Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1133–1145
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26669.3381

A study of functions involving $2j, k$-symmetrical points and associated with subordination

Syed Zakar Hussain Bukhari, Madiha Manzoor*

Department of Mathematics, Mirpur University of Science and Technology(MUST), Mirpur-10250 (AJK), Pakistan

(Communicated by Mugur Alexandru Acu)

Abstract

The analytic description of holomorphic mappings is coupled with the functions which map the unit disk to the right half plane. Also, a natural extension of the idea of inequalities between real-valued functions is the concept of subordination between functions of a complex variable. In this study, we make use of some first-order differential subordination and superordination conditions on functions associated with $2j, k$-symmetrical points, and determine some already known classes of analytic functions. We're also come up with some sandwich theorems based on specific assumptions about the parameters that are involved in our major findings.

Keywords: Functions with $2j, k$-symmetrical points, subordination, superordination
2020 MSC: Primary 30C45; Secondary 30C80

1 Introduction

Let $\mathbb{E}\ :\ \mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$ contained in $\mathbb{C}$ and let $\mathcal{H}(\mathbb{E})$ be the family of all mappings holomorphic in $\mathbb{E}$. A natural extension of the idea of inequalities between functions of a real variable is the concept of subordination between functions of a complex variable. This concept dates back to Lindelöf, but Littlewood and Rogosinski originally used this term of subordination instead or as an analogue of inequalities and defined its several properties, for the recent work, see [14]. For holomorphic mappings $f, g\in\mathcal{H}(\mathbb{E})$, $f\prec g$, if for a Schwarz mapping w, we write that $f(z)=g(w(z))$, for every $z\in\mathbb{E}$. If $g\in\mathcal{S}$, then $f\prec g \iff g(0)=f(0)$ and $g(\mathbb{E})\supset f(\mathbb{E})$. The analytic description of holomorphic mappings is coupled with the functions which map to the right half plane or have positive real part. Assume that $\mathcal{P}$ contains the family of holomorphic or analytic mappings p, in such a way that $p\in\mathcal{H}(\mathbb{E}):p(0)=1, \Re\left(p(z)\right)>0$ and

$$p(z)=1+c_1z^1+...,z\in\mathbb{E}.$$

For $-1\leq\alpha_2<\alpha_1\leq 1$, we define that a mapping $p\in\mathcal{P}\left[\alpha_1,\alpha_2\right]:p(0)=1$ and $p(z)\prec\frac{1+\alpha_1 z}{1+\alpha_2 z}$. We refer to [9] for more information. Consider that $\mathcal{A}$ denote the family or class of holomorphic or analytic mappings defined in $\mathbb{E}$ and meet the assumption that $f(0)=0$ and $f'(0)=1$. A mapping $f\in\mathcal{A}$ considers the following illustration:

$$f(z)=z+a_2z^2+...,z\in\mathbb{E}. \tag{1.1}$$

*Corresponding author
Email addresses: fatmi@must.edu.pk (Syed Zakar Hussain Bukhari), seharfatima21@yahoo.com (Madiha Manzoor)

Received: March 2022 *Accepted:* September 2022

A mapping $f \in \mathcal{A}$ is considered to be univalent, if it is one-to-one in $\mathbb{E}$. This family is abbreviated by $\mathcal{S}$. A mapping $f \in \mathcal{S}^*$ in $\mathbb{E}$ is said to be starlike assuming that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

This condition is due to Nevalinna, see [8]. A mapping f is convex, iff $zf' \in \mathcal{S}^*$. This condition was studied by Study. Lowner and many others also studied these families at breadth. In addition, Ma and Minda, see [13] also defined the following:

$$\mathcal{S}^*(\varphi) = \left\{f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec \varphi(z), \text{where } \varphi \in \mathcal{P}, z \in \mathbb{E}\right\}.$$

For detail, see [8, 18] with reference therein.

Definition 1.1. A mapping $f \in \mathcal{S}_{\mathrm{SP}}$ in $\mathbb{E}$ iff, it meets the condition:

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0.$$

Stankiewicz defined the related family $\mathcal{C}_{\mathrm{SP}}$ of convex mappings.

Definition 1.2. A mapping $f \in \mathcal{S}_{\mathrm{SCP}}$ iff, it observes the condition:

$$\Re\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0.$$

For detail see [7]. Also, $f \in \mathcal{C}_{\mathrm{SCP}}$ iff, $zf' \in \mathcal{S}_{\mathrm{SCP}}$. For more detail, see [10].

Definition 1.3. A mapping $f \in \mathcal{S}_{\mathrm{SCP}}(\varphi)$, If we have

$$\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}} \prec \varphi(z), \text{ where } z \in \mathbb{E} \text{ and } \varphi \in \mathcal{P}.$$

Also a mapping $f \in \mathcal{C}_{\mathrm{SCP}}(\varphi)$, iff, $zf' \in \mathcal{S}_{\mathrm{SCP}}(\varphi)$. These families were first studied by Ravichandran [17] in 2004. The families $\mathcal{S}^k_{\mathrm{SP}}$ and $\mathcal{C}^k_{\mathrm{SP}}$, investigated by Wang and Gao [22] are defined as:

$$\mathcal{S}^k_{\mathrm{SCP}}(\varphi) = \left\{f \in \mathcal{S}, \frac{zf'(z)}{f_k(z)} \prec \varphi(z), \text{where } \varphi \in \mathcal{P}, k \neq 1, k \in \mathbb{Z}^+, z \in \mathbb{E}\right\}$$

and $f \in \mathcal{C}^k_{\mathrm{SCP}}(\varphi)$ iff $zf' \in \mathcal{S}^k_{\mathrm{SCP}}(\varphi)$.

$$\mathcal{C}^k_{\mathrm{SCP}}(\varphi) = \left\{f \in \mathcal{S}, \frac{(zf'(z))'}{f'_k(z)} \prec \varphi(z), \text{where } \varphi \in \mathcal{P}, k \neq 1, k \in \mathbb{Z}^+, z \in \mathbb{E}\right\},$$

A mapping $f \in \mathcal{S}^{2k}_{\mathrm{SCP}}$, iff it meets the condition:

$$\mathcal{R}\frac{zf'(z)}{f_{2k}(z)} > 0, z \in \mathbb{E}.$$

where $k \neq 1$ and $k \in \mathbb{Z}^+$. Also, $f \in \mathcal{S}^{2k}_{\mathrm{SCP}}(\phi)$, iff it meets the condition:

$$\frac{zf'(z)}{f_{2k}(z)} \prec \phi(z)$$

where $\phi \in \mathcal{P}$. Also, $f \in \mathcal{C}^{2k}_{\mathrm{SCP}}(\phi)$, iff, $zf' \in \mathcal{S}^{2k}_{\mathrm{SCP}}(\phi)$. The families $\mathcal{S}^{2k}_{\mathrm{SCP}}(\phi)$ and $\mathcal{C}^{2k}_{\mathrm{SCP}}(\phi)$ were investigated by Wang and Gao. See [22].

Let $k \in \mathbb{N}$, $j = 0, 1, ..., k-1$. A mapping f is a j, k-symmetrical if for each $z \in \mathbb{E}$, $f(\varepsilon z) = \varepsilon^j f(z)$, where $\varepsilon = exp\frac{2\pi\iota}{k}$. For detail, see [11]. Observe that $f_{j,k}$ is defined by:

$$f_{j,k}(z) = \frac{1}{k}\sum_{\eta_2=0}^{k-1}\frac{f(\varepsilon^{\eta_2}z)}{\varepsilon^{\eta_2 j}}.$$

This mappings were first studied by Liczberski and Polubinski in [12].

Definition 1.4. A mapping $f \in \mathcal{S}_{\text{SCP}}^{j,k}$ in $\mathbb{E}$, iff it meets the condition:

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) > 0.$$

Similarly, a mapping $f \in \mathcal{C}_{\text{SCP}}^{j,k}$, iff, $zf' \in \mathcal{S}_{\text{SCP}}^{j,k}$. Also, we define the family $\mathcal{S}_{\text{SCP}}^{j,k}(\phi)$ as follows:

Definition 1.5. A mapping $f \in \mathcal{S}_{\text{SCP}}^{j,k}(\phi)$ in $\mathbb{E}$, iff it satisfy the condition:

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) \prec \phi(z).$$

Similarly, $f \in \mathcal{C}_{\text{SCP}}^{j,k}(\phi)$ iff, $zf' \in \mathcal{S}_{\text{SCP}}^{j,k}(\phi)$, see [12]. In 2013, Karthikeyan [11] investigated the classes $\mathcal{S}_{\text{SCP}}^{(2j,k)}(\phi)$ and $\mathcal{C}_{\text{SCP}}^{(2j,k)}(\phi)$ of starlike and convex functions with respect to $(2j,k)$-symmetric conjugate points, respectively, which are defined as follows.

Definition 1.6. A function f is said to be in the class $\mathcal{S}_{\text{SCP}}^{(2j,k)}(\phi)$ iff, we have

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \phi(z) \qquad (z \in \mathbb{E};\ \phi \in \mathcal{P}).$$

This class is related with the class $\mathcal{C}_{\text{SCP}}^{2j,k}(\phi)$ by a known Alexander type relation, as seen in [8], that is a function $f \in \mathcal{C}_{\text{SCP}}^{2j,k}(\phi)$ if and only if $zf'\ \mathcal{S}_{\text{SCP}}^{(2j,k)}(\phi)$. For fixed j and k, $f_{2j,k}$ is defined by

$$f_{2j,k}(z) = \frac{1}{2k}\sum_{\eta=0}^{k-1}\left[\varepsilon^{-\eta j} f(\varepsilon^{\eta} z) + \varepsilon^{\eta j}\overline{f(\varepsilon^{\eta}\overline{z})}\right],\ \varepsilon = e^{\frac{2\pi i}{k}}. \tag{1.2}$$

From (1.2), we obtain the following identities:

$$f'_{2j,k}(z) = \frac{1}{2k}\sum_{\eta=0}^{k-1}\left[\varepsilon^{-\eta j+\eta} f'(\varepsilon^{\eta} z) + \varepsilon^{\eta j-\eta}\overline{f(\varepsilon^{\eta}\overline{z})}\right],$$

$$f''_{2j,k}(z) = \frac{1}{2k}\sum_{\eta=0}^{k-1}\left[\varepsilon^{-\eta j+2\eta} f''(\varepsilon^{\eta} z) + \varepsilon^{\eta j-2\eta}\overline{f''(\varepsilon^{\eta}\overline{z})}\right],$$

$$f_{2j,k}(\varepsilon^{\eta} z) = \varepsilon^{\eta j} f_{2j,k}(z), \qquad f_{2j,k}(z) = \overline{f_{2j,k}(\overline{z})}$$

and

$$f'_{2j,k}(\varepsilon^{\eta} z) = \varepsilon^{\eta j-\eta} f'_{2j,k}(z) \quad \text{and} \qquad f'_{2j,k}(\overline{z}) = \overline{f'_{2j,k}(z)}.$$

Let $\Psi :\ \mathbb{C}^3 \times \mathbb{E} \longrightarrow \mathbb{C}$ and $p \in \mathcal{S}$ in $\mathbb{E}$. If h holomorphic in $\mathbb{E}$ in such a way that

$$\Psi(h, zh', h''; z) \prec p(z), \tag{1.3}$$

then h is the solution of (1.3) and the univalent function ℓ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $h(z) \prec \ell(z)$ for all h satisfying (1.3). A dominant $\widetilde{\ell}$ that satisfies $\widetilde{\ell}(z) \prec \ell(z)$, for every ℓ satisfying (1.3) is called the best dominant of (1.3). For detail, see [14]. On the other side, we also have

$$p(z) \prec \Psi(h, zh', h''; z) \tag{1.4}$$

then h is again a solution of (1.4). A mapping ℓ is subordinant to h, if $\ell(z) \prec h(z)$ for h satisfying (1.4). A subordinant $\widetilde{\ell}$ in such a way that $\ell(z) \prec \widetilde{\ell}(z)$ for every subordinant ℓ satisfying (1.4) is best subordinant of (1.4). For more detailed information, see [15].

Using similar results of Bulboaca [6], Aouf et al.[1] proved that

$$\ell_1(z) \prec \frac{zf'(z)}{f(z)} \prec \ell_2(z),$$

where $\ell_1 : \ell_1(0) = 1, \ell_2 : \ell_2(0) = 1$ are in $\mathcal{S}$. In 2006, Shanmugam et.al [20, 21] further extended these results. See [2, 3, 4, 5, 16, 19] for further information on these and other relevant families.

The main goal of this research is to look at certain analogues of differential inequalities for mappings to $2j,k$-symmetric points and come up with some sandwich results for these mappings.

2 Preliminary Results

Lemma 2.1. [14] For $\ell \in \mathcal{S}$ in the domain $\mathbb{E}$, and θ and ψ are holomorphic or analytic in $\mathbb{D} : \ell(\mathbb{E}) \subset \mathbb{D}$ and for $w \in \ell(\mathbb{E})$, $\psi(w) \neq 0$. We define and establish that

$$Q(z) = z\psi\left[\ell(z)\right]\ell'(z),$$

along with

$$h(z) = \theta\left[\ell(z)\right] + Q(z)$$

and further note that $h \in \mathcal{C}$ or $Q \in \mathcal{S}^*$ in $\mathbb{E}$ in such a way that $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. If p is holomorphic in $\mathbb{E} : p(0) = \ell(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and we have the subordination given by

$$zp'(z)\psi\left[p(z)\right] + \theta\left[p(z)\right] \prec z\ell'(z)\psi\left[\ell(z)\right] + \theta\left[\ell(z)\right] = h(z), \tag{2.1}$$

then $p(z) \prec \ell(z)$.

Let $\mathcal{Q}$ be the family of holomorphic or analytic and injective mappings f on $\overline{\mathbb{E}} \setminus \partial\mathbb{E}(f)$, where

$$\partial\mathbb{E}(f) = \left\{\zeta \in \partial\mathbb{E} \text{ such that } \lim_{z\to\ \zeta} f(z) = \infty,\ f'(\zeta) \neq 0 \text{ and } \zeta \in \partial\mathbb{E} \setminus \partial\mathbb{E}(f)\right\}.$$

Lemma 2.2. [14] For $\ell \in \mathcal{S}$ in the domain $\mathbb{E}$, and θ and ψ are holomorphic or analytic in a domain $\mathbb{D} : \ell(\mathbb{E}) \subset \mathbb{D}$ with $\psi(w) \neq 0, w \in \ell(\mathbb{E})$. Assume that

$$\Re\left(\frac{\theta'\left[\ell(z)\right]}{\psi\left[\ell(z)\right]}\right) > 0, z \in \mathbb{E}$$

and

$$h(z) = z\ell'(z)\psi\left[\ell(z)\right] \in \mathcal{S}^*.$$

If $p \in \mathcal{H}[\ell(0), 1] \cap \mathcal{Q}$ with $p(\mathbb{E}) \subset \mathbb{D}$,

$$zp'(z))\psi\left[p(z)\right] + \theta\left[p(z)\right] \in \mathcal{S}$$

and

$$z\ell'(z)\psi\left[\ell(z)\right] + \theta\left[\ell(z)\right] \prec \theta(p(z)) + zp'(z)\psi\left[p(z)\right], \tag{2.2}$$

then $\ell(z) \prec (p(z)$, and ℓ is the best subordinant of (2.2).

Lemma 2.3. [8] The mapping

$$\ell(z) = \frac{1}{(1-z)^{2ab}} \in \mathcal{S} \Longleftrightarrow |2ab \pm 1| \leq 1.$$

3 Subordination Results

Theorem 3.1. Assume that a mapping $\ell \in \mathcal{S} : \ell(0) = 1$ defined in $\mathbb{E}$ and fulfills the inequality

$$\Re\left(1 + \frac{z\ell''(z)}{\ell'(z)}\right) > \max\left\{0; -\Re\left(\frac{\eta}{\gamma}\right)\right\}, \gamma \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}, z \in \mathbb{E}. \tag{3.1}$$

If for $f \in \mathcal{A}$, the non-linear differential equation given by

$$(\eta+\rho)\frac{zf'(z)}{f_{2j,k}(z)} - \rho\frac{z^2f'(z)f'_{2j,k}(z)}{(f_{2j,k}(z))^2} + \rho\frac{z^2f''(z)}{f_{2j,k}(z)} \prec \eta\ell(z) + \rho z\ell'(z) \qquad (z \in \mathbb{E}), \tag{3.2}$$

then $\frac{zf'(z)}{f_{2j,k}(z)} \prec \ell(z), z \in \mathbb{E}$ and ℓ is the best dominant of (3.2).

Proof . In order to acquire the appropriate proof, we define and examine the following functional h in such a way that

$$h(z)=\frac{zf'(z)}{f_{2j,k}(z)}, \tag{3.3}$$

where $h:h(0)=1$ is holomorphic in $\mathbb{E}$. On taking derivative of (3.2), we can write

$$h'(z)=\frac{-zf'(z)f_{2j,k}(z)+f_{2j,k}(z)\{f'(z)+zf''(z)\}}{(f_{2j,k}(z))^2},\quad z\in\mathbb{E},$$

which further implies that

$$\eta h(z)+\gamma zh'(z)=\gamma\frac{z^2f''(z)}{f_{2j,k}(z)}+(\eta+\gamma)\frac{zf'(z)}{f_{2j,k}(z)}-\gamma\frac{z^2f'(z)f'_{2j,k}(z)}{(f_{2j,k}(z))^2}. \tag{3.4}$$

In view of (3.2), from (3.4) it is observed that

$$\eta h(z)+\gamma zh'(z)\prec\eta\ell(z)+\gamma z\ell'(z).$$

For the sack of the completeness of the condition of the Lemma 2.1, we set $\theta(w)=\eta w$ and $\psi(w)=\gamma$ which are holomorphic or analytic in $\mathbb{C}^*$ in such a way that $\ell(\mathbb{E})\subset\mathbb{C}^*$. Also, if we assume that

$$Q(z)=\psi\left[\ell(z)\right]z\ell'(z)=\gamma z\ell'(z)$$

alongwith

$$p(z)=Q(z)+\theta\left[\ell(z)\right]=\eta\ell(z)+\gamma z\ell'(z),$$

so that $Q(0)=0$ and $Q'(0)=\gamma\ell'(0)\neq 0$, then from (3.1), it is clear that the mapping $Q\in\mathcal{S}^*$ in $\mathbb{E}$ and

$$\Re\left(\frac{zp'(z)}{Q(z)}\right)=\Re\left(1+\frac{\eta}{\gamma}+\frac{z\ell''(z)}{\ell'(z)}\right)>0\qquad(z\in\mathbb{E}).$$

Making use of (3.2) alongwith Lemma 2.1, we find that $h(z)\prec\ell(z)$. This leads to the proof. □

Remark 3.2. If we consider $\ell(z)=\frac{1+\alpha_1 z}{1+\alpha_2 z}$ in Theorem 3.1, where $-1\leq\alpha_2<\alpha_1\leq 1$, then from the condition (3.1), we can write

$$\Re\left(\frac{1-\alpha_2 z}{1+\alpha_2 z}\right)>\max\left\{0;-\Re\left(\frac{\eta}{\gamma}\right)\right\}\qquad(z\in\mathbb{E}).$$

Also the mapping $w(z)=\frac{1-z}{1+z}\in\mathcal{C}$ for $|z|<|\alpha_2|$ in $\mathbb{E}$ and also $w(\overline{z})=\overline{w\left(z\right)}$ for all $|z|<|\alpha_2|$. It implies that $w(\mathbb{E})$ is convex as well as symmetric about the real axis. Hence

$$\inf\left\{\Re\left(\frac{1-\alpha_2 z}{1+\alpha_2 z}\right),\ z\in\mathbb{E}\right\}=\frac{1-|\alpha_2|}{1+|\alpha_2|}>0.$$

Thus, we have $\Re\left(\frac{\eta}{\gamma}\right)\geq\frac{|\alpha_2|-1}{|\alpha_2|+1}$.

In view of the Theorem 3.1, we may derive:

Corollary 3.3. Suppose that $\eta,\gamma\in\mathbb{C}:\gamma\neq 0$ and $-1\leq\alpha_2<\alpha_1\leq 1$ so that $\frac{1-|\alpha_2|}{1+|\alpha_2|}\geq\max\left\{0;-\Re\left(\frac{\eta}{\gamma}\right)\right\}$. If for the mapping $f\in\mathcal{A}$, we note the following condition

$$\gamma\frac{z^2f''(z)}{f_{2j,k}(z)}+(\eta+\gamma)\frac{zf'(z)}{f_{2j,k}(z)}-\gamma\frac{z^2f'(z)f'_{2j,k}(z)}{(f_{2j,k}(z))^2}\prec\eta\frac{1+\alpha_1 z}{1+\alpha_2 z}+\gamma\frac{(\alpha_1-\alpha_2)z}{(1+\alpha_2 z)^2},\ z\in\mathbb{E},$$

then $\frac{zf'(z)}{f_{2j,k}(z)}\in\mathcal{P}\left[\alpha_1,\alpha_2\right],z\in\mathbb{E}$.

For $\alpha_1=1$ and $\alpha_2=-1$, we may derive a related results as a special case.

Theorem 3.4. Suppose that a mapping $\ell \in \mathcal{S}$ defined in $\mathbb{E}$ in such a way that $\ell(0) = 1$ and we further assume that $\ell(z) \neq 0$. Consider that the condition

$$\Re\left(1 + \frac{z\ell''(z)}{\ell'(z)} - \frac{z\ell'(z)}{\ell(z)}\right) > 0 \qquad (z \in \mathbb{E}). \tag{3.5}$$

holds for ℓ. Assume that for the mapping f given by (1.1) and $\eta \in \mathbb{C}^*$ and $\eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \neq 0$, we have

$$\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)} \neq 0 \qquad (z \in \mathbb{E}). \tag{3.6}$$

If we see the subordination given by

$$2 + \frac{zf''(z)}{f'(z)} - \frac{\eta_1 z(zf'_{2j,k}(z))' + \eta_2 z f'_{2j,k}(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)} \prec 1 + \frac{z\ell'(z)}{\eta\ell(z)} \qquad (z \in \mathbb{E}), \tag{3.7}$$

holds, then we get $\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 z f'_{2j,k}(z)+\eta_2 f_{2j,k}(z)} \prec [\ell(z)]^{\frac{1}{\eta}}$.

Proof . In order to acquire the appropriate proof, we examine the functional h in such a way that

$$h(z) = \left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)}\right]^{\eta} \qquad (z \in \mathbb{E}), \tag{3.8}$$

where the power is assumed as the principal one. According to the supposition (3.6), the multivalued power mapping h has an holomorphic or analytic branch in $\mathbb{E}$, with $h(0) = 1$, and from (3.8), we can write that

$$2 + \frac{zf''(z)}{f'(z)} - \frac{\eta_1 z(zf'_{2j,k}(z))' + \eta_2 z f'_{2j,k}(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)} = \frac{zh'(z)}{\eta h(z)} + 1 \qquad (z \in \mathbb{E}). \tag{3.9}$$

In view of (3.7) and (3.9), we observe that

$$1 + \frac{zh'(z)}{\eta h(z)} \prec 1 + \frac{z\ell'(z)}{\eta\ell(z)} \qquad (z \in \mathbb{E}).$$

When it comes to Lemma 2.1, we assume that $\theta(w) = 1$ and $\psi(w) = 1/\eta w$ are holomorphic or analytic in $\mathbb{E}$. These are holomorphic mappings in $\mathbb{C}^*$ containing $\ell(\mathbb{E})$. Furthermore, if we assume that

$$Q(z) := z\ell'(z)\psi\left[\ell(z)\right] = \frac{z\ell'(z)}{\eta\ell(z)}$$

and

$$p(z) := \theta\left[\ell(z)\right] + Q(z) = 1 + \frac{z\ell'(z)}{\eta\ell(z)}$$

with $Q(0) = 0$ and $Q'(0) = \frac{1}{\eta}\frac{\ell'(0)}{\ell(0)} \neq 0$, then from our assumption (3.5), it is proved that $Q \in \mathcal{S}^*$ in $\mathbb{E}$ and

$$\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left(1 - \frac{z\ell'(z)}{\ell(z)} + \frac{z\ell''(z)}{\ell'(z)}\right) > 0 \qquad (z \in \mathbb{E}).$$

Lemma 2.1, and the assumption given by (3.7), prove that $h(z) \prec \ell(z)$. □

As a particular case, if we choose $\eta_1 = 0, \eta_2 = 1$ and $\ell(z) = \frac{1+\alpha_1 z}{1+\alpha_2 z}$ in the above theorem, then clearly the inequality (3.7) holds whenever $-1 \leq \alpha_1 < \alpha_2 \leq 1$. Hence, we can derive the following corollaries:

Corollary 3.5. Let $-1 \leq \alpha_1 < \alpha_2 \leq 1$ and $\eta \in \mathbb{C}^*$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{zf'(z)}{f_{2j,k}(z)} \neq 0 \qquad (z \in \mathbb{E}). \tag{3.10}$$

If we write

$$2 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} \prec 1 + \frac{1}{\eta}\frac{(\alpha_1 - \alpha_2)z}{(1 + \alpha_1 z)(1 + \alpha_2 z)} \qquad (z \in \mathbb{E}), \tag{3.11}$$

then we obtain $\frac{zf'(z)}{f_{2j,k}(z)} \prec \left[\frac{1+\alpha_1 z}{1+\alpha_2 z}\right]^{\frac{1}{\eta}}$.

Corollary 3.6. Let $-1 \le \alpha_1 < \alpha_2 \le 1$ with $\alpha_2 \ne 0$ and suppose that $|\frac{\alpha_1-\alpha_2}{\alpha_2-1}| \le \frac{1}{\eta}$ or $|\frac{\alpha_1-\alpha_2}{\alpha_2+1}| \le \frac{1}{\eta}$ where $\eta \in \mathbb{C}^*$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (3.10) in such a way that the subordination given by

$$1+\eta\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right] \prec \frac{1+\left[\eta(\alpha_1-\alpha_2)+\alpha_2\right]z}{1+\alpha_2 z} \qquad (z \in \mathbb{E}), \tag{3.12}$$

holds, then we observe that $\frac{zf'(z)}{f_{2j,k}(z)} \prec (1+\alpha_2 z)^{\frac{(\alpha_1-\alpha_2)}{\alpha_2}}$.

Using $\eta_1 = 0$, $\eta_2 = 1$ with $a, b \in \mathbb{C}^*$, $\eta = ab$ and $\ell(z) = \frac{1}{(1-z)^{2ab}}$ in the above Theorem 3.4 together with application of the Lemma 2.3, we arrive at the following conclusion as a corollary:

Corollary 3.7. Let $a, b \in \mathbb{C}^*$ in such a way that $|2ab-1| \le 1$ or $|2ab+1| \le 1$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (3.10) in such a way that

$$2+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} \prec \frac{1+z}{1-z}$$

then we see that

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \frac{1}{(1-z)^2}. \tag{3.13}$$

By putting $\eta_1 = 0$, $\eta_2 = 1$, $\eta = \frac{e^{\iota\pi}}{bcos\lambda}$, $a, b \in \mathbb{C}^*$, $|\lambda| < \frac{\pi}{2}$, and $\ell(z) = \frac{1}{(1-z)^{2abe^{-\iota\lambda}cos\lambda}}$ in the above Theorem 3.4, we obtain the following:

Corollary 3.8. Assume that $a, b \in \mathbb{C}^*$ and $|\lambda| < \frac{\pi}{2}$, and suppose that

$$|2abe^{-\iota\lambda}cos\lambda - 1| \le 1 \qquad or \qquad |2abe^{-\iota\lambda}cos\lambda + 1| \le 1.$$

Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (3.10), and

$$1+\frac{e^{\iota\pi}}{bcos\lambda}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right] \prec \frac{1+z}{1-z}$$

then,

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \frac{1}{(1-z)^{2ab}}.$$

Theorem 3.9. Suppose that the mapping $\ell \in \mathcal{S} : \ell(0) = 1$ and it meets the condition

$$\Re\left(1+\frac{z\ell''(z)}{\ell'(z)}\right) > \max\left\{0; -\Re\left(\frac{\delta}{\eta}\right)\right\}, \tag{3.14}$$

where $\eta, \gamma \in \mathbb{C}^*$ and $\eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \ne 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (1.1) and is in such a way that

$$\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)} \ne 0 \qquad (z \in \mathbb{E}),$$

and if the subordination relation given by

$$\left[\frac{(1+\frac{\eta_2}{\eta_1})f'(z)}{f'_{2j,k}(z)+\frac{\eta_2}{\eta_1 z}f_{2j,k}(z)}\right]^{\eta}(1+\delta+\frac{zf''(z)}{f'(z)}-\frac{(zf'_{2j,k}(z))'+\frac{\eta_2}{\eta_1}f'_{2j,k}(z)}{f'_{2j,k}(z)+\frac{\eta_2}{\eta_1 z}f_{2j,k}(z)}) \prec \delta\ell(z)+\frac{z\ell'(z)}{\eta} \tag{3.15}$$

holds, then $\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)} \prec [\ell(z)]^{\frac{1}{\eta}}$.

Proof . In order to acquire the appropriate proof, we examine the functional h as in (3.8), where $h : h(0) = 1$ is holomorphic or analytic in $\mathbb{E}$. From (3.8), we can see that

$$\left[\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\left[1+\frac{zf''(z)}{f'(z)}-\frac{\eta_1 z(zf'_{2j,k}(z))'+\eta_2 zf'_{2j,k}(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]=\frac{zh'(z)}{\eta}$$

or we can write

$$\left[\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\left[1+\delta-\frac{\eta_1 z(zf'_{2j,k}(z))'+\eta_2 zf'_{2j,k}(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]=\delta h(z)+\frac{1}{\eta}zh'(z).$$

On application of (3.15), we can also write

$$\delta h(z)+\frac{1}{\eta}zh'(z)\prec\delta\ell(z)+\frac{1}{\eta}z\ell'(z).$$

We let $\theta(w)=\delta w$ and $\psi(w)=\frac{1}{\eta}$ which are holomorphic or analytic on $\mathbb{C}$ containing $\ell(\mathbb{E})$. We also let

$$Q(z)=z\ell'(z)\psi\left[\ell(z)\right]=\frac{1}{\eta}z\ell'(z)\qquad(z\in\mathbb{E})$$

and

$$p(z)=\theta\left[\ell(z)\right]+Q(z)=\delta\ell(z)+\frac{1}{\eta}z\ell'(z)\qquad(z\in\mathbb{E}).$$

Since, $Q(0)=0$ and $Q'(0)=\frac{1}{\eta}\ell'(0)\neq 0$. From the assumption given in (3.11), we observe that $Q\in\mathcal{S}^*$ in $\mathbb{E}$ and also we note that

$$\Re\left(\frac{zp'(z)}{Q(z)}\right)=\Re\left(1+\frac{\delta}{\eta}+\frac{z\ell''(z)}{\ell'(z)}\right)>0\qquad(z\in\mathbb{E}).$$

Thus, Lemma 2.1 leads to the implication $h(z)\prec\ell(z)$. □

Taking $\ell(z)=\frac{1-\alpha_1 z}{1+\alpha_2 z}$ in the above Theorem 3.9, where $-1\le\alpha_2<\alpha_1\le 1, \eta_2=0$ and $\eta_1=1$, we arrived at the following conclusion:

Corollary 3.10. Let $-1\le\alpha_2<\alpha_1\le 1$ and $\eta\in\mathbb{C}^*$ and $\delta\in\mathbb{C}$ with $\max\left\{0;-\Re\left(\frac{\delta}{\eta}\right)\right\}\le\frac{1-|\alpha_2|}{1+|\alpha_2|}$. Furthermore, consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition in (3.14) and also the subordination results given by

$$\left[\frac{zf'(z)}{f'_{2j,k}(z)}\right]^{\eta}\left[1+\delta+\frac{zf''(z)}{f'(z)}-\frac{(zf'_{2j,k}(z))'}{f'_{2j,k}(z)}\right]\prec\delta\frac{1+\alpha_1 z}{1+\alpha_2 z}+\frac{1}{\eta}\frac{(\alpha_1-\alpha_2)z}{(1+\alpha_2 z)^2},z\in\mathbb{E}.$$

holds, then we obtain $\left[\frac{zf'(z)}{f'_{2j,k}(z)}\right]^{\eta}\prec\frac{1+\alpha_1 z}{1-\alpha_2 z}$.

Taking $\eta_2=\gamma=1$, $\eta_1=0$ and $\ell(z)=\frac{1+z}{1-z}$ in the above Theorem 3.9, then we get the following corollary:

Corollary 3.11. Assume that $\delta\in\mathbb{C}$ with $\Re(\delta)\ge 0$. Furthermore, consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition in (3.10), and moreover the subordination condition given by

$$\left[\frac{zf'(z)}{f_{2j,k}(z)}\right]^{\eta}\left[1+\delta-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}+\frac{zf''(z)}{f'(z)}\right]\prec\delta\frac{1+z}{1-z}+\frac{2z}{(1-z)^2}$$

holds, then we have

$$\left[\frac{zf'(z)}{f_{2j,k}(z)}\right]^{\eta}\prec\frac{1+z}{1-z}.$$

4 Superordination Results

Theorem 4.1. Suppose that $\ell : \ell(0) = 1$ is convex in $\mathbb{E}$, and $\gamma \in \mathbb{C}$ with $\Re\left(\frac{\eta}{\gamma}\right) > 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{zf'(z)}{f_{2j,k}(z)} \in \mathcal{Q}$$

and the mapping ϕ in such a way that

$$\frac{zf'(z)}{f_{2j,k)}(z)}\left[\eta + \gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right] \in \mathcal{S}.$$

If the subordination result given by

$$\eta\ell(z) + \gamma z\ell'(z) \prec \frac{zf'(z)}{f_{2j,k}(z)}\left[\eta + \gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right], \tag{4.1}$$

holds, then $\ell(z) \prec \frac{zf'(z)}{f_{2j,k}(z)}$.

Proof . In order to acquire the appropriate proof, we examine the following functional:

$$h(z) = \frac{zf'(z)}{f_{2j,k}(z)} \qquad (z \in \mathbb{E}),$$

where h is holomorphic or analytic in $\mathbb{E}$ with $h(0) = 1$. Based on the simple calculations of the above result, we may have

$$\eta h(z) + \gamma zh'(z) = \frac{zf'(z)}{f_{2j,k}(z)}\left[\eta + \gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right].$$

This assumption (5.1) is similar to

$$\eta\ell(z) + \gamma z\ell'(z) \prec \eta h(z) + \gamma zh'(z).$$

The mappings $\theta : \theta(w) = \eta w$ and $\psi : \psi(w) = \gamma$ are holomorphic or analytic in $\mathbb{C}$. Set $p(z) = z\ell'(z)\psi[\ell(z)] = \gamma z\ell'(z) :$ $p(0) = 0$ and $p'(0) = \gamma\ell'(0) \neq 0$ for $\ell \in \mathcal{C}$. As a result of some calculation, we observe that $p \in \mathcal{S}^*$ in $\mathbb{E}$ and also we note that

$$\Re\left(\frac{\theta'[\ell(z)]}{\psi[\ell(z)]}\right) = \Re\left(\frac{\eta}{\gamma}\right) > 0, z \in \mathbb{E}.$$

The above condition and our deduction from Lemma 2.2 along with the assumption laid down in (5.1) implies that $\ell(z) \prec p(z)$. □

Setting $\ell(z) = \frac{1-\alpha_1 z}{1+\alpha_2 z}$ in the above Theorem 5.1, where $-1 \leq \alpha_2 < \alpha_1 \leq 1$, we may have

Corollary 4.2. Assume a holomorphic mapping $\ell \in \mathcal{C} : \ell(0) = 1$ in $\mathbb{E}$, and $\gamma \in \mathbb{C}$ with $\Re\left(\frac{\eta}{\gamma}\right) > 0$. In addition, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition given by

$$\frac{zf'(z)}{f_{2j,k}(z)} \in \mathcal{Q}$$

and moreover suppose that the functional

$$\frac{zf'(z)}{f_{2j,k}(z)}\left[\eta + \gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right] \in \mathcal{S}.$$

If

$$\eta\frac{1-\alpha_1 z}{1+\alpha_2 z} + \gamma\frac{(\alpha_1 - \alpha_2)z}{(1+\alpha_2 z)^2} \prec \frac{zf'(z)}{f_{2j,k}(z)}\left[\eta + \gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right],$$

then $\frac{1-\alpha_1 z}{1+\alpha_2 z} \prec \frac{zf'(z)}{f_{2j,k}(z)}$.

Theorem 4.3. For the complex number $\eta \in \mathbb{C}^*$ and $\delta, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \neq 0$ and $\Re(\delta\eta) > 0$, suppose that $\ell \in \mathcal{C} : \ell(0) = 1$ in $\mathbb{E}$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)} \neq 0, z \in \mathbb{E}$$

and also assume that

$$\left[\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)}\right]^{\eta} \in \mathcal{Q}.$$

If the mapping ϕ is given in such a way that

$$\phi(z) = (1 + \delta + \frac{z f''(z)}{f'(z)} - \frac{\eta_1 z (z f'_{2j,k}(z))' + \eta_2 z f'_{2j,k}(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)}) \left[\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{2j,k}(z)}\right]^{\eta} \in \mathcal{S},$$

and we have

$$\delta \ell(z) + \frac{1}{\eta} z \ell'(z) \prec \phi(z),$$

then, $\ell(z) \prec \left[\frac{(\eta_1+\eta_2) z f'(z)}{\eta_1 z f'_{2j,k}(z) + \eta_2 f_{k)}(z)}\right]^{\eta}$.

5 Sandwich Results

Theorem 5.1. Suppose that $\ell : \ell(0) = 1$ is convex in $\mathbb{E}$, and $\gamma \in \mathbb{C}$ with $\Re\left(\frac{\eta}{\gamma}\right) > 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{z f'(z)}{f_{2j,k}(z)} \in \mathcal{Q}$$

and the mapping ϕ in such a way that

$$\frac{z f'(z)}{f_{2j,k)}(z)} \left[\eta + \gamma \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right] \in \mathcal{S}.$$

If the subordination result given by

$$\eta \ell(z) + \gamma z \ell'(z) \prec \frac{z f'(z)}{f_{2j,k}(z)} \left[\eta + \gamma \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right], \tag{5.1}$$

holds, then $\ell(z) \prec \frac{z f'(z)}{f_{2j,k}(z)}$.

Proof . In order to acquire the appropriate proof, we examine the following functional:

$$h(z) = \frac{z f'(z)}{f_{2j,k}(z)} \qquad (z \in \mathbb{E}),$$

where h is holomorphic or analytic in $\mathbb{E}$ with $h(0) = 1$. Based on the simple calculations of the above result, we may have

$$\eta h(z) + \gamma z h'(z) = \frac{z f'(z)}{f_{2j,k}(z)} \left[\eta + \gamma \left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right].$$

This assumption (5.1) is similar to

$$\eta \ell(z) + \gamma z \ell'(z) \prec \eta h(z) + \gamma z h'(z).$$

The mappings $\theta : \theta(w) = \eta w$ and $\psi : \psi(w) = \gamma$ are holomorphic or analytic in $\mathbb{C}$. Set $p(z) = z \ell'(z) \psi[\ell(z)] = \gamma z \ell'(z) :$ $p(0) = 0$ and $p'(0) = \gamma \ell'(0) \neq 0$ for $\ell \in \mathcal{C}$. As a result of some calculation, we observe that $p \in \mathcal{S}^*$ in $\mathbb{E}$ and also we note that

$$\Re\left(\frac{\theta'\left[\ell(z)\right]}{\psi\left[\ell(z)\right]}\right)=\Re\left(\frac{\eta}{\gamma}\right)>0, z\in\mathbb{E}.$$

The above condition and our deduction from Lemma 2.2 along with the assumption laid down in (5.1) implies that $\ell(z)\prec p(z)$. □

Setting $\ell(z)=\frac{1-\alpha_1 z}{1+\alpha_2 z}$ in the above Theorem 5.1, where $-1\leq\alpha_2<\alpha_1\leq 1$, we may have

Corollary 5.2. Assume a holomorphic mapping $\ell\in\mathcal{C}:\ell(0)=1$ in $\mathbb{E}$, and $\gamma\in\mathbb{C}$ with $\Re\left(\frac{\eta}{\gamma}\right)>0$. In addition, consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition given by

$$\frac{zf'(z)}{f_{2j,k}(z)}\in\mathcal{Q}$$

and moreover suppose that the functional

$$\frac{zf'(z)}{f_{2j,k}(z)}\left[\eta+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right]\in\mathcal{S}.$$

If

$$\eta\frac{1-\alpha_1 z}{1+\alpha_2 z}+\gamma\frac{(\alpha_1-\alpha_2)z}{(1+\alpha_2 z)^2}\prec\frac{zf'(z)}{f_{2j,k}(z)}\left[\eta+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right],$$

then $\frac{1-\alpha_1 z}{1+\alpha_2 z}\prec\frac{zf'(z)}{f_{2j,k}(z)}$.

Theorem 5.3. For the complex number $\eta\in\mathbb{C}^*$ and $\delta,\eta_1,\eta_2\in\mathbb{C}$ with $\eta_1+\eta_2\neq 0$ and $\Re(\delta\eta)>0$, suppose that $\ell\in\mathcal{C}:\ell(0)=1$ in $\mathbb{E}$. Furthermore, consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition

$$\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\neq 0, z\in\mathbb{E}$$

and also assume that

$$\left[\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\in\mathcal{Q}.$$

If the mapping ϕ is given in such a way that

$$\phi(z)=(1+\delta+\frac{zf''(z)}{f'(z)}-\frac{\eta_1 z(zf'_{2j,k}(z))'+\eta_2 zf'_{2j,k}(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)})\left[\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\in\mathcal{S},$$

and we have

$$\delta\ell(z)+\frac{1}{\eta}z\ell'(z)\prec\phi(z),$$

then, $\ell(z)\prec\left[\frac{(\eta_1+\eta_2)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{k)}(z)}\right]^{\eta}$.

6 Sandwich Results

Combining the results describe in the Theorem 3.1 with Theorem 5.1 yields the sandwich results seen below.

Theorem 6.1. Assume that $\ell_1:\ell_1(0)=1$ and $\ell_2:\ell_2(0)=1$ are convex mappings with $\gamma\in\mathbb{C}^*$ and $\delta,\eta_1,\eta_2\in\mathbb{C}:\eta_1+\eta_2\neq 0$. Further consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition

$$\frac{zf'(z)}{f_{2j,k}(z)}\neq 0, z\in\mathbb{E}.$$

and,

$$\frac{zf'(z)}{f_{2j,k}(z)}\in\mathcal{Q}$$

If the functional ϕ defined by

$$\phi(z)=\frac{zf'(z)}{f_{2j,k}(z)}\left[\eta+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right)\right]\in\mathcal{S}$$

then, we may write that

$$\gamma z\ell_1'(z)+\eta\ell_1(z)\prec\phi(z)\prec\gamma z\ell_2'(z)+\eta\ell_2\left(z\right).$$

Combining the results describe in the Theorem 3.9 with Theorem 5.3 yields the sandwich results seen below.

Theorem 6.2. Consider that $\ell_1:\ell_1(0)=1$ and $\ell_2:\ell_2(0)=1$ are convex mappings with $\eta\in\mathbb{C}^*$ and $\delta,\eta_1,\eta_2\in\mathbb{C}:\eta_1+\eta_2\neq 0$. Further consider that a holomorphic mapping $f\in\mathcal{A}$ meets the condition

$$\frac{(\eta_2+\eta_1)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\neq 0, z\in\mathbb{E}.$$

and,

$$\left[\frac{(\eta_2+\eta_1)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\in\mathcal{Q}$$

If the mapping ϕ defined by

$$\phi(z)=\left[\frac{(\eta_2+\eta_1)zf'(z)}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]^{\eta}\left[1+\delta+\frac{zf''(z)}{f'(z)}-\frac{\eta_2 zf'_{2j,k}(z)+\eta_1 z(zf'_{2j,k}(z))'}{\eta_1 zf'_{2j,k}(z)+\eta_2 f_{2j,k}(z)}\right]$$

is such that $\phi\in\mathcal{S}$ in $\mathbb{E}$, and

$$\frac{1}{\eta}z\ell_1'(z)+\delta\ell_1(z)\prec\phi(z)\prec\frac{1}{\eta}z\ell_2'(z)+\delta\ell_2\left(z\right).$$

7 Concluding Remarks

In this research, by making use of first order differential subordination and superordination conditions, we determined some previously known families of mappings associated with $2j,k$-symmetrical points. We also proved some sandwich theorems based on specific assumptions about the parameters that are involved in our major findings. From our discussion, it is obvious that the research is related with the existing literature of the subject. It may be kept updated with the classical and emerging trends of geometric functions theory

References

[1] M. Aouf, F. Al-Oboudi and M. Haidan, *On some results for λ-spirallike and λ-robertson functions of complex order*, Publ. Instit. Math. Belgrade **77** (2005), no. 97, 93–98.

[2] M. Aouf and T. Bulboacă, *Subordination and superordination properties of multivalent functions defined by certain integral operator*, J. Franklin Inst. **347** (2010), no. 3, 641–653.

[3] S.Z.H. Bukhari, T Bulboacă and M.S Shabbir, *Subordination and superordination results for analytic functions with respect to symmetrical points*, Quaest. Math. **41** (2018), no. 1, 65–79.

[4] S.Z.H. Bukhari, M. Nazir and M. Raza, *Some generalizations of analytic functions with respect to 2k-symmetric conjugate points*, Maejo Int. J. Sci. Technol. **10** (2016), no. 1, 1–12.

[5] S.Z.H. Bukhari, M. Raza and M. Nazir, *Some generalizations of the class of analytic functions with respect to k-symmetric points*, Hacet. J. Math. Stat. **45** (2016), 1–14.

[6] T. Bulboaca, *Classes of first order differential superordinations*, Demonst. Math. Warsaw Tech.Univ. Inst. Math. **35** (2002), no. 2, 287–292.

[7] R.M. El-Ashwah and D. Thomas, *Some subclasses of close-to-convex functions*, J. Ramanujan Math. Soc. **2** (1987), no. 1, 85–100.

[8] A.W. Goodman, *Univalent functions*, Vol. 1, Mariner Publishing House, Tampa, Florida, 1983.

[9] W. Janowski, *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math. **28** (1973), 297–326.

[10] A. Janteng, S.A. Halim and M. Darus, *Functions close-to-convex and quasi-convex with respect to other points*, Int. J. Pure App, Math. **30** (2006), no. 2, 225.

[11] KR. Karthikeyan., *Some classes of analytic functions with respect to j, k-symmetric points*, ROMAI J. **9** (2013), 51–60.

[12] P. Liczberski and J. Połubiński, *On $2j, k$-symmetrical functions*, Math. Bohemica 120 (1995), no. 1, 13–28.

[13] W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proc. Conf. Complex Anal. Tianjin, Internat. Press, Cambridge, MA, 1994, pp. 157–169.

[14] S.S. Miller and P.T. Mocanu, *Differential subordinations: theory and applications.* CRC Press, 2000.

[15] S.S. Miller and P.T. Mocanu, *Subordinants of differential superordinations*, Complex Var. **48** (2003), no. 10, 815–826.

[16] M. Obradović and S. Owa, *On certain properties for some classes of starlike functions*, J. Math. Anal. Appl. **145** (1990), no. 2, 357–364.

[17] V. Ravichandran, *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi.(NS) **20** (2004), no. 1, 31–37.

[18] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soci. Japan **11** (1959), no. 1, 72–75.

[19] S. Shams, S. Kulkarni and J.M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Int. J. Math. Math. Sci. **2006** (2006), Art ID 94572, 3 pages.

[20] T. Shanmugam, C. Ramachandran, M. Darus and S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions involving a linear operator*, Acta Math. Univ. Comenianae **74** (2007), no. 2, 287–294.

[21] T. Shanmugam, S. Sivasubramanian and H. Silverman, *On sandwich theorems for some classes of analytic functions*, Int. J. Math. Math. Sci. **2006** (2006), Art ID 029684, 13 pages.

[22] Z-G. Wang and Y-P. Jiang, *Some properties of certain subclasses of close-to-convex and quasi-convex functions with respect to $2k$-symmetric conjugate points*, Bull. Iran. Math. Soc. **36** (2011), 217–238.