)\big) = 2 = \mathcal{D}(\mathcal{V}, \mathcal{W})$,
where $\mathcal{V}_0 = \{(0,0), (0,4), (0,7), (0,9)\}$, and $\mathcal{W}_0 = \{(-1,1), (1,3), (1,8)\}$.
Let $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ be defined as

$$\mathcal{A}\mathfrak{v} = \begin{cases} \{(1,3), (1,8)\}, & \text{if } \mathfrak{v} = (0,0) \\ \{(1,8)\}, & \text{otherwise} \end{cases}$$

and $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = \frac{1}{2}\mathfrak{t}$. Let $\mathfrak{m}_1 = \frac{1}{4} = \mathfrak{m}_2 = \mathfrak{m}_3$ so that $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 < 1$ and for all comparable $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, one may verify that $\mathcal{A}$ satisfies inequality (2.1) and condition (i).

(i) $(0,0) \preceq (0,4)$ and $(1,3) \in \mathcal{A}(0,0) \Longrightarrow d((0,4), (1,3)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$,
$(0,0) \preceq (0,7)$, and $(1,8) \in \mathcal{A}(0,0) \Longrightarrow d((0,7), (1,8)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$,
$(0,4) \preceq (0,7)$, and $(1,8) \in \mathcal{A}(0,4) \Longrightarrow d((0,7), (1,8)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$,
$(0,7) \preceq (0,9)$, and $(1,8) \in \mathcal{A}(0,7) \Longrightarrow d((0,9), (1,8)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.
Hence, there exist $\mathfrak{v}_0$, $\mathfrak{v}_1 \in \mathcal{V}_0$, and $\mathfrak{w}_0 \in \mathcal{A}\mathfrak{v}_0$ satisfying $d(\mathfrak{v}_1, \mathfrak{w}_0) = \mathcal{D}(\mathcal{V}, \mathcal{W})$, and $\mathfrak{v}_0 \preceq \mathfrak{v}_1$.

(ii) One may verify that $(\mathcal{V}, \mathcal{W})$ satisfies weak $P-$property and condition (iv).

(iii) (a) $\{\mathfrak{v}_n\} = \{(-\frac{1}{n}, 7 - \frac{1}{n})\}$ is an increasing sequence such that $\mathfrak{v}_n \longrightarrow (0,7)$, $\mathfrak{v}_n \prec (0,7)$, $n \in \mathbb{N}$, and $d((0,7), \mathcal{A}(0,7)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.
(b) $\{\mathfrak{v}_n\} = \{(-\frac{3}{2n}, 9 - \frac{2}{3n})\}$ is an increasing sequence such that $\mathfrak{v}_n \longrightarrow (0,9)$, $\mathfrak{v}_n \prec (0,9)$, $n \in \mathbb{N}$, and $d((0,9), \mathcal{A}(0,9)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.

Hence, $(0,7)$ and $(0,9)$ are the two best proximity points of $\mathcal{A}$. Clearly, sets $\mathcal{V}$ and $\mathcal{W}$ are closed and partially ordered space $(\mathcal{Z}, d, \preceq)$ is not complete.

Theorem 2.7. Inference of Theorem 2.5 is correct even if we substitute $\mathcal{AM}-$contraction (2.1) with generalized $\mathcal{AM}-$contraction (2.2).

Proof . The proof adheres to the pattern of Theorem 2.1. □

Remark 2.8. If we put $\mathfrak{m}_2 = \mathfrak{m}_3 = 0$ in Theorems 2.5 and 2.7, we get the improved proximal version of Nadler's Theorem [9] in non-complete partially ordered metric space. Moreover, Theorems 2.5 and 2.7 are extensions and improvements of Theorem 2.1 and 3.2 of Pragadeeswarar et al. [11] without exploiting the completeness of the space and replacing $P-$Property introduced by Raj [13], by weak $P-$property.

Now, we propose an idea of $\alpha-$proximal admissibility for a pair of set-valued maps to prove common best proximant using $\mathcal{SAM}-$contraction.

Definition 2.9. Let $\alpha : \mathcal{V}^2 \to [0,\infty)$. A pair $(\mathcal{S}, \mathcal{A})$ of set-valued non-self maps $\mathcal{S},\ \mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ is α-proximal admissible if

$$\begin{cases} \alpha(\mathfrak{v}_1, \mathfrak{v}_2) \geq 1 \\ d(\mathfrak{w}_1, \mathfrak{c}_1) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \\ d(\mathfrak{w}_2, \mathfrak{c}_2) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \implies \alpha(\mathfrak{w}_1, \mathfrak{w}_2) \geq 1,$$

$\mathfrak{v}_1,\ \mathfrak{v}_2,\ \mathfrak{w}_1,\ \mathfrak{w}_2 \in \mathcal{V}, \mathfrak{c}_1 \in \mathcal{S}\mathfrak{v}_1$ and $\mathfrak{c}_2 \in \mathcal{A}\mathfrak{v}_2$.

Example 2.10. Let partially ordered metric space $(\mathcal{Z}, d, \preceq)$ be defined as:
$d\big((\mathfrak{v}_1, \mathfrak{w}_1), (\mathfrak{v}_2, \mathfrak{w}_2)\big) = |\mathfrak{v}_1 - \mathfrak{v}_2| + |\mathfrak{w}_1 - \mathfrak{w}_2|$, $(\mathfrak{v}_1, \mathfrak{w}_1), (\mathfrak{v}_2, \mathfrak{w}_2) \in \mathcal{Z}$ and $\mathcal{Z} = (-11, 11) \subseteq \mathbb{R}^2$ equipped with partial order so that $\preceq$ be $(\mathfrak{v}_1, \mathfrak{w}_1) \preceq (\mathfrak{v}_2, \mathfrak{w}_2)$ iff $\mathfrak{v}_1 \leq \mathfrak{v}_2$ and $\mathfrak{w}_1 \leq \mathfrak{w}_2$.
Let $\mathcal{V} = \big\{(0,0), (0,4), (0,7), (0,10)\big\}$, $\mathcal{W} = \big\{(-1,1), (1,3), (1,8), (1,9)\big\}$, and $\mathcal{D}(\mathcal{V}, \mathcal{W}) = 2$. Let $\mathcal{S},\ \mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ be defined as

$$\mathcal{S}\mathfrak{v} = \begin{cases} \{(-1,1)\}, & \text{if, } \mathfrak{v} = (0,0) \\ \{(1,8), (-1,1)\}, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathcal{A}\mathfrak{v} = \begin{cases} \{(1,3)\}, & \text{if } \mathfrak{v} = (0,4) \\ \{(1,5), (1,3)\}, & \text{otherwise.} \end{cases}$$

Now, define an $\alpha : \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$ as

$$\alpha(\mathfrak{v}, \mathfrak{w}) = \begin{cases} 1, & \text{if } \mathfrak{v},\ \mathfrak{w} \in \mathcal{V} \\ 0, & \text{if otherwise.} \end{cases}$$

One may verify that $(\mathcal{S}, \mathcal{A})$ is α-proximal admissible.

Remark 2.11. Noticeably, it reduces to $\alpha-$proximal admissibility introduced by Ali et al. [2] for a set-valued map if $\mathcal{S} = \mathcal{A}$ and $\alpha-$proximal admissibility introduced by Jleli et al. [8], if in addition to $\mathcal{S} = \mathcal{A}$, $\mathcal{A}$ is taken to be a single valued self-map.

Theorem 2.12. Let $\mathcal{S}, \mathcal{A} : \mathcal{V} \longrightarrow 2^{\mathcal{W}}$ be a $\mathcal{SAM}-$contraction (2.3) in a partially ordered metric space $(\mathcal{Z}, d, \preceq)$ satisfying

- **(i)** $\mathcal{S}(\mathfrak{v}_0) \subseteq \mathcal{W}_0$, $\mathcal{A}(\mathfrak{v}_0) \subseteq \mathcal{W}_0$, $\mathfrak{v}_0 \in \mathcal{V}_0$ and $(\mathcal{V}, \mathcal{W})$ satisfies the weak P-property,
- **(ii)** a pair $(\mathcal{S}, \mathcal{A})$ is α-proximal admissible,
- **(iii)** $\mathfrak{v}_0 \in \mathcal{V}_0$ satisfying $d(\mathfrak{v}_0, \mathfrak{w}_0) = \mathcal{D}(V, W)$, $\{\mathfrak{w}_0\} \prec_1 \mathcal{A}\mathfrak{v}_0$, and $\alpha(\mathfrak{v}_0, \mathfrak{v}_1) \geq 1$,
- **(iv)** $\mathfrak{v},\ \mathfrak{w} \in \mathcal{V}_0$, $\mathfrak{v} \preceq \mathfrak{w}$ implies that $S\mathfrak{w} \prec_{(3)} \mathcal{A}\mathfrak{v}$,
- **(v)** If $\{\mathfrak{v}_n\}$ is an increasing sequence in $\mathcal{V}$ so that $\mathfrak{v}_n \to \mathfrak{v}$ then $\mathfrak{v}_n \preceq \mathfrak{v}$, $n \in \mathbb{N}$.

Then $\mathfrak{v} \in \mathcal{V}$ is a common best proximity of $\mathcal{S}$ and $\mathcal{A}$.

Proof . Assuming (iii), there exists $\mathfrak{v}_0 \in \mathcal{V}_0$ with

$$d(\mathfrak{v}_0, \mathfrak{w}_0) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \ \text{ so that } \ \{\mathfrak{w}_0\} \prec_1 \mathcal{A}\mathfrak{v}_0 \quad \text{and} \quad \alpha(\mathfrak{v}_0, \mathfrak{v}_1) \geq 1. \tag{2.8}$$

Now, for $\mathfrak{w}_0 \in \{\mathfrak{w}_0\}$ there exists $\mathfrak{w}_1 \in \mathcal{A}\mathfrak{v}_0$ with

$$d(\mathfrak{v}_1, \mathfrak{w}_1) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \quad \text{so that} \quad \mathfrak{v}_0 \preceq \mathfrak{v}_1. \tag{2.9}$$

Using (iv), $\mathcal{S}\mathfrak{v}_1 \prec_3 \mathcal{A}\mathfrak{v}_0 \Longrightarrow \mathcal{S}\mathfrak{v}_1 \prec_2 \mathcal{A}\mathfrak{v}_0$. Therefore, for $\mathfrak{w}_1 \in \mathcal{A}\mathfrak{v}_0$ there exists $\mathfrak{w}_2 \in \mathcal{S}\mathfrak{v}_1$ with

$$d(\mathfrak{v}_2, \mathfrak{w}_2) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \quad \text{so that} \quad \mathfrak{v}_1 \preceq \mathfrak{v}_2. \tag{2.10}$$

Since, $(\mathcal{S}, \mathcal{A})$ is α-proximal admissible, so utilizing inequalities (2.8), (2.9), (2.10), and $\alpha(\mathfrak{v}_1, \mathfrak{v}_2) \geq 1$.

Again by using inequality (iv), $\mathcal{S}\mathfrak{v}_2 \prec_3 \mathcal{A}\mathfrak{v}_1 \Longrightarrow \mathcal{S}\mathfrak{v}_2 \prec_1 \mathcal{A}\mathfrak{v}_1$. So for $\mathfrak{w}_2 \in \mathcal{S}\mathfrak{v}_1$ there exists $\mathfrak{w}_3 \in \mathcal{A}\mathfrak{v}_1$ with

$$d(\mathfrak{v}_3, \mathfrak{w}_3) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \quad \text{so that} \quad \mathfrak{v}_2 \preceq \mathfrak{v}_3. \tag{2.11}$$

Following this pattern, we construct a sequence $\{\mathfrak{v}_n\} \in \mathcal{V}_0$ so that

(i)
$$\alpha(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \geq 1, \qquad \mathfrak{n} \geq 0, \tag{2.12}$$

(ii) $\mathfrak{w}_{2n+1} \in \mathcal{A}\mathfrak{v}_{2n}$, $\mathfrak{w}_{2n+2} \in \mathcal{S}\mathfrak{v}_{2n+1}$, and $\mathfrak{v}_{2n} \preceq \mathfrak{v}_{2n+1}$, $\mathfrak{v}_{2n+2} \preceq \mathfrak{v}_{2n+1}$ with $d(\mathfrak{v}_n, \mathfrak{w}_n) = \mathcal{D}(\mathcal{V}, \mathcal{W})$, $n \geq 0$.

Now, we assert that a best proximity point of $\mathcal{A}$ is also a best proximity point of $\mathcal{S}$ and vice-versa. Let $\mathfrak{u}$ be a best proximity point of $\mathcal{A}$ however, it is not a best proximity point of $\mathcal{S}$. We have

$$\mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) \leq \mathcal{D}(\mathfrak{u}, \mathcal{A}\mathfrak{u}) + \delta(\mathcal{A}p, \mathcal{S}\mathfrak{u}) = \mathcal{D}(\mathcal{V}, \mathcal{W}) + \delta(\mathcal{A}p, \mathcal{S}\mathfrak{u}).$$

So, $\mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \leq \delta(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u}) \leq \alpha(\mathfrak{u}, \mathfrak{u})\ \delta(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u})$. Then from inequality (2.3),

$$\begin{aligned}
\mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) &\leq \alpha(\mathfrak{u}, \mathfrak{u})\ \delta(\mathcal{S}\mathfrak{u}, \mathcal{A}\mathfrak{u}) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{u}, \mathfrak{u}) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{u}, \mathfrak{u}), \\ \mathcal{D}(\mathfrak{u}, \mathcal{A}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{u}, \mathfrak{u})[1 + \sqrt{d(\mathfrak{u}, \mathfrak{u})(\mathcal{D}(\mathfrak{u}, \mathcal{A}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{(1 + d(\mathfrak{u}, \mathfrak{u}))^2} \right) \\
&= \mathfrak{m}_1\ \cdot 0 + \mathfrak{m}_2\, \phi\big(\max\{0, 0, \mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W})\} \big) + \mathfrak{m}_3\, \phi(0) \\
&= \mathfrak{m}_2\, \phi(\mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W})) \\
&\leq \mathfrak{m}_2\, \mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{A}\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \\
&\leq \mathfrak{m}_2\, \mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) - \mathcal{D}(\mathcal{V}, \mathcal{W}),
\end{aligned}$$

a contradiction, except if $\mathcal{D}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.
Similarly, we assert that a best proximity point of $\mathcal{S}$ is a best proximity point of $\mathcal{A}$. If $\mathfrak{v}_{2\mathcal{N}} = \mathfrak{v}_{2\mathcal{N}+1}$ then $\mathfrak{v}_{2\mathcal{N}}$ is a common best proximity point and the same inference may be drawn if $\mathfrak{v}_{2\mathcal{N}+1} = \mathfrak{v}_{2N+2}$, $\mathcal{N} \in \mathbb{N}$. So $\mathfrak{v}_n \neq \mathfrak{v}_{n+1}$, $\mathfrak{n} \geq 0$. Since, $d(\mathfrak{v}_n, \mathfrak{w}_n) = d(\mathfrak{v}_{n+1}, \mathfrak{w}_{n+1}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$. Then using weak P-property,

$$d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq d(\mathfrak{w}_n, \mathfrak{w}_{n+1}), \qquad \mathfrak{n} \in \mathbb{N}. \tag{2.13}$$

Now, from inequalities (2.3) and (2.13),

$$
\begin{aligned}
d(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2}) &\le d(\mathfrak{w}_{2n+1},\mathfrak{w}_{2n+2}) \le \delta(\mathcal{A}\mathfrak{v}_{2n}, S\mathfrak{v}_{2n+1}) \le \alpha(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})\delta(\mathcal{A}\mathfrak{v}_{2n},\mathcal{S}\mathfrak{v}_{2n+1})\\
&\le \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\,\phi\Bigg(\max\left\{\begin{cases} d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}),\\ \mathcal{D}(\mathfrak{v}_{2n+1},\mathcal{A}\mathfrak{v}_{2n}) - \mathcal{D}(\mathcal{V},\mathcal{W}),\\ \mathcal{D}(\mathfrak{v}_{2n+1}, S\mathfrak{v}_{2n+1}) - \mathcal{D}(\mathcal{A}\mathfrak{v}_{2n},\mathcal{S}\mathfrak{v}_{2n+1}) - \mathcal{D}(\mathcal{V},\mathcal{W})\end{cases}\right.\Bigg)\\
&\qquad + \mathfrak{m}_3\,\phi\left(\frac{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})[1+\sqrt{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})(\mathcal{D}(\mathfrak{a}_{2n+1},\mathcal{A}\mathfrak{v}_{2n}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}\,]^2}{(1+d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}))^2}\right)\\
&= \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\,\phi\Bigg(\max\left\{\begin{cases} d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}),\\ d(\mathfrak{v}_{2n+1},\mathfrak{w}_{2n+1}) - \mathcal{D}(\mathcal{V},\mathcal{W}),\\ d(\mathfrak{v}_{2n+1},\mathfrak{w}_{2n+2}) - d(\mathfrak{w}_{2n+1},\mathfrak{w}_{2n+2}) - \mathcal{D}(\mathcal{V},\mathcal{W})\end{cases}\right.\Bigg)\\
&\qquad + \mathfrak{m}_3\,\phi\left(\frac{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})[1+\sqrt{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})(d(\mathfrak{v}_{2n+1},\mathfrak{w}_{2n+1}) - \mathcal{D}(\mathcal{V},\mathcal{W}))}\,]^2}{(1+d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}))^2}\right)\\
&= \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\,\phi\Bigg(\max\left\{\begin{cases} d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}),\\ \mathcal{D}(\mathcal{V},\mathcal{W}) - \mathcal{D}(\mathcal{V},\mathcal{W}),\\ d(\mathfrak{v}_{2n+1},\mathfrak{w}_{2n+1}) + d(\mathfrak{w}_{2n+1},\mathfrak{w}_{2n+2}) - d(\mathfrak{w}_{2n+1},\mathfrak{w}_{2n+2}) - \mathcal{D}(\mathcal{V},\mathcal{W})\end{cases}\right.\Bigg)\\
&\qquad + \mathfrak{m}_3\,\phi\left(\frac{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})[1+\sqrt{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})\big(\mathcal{D}(\mathcal{V},\mathcal{W}) - \mathcal{D}(\mathcal{V},\mathcal{W})\big)}\,]^2}{(1+d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}))^2}\right)\\
&= \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\,\phi\Big(\max\{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}),0,0\}\Big) + \mathfrak{m}_3\,\phi\left(\frac{d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})}{(1+d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}))^2}\right)\\
&\le \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\,\phi\big(d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})\big) + \mathfrak{m}_3\,\phi\big(d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})\big).
\end{aligned}
$$

Since, $\phi(t) < t$, so

$$
\begin{aligned}
d(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2}) &\le \mathfrak{m}_1\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_2\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) + \mathfrak{m}_3\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1})\\
&= \Big(\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3\Big)\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}).
\end{aligned}
$$

Let $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 = \eta < 1$. Then we have

$$d(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2}) \le \eta\, d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}).$$

Similarly,

$$d(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1}) \le \eta\, d(\mathfrak{v}_{2n-1},\mathfrak{v}_{2n}).$$

Following this pattern, we attain

$$d(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2}) \le \eta^n\, d(\mathfrak{v}_0,\mathfrak{v}_1).$$

Since, $\eta \in (0,1)$. As $n \to \infty$, $\eta^n \to 0$ we obtain

$$\lim_{n\to\infty} d(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2}) = 0. \tag{2.14}$$

Hereafter, we assert that $\{\mathfrak{v}_n\}$ is a Cauchy sequence. Let $m > n$, we obtain

$$
\begin{aligned}
d(\mathfrak{v}_m,\mathfrak{v}_n) &\le d(\mathfrak{v}_m,\mathfrak{v}_{m-1}) + d(\mathfrak{v}_{m-1},\mathfrak{v}_{m-2}) + \cdots + d(\mathfrak{v}_{n+1},\mathfrak{v}_n)\\
&\le [\eta^{m-1} + \eta^{m-2} + \cdots + \eta^n]\, d(\mathfrak{v}_0,\mathfrak{v}_1)\\
&\le \left(\frac{\eta^n(1-\eta^{m-n})}{1-\eta}\right) d(\mathfrak{v}_0,\mathfrak{v}_1)\\
&\le \left(\frac{\eta^n}{1-\eta}\right) d(\mathfrak{v}_0,\mathfrak{v}_1) \to 0, \quad \text{as} \quad n\to\infty,
\end{aligned}
$$

i.e., $\{\mathfrak{v}_n\}$ is a Cauchy sequence in a closed set $\mathcal{V}$, therefore it converges to $\mathfrak{v} \in \mathcal{V}$. As $d(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \leq d(\mathfrak{w}_n, \mathfrak{w}_{n+1})$, the sequence $\{\mathfrak{w}_n\}$ is Cauchy in closed set $\mathcal{W}$. Thus it converges to $\mathfrak{w} \in \mathcal{W}$. Since, $d(\mathfrak{v}_n, \mathfrak{w}_n) = \mathcal{D}(V, W)$, $n \in \mathbb{N}$, as $n \longrightarrow \infty$ we inference that $d(\mathfrak{v}, \mathfrak{w}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$. Next, we establish that $\mathfrak{w} \in \mathcal{A}\mathfrak{v}$.

$$\begin{aligned}
\mathcal{D}(\mathcal{A}\mathfrak{v}, \mathfrak{w}_{2n+2}) &\leq \delta(\mathcal{A}\mathfrak{v}, \mathcal{S}\mathfrak{v}_{2n+1}) \leq \alpha(\mathfrak{v}, \mathfrak{v}_{2n+1})\delta(\mathcal{A}\mathfrak{v}, \mathcal{S}\mathfrak{v}_{2n+1}) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}, \mathfrak{v}_{2n+1}) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}, \mathfrak{v}_{2n+1}), \\ \mathcal{D}(\mathfrak{v}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathfrak{v}_{2n+1}, \mathcal{S}\mathfrak{v}_{2n+1}) - \mathcal{D}(\mathcal{A}\mathfrak{v}, \mathcal{S}\mathfrak{v}_{2n+1}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}, \mathfrak{v}_{2n+1})[1 + \sqrt{d(\mathfrak{v}, \mathfrak{v}_{2n+1})(\mathcal{D}(\mathfrak{a}_{2n+1}, \mathcal{A}\mathfrak{a}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{(1 + d(\mathfrak{v}, \mathfrak{v}_{2n+1}))^2} \right) \\
&\leq \mathfrak{m}_1\, d(\mathfrak{v}, \mathfrak{v}_{2n+1}) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}, \mathfrak{v}_{2n+1}), \\ \mathcal{D}(\mathfrak{v}_{2n+1}, \mathfrak{w}_{2n+1}) + \mathcal{D}(\mathfrak{w}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathfrak{v}_{2n+1}, \mathfrak{w}_{2n+2}) - \mathcal{D}(\mathcal{A}\mathfrak{v}, \mathfrak{w}_{2n+1}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}, \mathfrak{v}_{2n+1})[1 + \sqrt{d(\mathfrak{v}, \mathfrak{v}_{2n+1})(\mathcal{D}(\mathfrak{v}_{2n+1}, \mathfrak{w}_{2n+1}) + \mathcal{D}(\mathfrak{w}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{(1 + d(\mathfrak{v}, \mathfrak{v}_{2n+1}))^2} \right) \\
&= \mathfrak{m}_1\, d(\mathfrak{v}, \mathfrak{v}_{2n+1}) + \mathfrak{m}_2\, \phi\left(\max \left\{ \begin{cases} d(\mathfrak{v}, \mathfrak{v}_{2n+1}), \\ \mathcal{D}(\mathcal{V}, \mathcal{W}) + \mathcal{D}(\mathfrak{w}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}), \\ \mathcal{D}(\mathcal{V}, \mathcal{W}) + \mathcal{D}(\mathfrak{w}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}) \end{cases} \right. \right) \\
&\quad + \mathfrak{m}_3\, \phi\left(\frac{d(\mathfrak{v}, \mathfrak{v}_{2n+1})[1 + \sqrt{d(\mathfrak{v}, \mathfrak{v}_{2n+1})(\mathcal{D}(\mathcal{V}, \mathcal{W}) + \mathcal{D}(\mathfrak{w}_{2n+1}, \mathcal{A}\mathfrak{v}) - \mathcal{D}(\mathcal{V}, \mathcal{W}))}]^2}{(1 + d(\mathfrak{v}, \mathfrak{v}_{2n+1}))^2} \right)
\end{aligned}$$

Taking $\mathfrak{n} \to \infty$,

$$\begin{aligned}
\mathcal{D}(\mathcal{A}\mathfrak{v}, \mathfrak{w}) &\leq \mathfrak{m}_1 d(\mathfrak{v}, \mathfrak{v}) + \mathfrak{m}_2 \phi\big(\max\{d(\mathfrak{v}, \mathfrak{v}), \mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}), v(\mathfrak{w}, \mathcal{A}\mathfrak{v})\}\big) + \mathfrak{m}_3 \phi\left(\frac{d(\mathfrak{v}, \mathfrak{v})[1 + \sqrt{d(\mathfrak{v}, \mathfrak{v})(\mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}))}]^2}{(1 + d(\mathfrak{v}, \mathfrak{v}))^2} \right) \\
&= \mathfrak{m}_1\, 0 + \mathfrak{m}_2\, \phi\big(\max\{0, \mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v})\}\big) + \mathfrak{m}_3 \phi(0) \\
&\leq \mathfrak{m}_2\, \phi\big(\mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v})\big),
\end{aligned}$$

i.e., $\mathcal{D}(\mathfrak{w}, \mathcal{A}\mathfrak{v}) = 0$. Therefore $\mathfrak{w} \in \mathcal{A}\mathfrak{v}$. Hence, $\mathcal{D}(\mathfrak{v}, \mathcal{S}\mathfrak{v}) = \mathcal{D}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) = \mathcal{D}(\mathcal{V}, \mathcal{W})$. □

Example 2.13. Let partially ordered metric space $(\mathcal{Z}, d, \preceq)$ be defined as:
$d\big((\mathfrak{v}_1, \mathfrak{w}_1), (\mathfrak{v}_2, \mathfrak{w}_2)\big) = |\mathfrak{v}_1 - \mathfrak{v}_2| + |\mathfrak{w}_1 - \mathfrak{w}_2|$, $(\mathfrak{v}_1, \mathfrak{w}_1)$, $(\mathfrak{v}_2, \mathfrak{w}_2) \in \mathcal{Z}$ and $\mathcal{Z} = (0, 20) \times (-20, 20) \subseteq \mathbb{R}^2$ equipped with partial order $\preceq$ so that $(\mathfrak{v}_1, \mathfrak{w}_1) \leq (\mathfrak{v}_2, \mathfrak{w}_2)$ if and only if $\mathfrak{v}_1 \leq \mathfrak{v}_2$ and $\mathfrak{v}_1 \leq \mathfrak{w}_2$.
Consider $\mathcal{V} = \big\{(0,0), (0,3), (0,4), (0,6), (0,9), (0,12), (0,15)\big\}$, $\mathcal{W} = \big\{(1,-1), (1,2), (1,5), (1,10), (1,15)\big\}$ and $\mathcal{D}(\mathcal{V}, \mathcal{W}) = 2$. Now, $d\big((0,0), (-1,1)\big) = d\big((0,3), (1,2)\big) = d\big((0,6), (1,5)\big) = d\big((0,9), (1,8)\big) = 2 = \mathcal{D}(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}_0 = \{(0,0), (0,3), (0,4), (0,6), (0,9)\}$ and $\mathcal{W}_0 = \{(1,-1), (1,2), (1,5)\}$.
Let $\mathcal{S}$, $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{W}}$ be defined as

$$\mathcal{A}\mathfrak{v} = \begin{cases} \{(1,5), (1,8)\}, & \text{if } \mathfrak{v} = (0,0) \\ \{(1,5)\}, & \text{otherwise} \end{cases} \qquad \text{and} \qquad S\mathfrak{v} = \{(1,5)\}, \text{ for all } \mathfrak{v} \in \mathcal{V}.$$

Let $\alpha : \mathcal{Z} \times \mathcal{Z} \to (0, \infty)$ be defined as

$$\alpha(\mathfrak{v}, \mathfrak{w}) = \begin{cases} 1, & \text{if } (\mathfrak{v},\ \mathfrak{w}) \in \mathcal{V} \\ 0, & \text{if otherwise} \end{cases},$$

and $\phi : [0, \infty) \to [0, \infty)$ as $\phi(\mathfrak{t}) = \frac{1}{2}\mathfrak{t}$. Let $\mathfrak{m}_1 = \frac{1}{4} = \mathfrak{m}_2 = \mathfrak{m}_3$ so that $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 < 1$ and for all comparable $\mathfrak{v}$, $\mathfrak{w} \in \mathcal{V}$, one may verify that $\mathcal{A}$ satisfies inequality (2.3) and conditions $(i), (ii)$, and (iii).

(i) $d((0,0), (1,-1)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$ and $(1,-1) \leq \mathcal{A}(0,0) = \{(1,5), (1,8)\}$
$d((0,3), (1,2)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$ and $(1,2) \leq \mathcal{A}(0,3) = \{(1,5)\}$
$d((0,6), (1,5)) = D(\mathcal{V}, \mathcal{W})$ and $(1,5) \leq \mathcal{A}(0,6) = \{(1,5)\}$
$d((0,4), (1,5)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$ and $(1,5) \leq \mathcal{A}(0,4) = \{(1,5)\}$.
Hence, there exists $\mathfrak{v}_0 \in \mathcal{V}_0$, $\mathfrak{w}_0 \in \mathcal{W}_0$, $d(\mathfrak{v}_0, \mathfrak{w}_0) = \mathcal{D}(\mathcal{V}, \mathcal{W}) \Longrightarrow \{\mathfrak{w}_0\} \prec_{(1)} \mathcal{A}\mathfrak{v}_0$.

(ii) One may verify that $(\mathcal{V}, \mathcal{W})$ satisfies weak $P-$property and condition (iv).

(iii) $(0,0) \leq (0,3) \Longrightarrow \mathcal{S}(0,3) = (1,5) \leq \mathcal{A}(0,0) = \{(1,5),(1,8)\}$,
$(0,3) \leq (0,6) \Longrightarrow \mathcal{S}(0,6) = (1,5) \leq \mathcal{A}(0,3) = \{(1,5)\}$,
$(0,6) \leq (0,9) \Longrightarrow \mathcal{S}(0,9) = (1,5) \leq \mathcal{A}(0,6) = \{(1,5)\}$.
Hence, there exists $\mathfrak{v}$, $\mathfrak{w} \in \mathcal{V}_0$, $\mathfrak{v} \preceq \mathfrak{w}$ implies that $S\mathfrak{w} \prec_{(3)} \mathcal{A}\mathfrak{v}$,

(iv) (a) $\{\mathfrak{v}_n\} = \{(-\frac{1}{n}, 4 - \frac{1}{n})\}$ is an increasing sequence such that $\mathfrak{v}_n \longrightarrow (0,4)$, $\mathfrak{v}_n \prec (0,7)$, $n \in \mathbb{N}$ and $d((0,4), \mathcal{T}(0,4)) = d((0,4), \mathcal{S}(0,4)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.

(b) $\{\mathfrak{v}_n\} = \{(-\frac{4}{5n}, 6 - \frac{4}{5n})\}$ is an increasing sequence such that $\mathfrak{v}_n \longrightarrow (0,6)$, $\mathfrak{v}_n \prec (0,6)$, $n \in \mathbb{N}$ and $d((0,6), \mathcal{A}(0,6)) = d((0,4), \mathcal{S}(0,6)) = \mathcal{D}(\mathcal{V}, \mathcal{W})$.

Hence, $(0,4)$ and $(0,7)$ are the two common best proximity points of $\mathcal{A}$ and $\mathcal{S}$. Clearly, sets $\mathcal{V}$ and $\mathcal{W}$ are closed and partially ordered metric space $(\mathcal{Z}, d, \preceq)$ is non-complete.

Theorem 2.14. The conclusion of Theorem 2.12 remains true if $\mathcal{SAM}-$contraction (2.13) is replaced by generalized $\mathcal{SAM}-$contraction (2.4).

Proof . The proof adheres to the pattern of Theorem 2.12. □

Remark 2.15. **(i)** Examples 2.6 and 2.13 demonstrate that set-valued proximal contractions (in particular, $\mathcal{AM}$ and $\mathcal{SAM}-$contractions and their generalized versions) in a partially ordered metric space are not essentially continuous. However, the closedness of the subset is essential. It is interesting to see that the contractive conditions considered in Theorem 2.5, 2.7, 2.12, 2.14 are not symmetric in the variables.

(ii) Conclusions of above Theorems 2.5, 2.7, 2.12, and 2.14 transforms from a best proximant point/common best proximant point to a fixed point/common fixed point if $\mathcal{V} = \mathcal{W}$ as $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{V}}$ is a set-valued map and the pair $(\mathcal{V}, \mathcal{V})$ has a weak P-property.

(iii) Noticeably, Examples 2.6 and 2.13 are not covered by proximal point theorems existing in the literature involving any one of $P-$property, completeness of space, continuity of involved map, or distance function which is not equipped with ordered relations. Consequently, obtained results are improvements, generalizations, extensions, and unifications of results existing in the literature (for details refer to references given in the end). In particular, Theorems 2.5, 2.7, 2.12, and 2.14 are improvements and extensions of Ali et al. [2], and Theorems 2.12 and 2.14 are improvements of Aydi et al. [3] to non-complete ordered partial metric space.

3 Open Problem

Recently Tomar et al. [14] exploited an arbitrary binary relation to determine a fixed point in a partial Pompeiu-Hausdorff metric space via relation theoretic non-linear contractions utilizing weaker assertions and resolved an integral inclusion of Fredholm type. Now, we pronounce an open problem: Under what conditions the main result of Tomar et al. [14] may be converted to a best proximity result?

4 Conclusion

Motivated by the idea that a best proximity point plays a significant role in providing an optimal approximate solution when a map does not possess a fixed point, we have obtained the necessary conditions for $\min_{\mathfrak{v}\in\mathcal{V}}, \mathcal{D}(\mathfrak{v}, \mathcal{A}\mathfrak{v})$ to have at least one solution. In particular, we have established best proximants for set-valued proximal maps involving rational type contractive conditions exploiting weak $P-$property via proximal relations in non-complete partially ordered metric spaces. Further, we have given suitable exemplification to validate our results. Our results may be utilized to find a minimum distance between two sets in real world optimization problems.

Acknowledgment

The authors are grateful to the anonymous referee for his remarks which led to the improvement of the paper.

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