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Numerical solution for solving inverse telegraph equation by extended cubic B-spline

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Abstract

In this paper, we consider a numerical method based on extended cubic B-spline basis functions for the determination of an unknown boundary condition in the inverse second-order one-dimensional hyperbolic telegraph equation. Extended cubic B-spline (ExCuBs) is an extension of cubic B-spline consisting of a parameter, we combined it with the Tikhonov regularization method to obtain a numerically stable solution. The convergence and stability of the technique are proved and shown that it is established under suitable assumptions and accurate order $O(k + h^2)$. The numerical results have been compared with those obtained by the cubic B-spline method to verify the accurate nature of our method.

Keywords: Extended Cubic B-spline Collocation Method, Stability, Convergence Analysis, Telegraph Equation
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1 Introduction

The telegraph equation is one of the important equations of engineering and science with applications in many different fields such as modeling of anomalous diffusive and wave propagation phenomenon, modeling of anomalous diffusion and sub-diffusive systems, continuous-time random walks, transmission and propagation of electrical signals, digital image processing, thermodynamics, hydrodynamics, elasticity, fluid dynamics, etc[5, 11, 9, 18, 10, 20, 3, 27, 2]. Recently, author of [19] presented a mathematical model of oncolytic virus spread using a reaction-telegraph equation.

The inverse wave equation is known as one of the fundamental equations in mathematical physics is occur in many excellent research projects done in this area[4, 12]. In [15], Kozhanov made investigations on the parabolic and hyperbolic inverse problems of finding a solution together with an unknown right-hand side. Small perturbations from observations of a scattered field generated by probing the medium with a known signal is discussed via [7]. The wave-splitting, layer-stripping approach to the time-dependent inverse problem are arisen in the wave equation and the telegraph equation [17, 25] In recent years, inverse problems for hyperbolic equations are studied to determine unknown coefficient of the telegraph equation [16]. Mathematically, the problem of signal recovery is referred to as solving an inverse problem. Inverse telegraph equation deals with the problem of state estimation for a hyperbolic equation in the presence of unknown, but bounded disturbances. In this paper, we consider the inverse generalized Telegraph equation to the following form

$$u_{tt} + 2\alpha u_t - u_{ss} + \beta^2 u = f(s,t), \qquad (s,t) \in [0,1] \times [0,t_f], \tag{1.1}$$

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with initial and boundary conditions

$$u(s,0) = f_1(s), \qquad s \in [0,1], \tag{1.2}$$

$$u_t(s,0) = f_2(s), \qquad s \in [0,1], \tag{1.3}$$

$$u(0,t) = g_1(t), \qquad t \in [0,t_f], \tag{1.4}$$

$$u(1,t) = g_2(t), \qquad t \in [0,t_f], \tag{1.5}$$

where α, β are arbitrary positive constants and t_f represents the final time, while the functions $f_1(s)$, $f_2(s)$, $g_1(t)$ and $f(s,t)$ are known functions, $g_2(t)$ and $u(s,t)$ are unknown.

The rest of the paper is organized as follows: In Section 2, estened cubic B-spline collocation scheme is explained and in Subsections 2.1 and 2.2 the method is applied to solve problem (1.1)-(1.5). we prove the stability and convergence of the method in Subsections 2.3 and 2.4. In Section 3 numerical experiment is conducted to demonstrate the viability and the efficiency of the proposed methods computationally. A summary is given at the end of the paper in Section 4.

2 Extended B-spline collocation method

In this part, we solve the inverse problem (1.1)-(1.5) with the over-specified condition

$$u(a,t) = h_1(t), \qquad t \in [0,t_f], \tag{2.1}$$

where $0 < a < 1$ is a fixed point.

The solution domain $s \in [0,1]$ is partitioned into a mesh of uniform length $h = s_{i+1} - s_i$ by the knots s_i where $i = 0, 1, \ldots, N-1$ such that $0 = s_0 < s_1 < \cdots < s_N = 1$ be the partition in $[0,1]$. The ExCuBs basis functions at the nodal point s_i can be presented as[23]

$$B_i(s,\eta) = \frac{1}{24h^4} \begin{cases} 4h(1-\eta)(s-s_{i-2})^3 + 3\eta(s-s_{i-2})^4, & s \in [s_{i-2}, s_{i-1}), \\ (4-\eta)h^4 + 12h^3(s-s_{i-1}) + 6h^2(2+\eta)(s-s_{i-1})^2 & \\ -12h(s-s_{i-1})^3 - 3\eta(s-s_{i-1})^4, & s \in [s_{i-1}, s_i), \\ (4-\eta)h^4 + 12h^3(s_{i+1}-s) + 6h^2(2+\eta)(s_{i+1}-s)^2 & \\ -12h(s_{i+1}-s)^3 - 3\eta(s_{i+1}-s)^4, & s \in [s_i, s_{i+1}), \\ 4h(1-\eta)(s_{i+2}-s)^3 + 3\eta(s_{i+2}-s)^4, & s \in [s_{i+1}, s_{i+2}), \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

where $\eta \in \mathbb{R}$ is a free parameter and $s \in \mathbb{R}$ is a variable. For $-8 \le \eta \le 1$, the ExCuBs basis and cubic B-spline possess the same properties. The basis function will be reduced to the basis function of cubic B-spline for $\eta = 0$. The set of functions $\{B_{-1}, B_0, \ldots, B_{N+1}\}$ forms a basis over the considered domain on $[0,1]$. The approximated solution $U(s,t)$ to $u(s,t)$ can be written in terms of the extended B-splines as[1]

$$U(s,t) = \sum_{i=-1}^{N+1} c_i(t) B_i(s,\eta), \tag{2.3}$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary and over-specified conditions and collocation from of the differential equations. The values of $B_i(s,\eta)$ and its derivatives may be tabulated as in Table 1. Using approximate function (2.3) and ExCuBs (2.2), the approximate values at the knots of $U(s)$ and its derivatives up to third order are determined in terms of the time parameters c_m as

$$U_m = \frac{4-\eta}{24} c_{m-1} + \frac{8+\eta}{12} c_m + \frac{4-\eta}{24} c_{m+1}, \tag{2.4}$$

$$U'_m = -\frac{1}{2h} c_{m-1} + \frac{1}{2h} c_{m+1}, \tag{2.5}$$

$$U''_m = \frac{2+\eta}{2h^2} c_{m-1} - \frac{2+\eta}{h^2} c_m + \frac{2+\eta}{2h^2} c_{m+1}. \tag{2.6}$$

Table 1: Coefficient of extended cubic B-splines and its derivatives at knots s_i.

s	s_{i-1}	s_i	s_{i+1}
$B_i(s,\eta)$	$\frac{4-\eta}{24}$	$\frac{8+\eta}{12}$	$\frac{4-\eta}{24}$
$B_i'(s,\eta)$	$\frac{1}{2h}$	0	$-\frac{1}{2h}$
$B_i''(s,\eta)$	$\frac{2+\eta}{2h^2}$	$-\frac{2+\eta}{h^2}$	$\frac{2+\eta}{2h^2}$

2.1 Temporal discretization

Let us consider a uniform mesh (s_i, t_j) to discretize the region $[0,1]\times[0,t_f]$ where $s_i = ih, i = 0,1,2,\dots,N$ and $t_j = jk, j = 0,1,\dots,$ where h and k are mesh sizes in the space and time directions respectively.

At first we discretize the problem in time variable using the following finite difference approximation with uniform step size k thus, we have

$$\theta_1 u^{j+1} = \theta_2 u^j - u^{j-1} + k^2 u^j_{ss} + k^2 f(s,t_j), \tag{2.7}$$

where

$$\theta_1 = 1 + 2\alpha k, \quad \theta_2 = 2 + 2\alpha k - \beta^2 k^2. \tag{2.8}$$

Substituting the approximate solution U for u and putting the values of the nodal values U, its derivatives using equations (2.4)-(2.6) at the knots in equation (2.7) yields the following difference equation with the variable c

$$A^* c^{j+1}_{i-1} + B^* c^{j+1}_{i-1} + A^* c^{j+1}_{i} = C^* c^j_{i-1} + D^* c^j_i + C^* c^j_{i+1} - \delta_1 c^{j-1}_{i-1} - \delta_2 c^{j-1}_i - \delta_1 c^{j-1}_{i+1} + k^2 f(s_i, t_{j+1}) = \rho_{ij}, \quad 0 \le i \le N, \tag{2.9}$$

where

$$A^* = \theta_1\delta_1, \qquad B^* = \theta_1\delta_2, \qquad C^* = \theta_2\delta_1 + k^2\delta_3, \qquad D^* = \theta_2\delta_2 - 2k^2\delta_3,$$
$$\delta_1 = \frac{4-\eta}{24}, \qquad \delta_2 = \frac{8+\eta}{12}, \qquad \delta_3 = \frac{2+\eta}{2h^2}.$$

The system (2.9) consist of $(N+1)$ linear equations in $(N+3)$ unknowns

$$(c_{-1}, c_0, \dots, c_N, c_{N+1})^T.$$

To obtain a unique solution to this system we need two additional equations which will come from boundary condition (1.4) and the over-specified conditions (2.1) is required.
Let $a = s_z$, $1 \le z \le N-1$, so we have

$$u(s_z, t) = h_1(t), \qquad t \in [0, t_f], \tag{2.10}$$

expanding u in terms of approximate Extended B-spline formula from (2.4) at s_z putting $m = z$ we get

$$\frac{4-\eta}{24} c^j_{z-1} + \frac{8+\eta}{12} c^j_z + \frac{4-\eta}{24} c^j_{z+1} = h_1(t_{j+1}), \tag{2.11}$$
$$\frac{4-\eta}{24} c^j_{-1} + \frac{8+\eta}{12} c^j_0 + \frac{4-\eta}{24} c^j_1 = g_1(t_{j+1}), \quad j = 0,1,\dots, \tag{2.12}$$

thus, the system (2.9) is changed to a system of $(N+3)$ linear equations in $(N+3)$ unknowns, given by

$$AC = D, \tag{2.13}$$

where

$$A[1,1] = A[1,3] = \delta_1, A[1,2] = \delta_2, \quad A[N+3, s+1] = A[N+3, s+3] = \delta_1, \quad A[N+3, s+2] = \delta_2,$$

thus

$$A=\begin{pmatrix}\delta_1&\delta_2&\delta_1&0&0&0&&\cdots&0\\A^*&B^*&A^*&&&&&&\vdots\\&A^*&B^*&A^*&&&&&\\&&\cdots&\cdots&\cdots&&&&\\&&&\cdots&\cdots&\cdots&&&\\&&&&&A^*&B^*&A^*&\\\vdots&&&&&&A^*&B^*&A^*\\0&\cdots&\cdots&0&\delta_1&\delta_2&\delta_1&\cdots&0\end{pmatrix},\quad C=\begin{pmatrix}c_{-1}^{(j+1)}\\c_0^{(j+1)}\\c_1^{(j+1)}\\\vdots\\c_z^{(j+1)}\\\vdots\\c_{N-1}^{(j+1)}\\c_N^{(j+1)}\\c_{N+1}^{(j+1)}\end{pmatrix},\quad D=\begin{pmatrix}D_{-1}^{(j)}\\{D_0}^{(j)}\\D_1^{(j)}\\\vdots\\D_z^{(j)}\\\vdots\\{D_{N-1}}^{(j)}\\{D_N}^{(j)}\\{D_{N+1}}^{(j)}\end{pmatrix},$$

$$D_{-1}^{(j)}=g_1(t_{j+1}),$$

$$D_i^{(j)}=\rho_{ij},\quad 0\le i\le N,$$

$$D_{N+1}^{(j)}=h_1(t_{j+1}).$$

A is ill-conditioned matrix, thus we solved this system (2.13) by the Tikhonov regularization method [21].

2.2 The initial state

The initial vector c^0 and c^1 can be obtained from the initial conditions (1.2), (1.3), boundary and over-specified condition (1.4) and (2.1) as the following expressions:

$$\begin{aligned}u(s_i,t_0)&=\frac{4-\eta}{24}c_{i-1}^0+\frac{8+\eta}{12}c_i^0+\frac{4-\eta}{24}c_{i+1}^0=f_1(s_i)=u^0, && 0\le i\le N,\\u(s_z,t_0)&=\frac{4-\eta}{24}c_{z-1}^0+\frac{8+\eta}{12}c_z^0+\frac{4-\eta}{24}c_{z+1}^0=h_1(t_0),\\u(0,t_0)&=\frac{4-\eta}{24}c_{-1}^0+\frac{8+\eta}{12}c_0^0+\frac{4-\eta}{24}c_1^0=g_1(t_0).\end{aligned}$$

This yields a $(N+3)\times(N+3)$ system of equations, of the form

$$\Lambda C^0=\rho, \tag{2.14}$$

where

$$\begin{aligned}&\Lambda[1,1]=\Lambda[1,3]=\delta_1, && \Lambda[1,2]=\delta_2,\\&\Lambda[N+3,s+1]=\Lambda[N+3,s+3]=\delta_1, && \Lambda[N+3,s+2]=\delta_2,\end{aligned}$$

thus

$$\Lambda=\begin{pmatrix}\delta_1&\delta_2&\delta_1&0&0&0&&\cdots&0\\\delta_1&\delta_2&\delta_1&&&&&&\vdots\\&\delta_1&\delta_2&\delta_1&&&&&\\&&\cdots&\cdots&\cdots&&&&\\&&&\cdots&\cdots&\cdots&&&\\&&&&&\delta_1&\delta_2&\delta_1&\\\vdots&&&&&&\delta_1&\delta_2&\delta_1\\0&\cdots&\cdots&0&\delta_1&\delta_2&\delta_1&\cdots&0\end{pmatrix},\quad C^0=\begin{pmatrix}c_{-1}^0\\{c_0}^0\\{c_1}^0\\\vdots\\{c_z}^0\\\vdots\\{c_{N-1}}^0\\{c_N}^0\\{c_{N+1}}^0\end{pmatrix},\quad \rho=\begin{pmatrix}g_1(t_0)\\f_1(s_0)\\f_1(s_1)\\\vdots\\f_1(s_z)\\\vdots\\f_1(s_{N-1})\\f_1(s_N)\\h_1(t_0)\end{pmatrix}.$$

The solution of (2.14) can be found by the Tikhonov regularization method. Similarly from (1.3), the initial vector c^1 can be determined.

$$\begin{aligned} u_t(s,0) &= \frac{u^1-u^0}{k} = f_2(s), \\ u^1 &= u^0 + kf_2(s), \end{aligned}$$

we have

$$\begin{aligned} u(s_i,t_1) &= \frac{4-\eta}{24}c^1_{i-1} + \frac{8+\eta}{12}c^1_i + \frac{4-\eta}{24}c^1_{i+1} = f_1(s_i) + kf_2(s_i), \qquad 0 \le i \le N, \\ u(s_z,t_1) &= \frac{4-\eta}{24}c^1_{z-1} + \frac{8+\eta}{12}c^1_z + \frac{4-\eta}{24}c^1_{z+1} = h_1(t_1), \\ u(0,t_1) &= \frac{4-\eta}{24}c^1_{-1} + \frac{8+\eta}{12}c^1_0 + \frac{4-\eta}{24}c^1_1 = g_1(t_1). \end{aligned} \tag{2.15}$$

We solve system (2.15) for vector c^1, by using the Tikhonov regularization method.

2.3 Stability analysis

In this section, we use the Von Neumann method[14, 24]. Thus using $c^j_i = \xi^j exp(\iota\phi ih)$ into the homogeneous part of (2.9) where ϕ is the mode number, $\iota = \sqrt{-1}$, h is the element size and ξ is the amplification factor of the scheme. we get

$$\Upsilon(\xi) = v\xi^2 - \nu\xi + \vartheta = 0, \tag{2.16}$$

where

$$\begin{aligned} v &= \theta_1(2\delta_1\cos\chi + \delta_2), \qquad \vartheta = (2\delta_1\cos\chi + \delta_2), \\ \nu &= \theta_2\delta_2 - 2k^2\delta_3 + 2(\theta_2\delta_1 + k^2\delta_3)\cos\chi. \end{aligned}$$

We apply Routh–Hurwitz criterion and transformation $\xi = \frac{1+z}{1-z}$ in (2.16). Thus, $(1-z)^2\Upsilon(\frac{1+z}{1-z}) = (v+\nu+\vartheta)z^2 + 2(v-\vartheta)z + (v-\nu+\vartheta)$. The necessary and sufficient condition for $|\xi| \le 1$ is that $(v+\nu+\vartheta) > 0$, $(v-\vartheta) > 0$, $(v-\nu+\vartheta) > 0$. The condition $v-\vartheta > 0$ is always satisfied for all real variable angle χ.
But the conditions $(v+\nu+\vartheta)$ and $(v-\nu+\vartheta) > 0$ are always satisfied for $(v+\vartheta) > \nu, \nu > 0$. It can be easily verified that $(v+\vartheta) > \nu$. But for $\nu > 0$, it is satisfied when $(\delta_3\cos\chi - 2\beta^2\delta_1\cos\chi - 2\delta_3 - \beta^2\delta^2)k^2 + 2(\alpha\delta_2 + \alpha\delta_1\cos\chi)k + 2\delta_2 + 4\delta_1\cos\chi > 0$. Since, for all $k > k_1$, $k < k_2$ where

$$\begin{aligned} &k_1, k_2 = -b_1 \pm \sqrt{(b_1^2 - b_2)}, \quad b_1 = \frac{\varepsilon_1}{\varepsilon_2}, \quad b_2 = \frac{\varepsilon_3}{\varepsilon_2}, \quad \varepsilon_1 = \alpha\delta_2 + \alpha\delta_1\cos\chi, \\ &\varepsilon_2 = \delta_3\cos\chi - 2\beta^2\delta_1\cos\chi - 2\delta_3 - \beta^2\delta^2, \quad \varepsilon_3 = 2\delta_2 + 4\delta_1\cos\chi, \end{aligned}$$

we have $(v+\nu+\vartheta)$ and $(v-\nu+\vartheta) > 0$. Consequently, The ExCuBs technique for model problem is stable for $k > k_1$, $k < k_2$. □

2.4 Convergence analysis

Let $u(s) = u(s,t_{j+1})$ be the exact solution of the equation (1.1) in $t = t_{j+1}$ with the over-specified condition (2.1) and initial condition (1.2) and also $U(s) = \sum_{i=-1}^{N+1} c_i B_i(s,\eta)$ be the extened cubic B-spline collocation approximation to $u(s)$. Due to round off errors in computations we assume that $\widehat{U}(x)$ be the computed spline for $U(x)$ so that $\widehat{U}(x) = \sum_{i=-2}^{N+1} \hat{c}_i B_i(s,\eta)$ where $\widehat{C} = (\hat{c}_{-1}, \hat{c}_0, \hat{c}_1, \ldots, \hat{c}_N, \hat{c}_{N+1})$. To estimate the error $\|u(s) - U(s)\|_\infty$ we must estimate the errors $\|u(s) - \widehat{U}(s)\|_\infty$ and $\|\widehat{U}(s) - U(s)\|_\infty$ separately. Following (2.13) for $\widehat{U}$ we have

$$A\widehat{C} = \widehat{D}, \tag{2.17}$$

where $\widehat{D} = (g_1(t_{j+1}), \hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_N, h_1(t_{j+1}))$, and

$$\hat{\omega}_i = C^* c^j_{i-1} + D^* c^j_i + C^* c^j_{i+1} - \delta_1 c^{j-1}_{i-1} - \delta_2 c^{j-1}_i - \delta_1 c^{j-1}_{i+1} + k^2 f(s_i, t_{j+1}).$$

Subtracting (2.17) and (2.13) we have

$$A(C - \widehat{C}) = (D - \widehat{D}), \tag{2.18}$$

first we need to recall a Theorem.

Theorem 2.1. Suppose that $u^* \in C^4[0,1]$ and $\|(u^*)^{(4)}(s)\| \leq L$, $\forall s \in [0,1]$ and $\Delta = \{0 = s_0 < s_1 < \cdots < s_N = 1\}$ be the equality spaced partition of $[0,1]$ with step size h. If $U\hat{(s)}$ be the unique spline function interpolate $u^*(s)$ at nodes $s_0, s_1, \ldots, s_N \in \Delta$, then there exist a constant λ_i where it is independent of h, such that

$$\|(u^*)^i(s) - (\hat{U})^i(s)\| \leq \lambda_i L h^{4-i}, \quad i = 0,1,2,3, \tag{2.19}$$

where $\|.\|$ represents the ∞-norm.

Proof . For the proof see [13, 8]. □ Now, we want to find a bound on $\|D - \widehat{D}\|_\infty$ first. We have

$$|D(s_i) - \widehat{D}(s_i)| = \left|\omega\Big(s_i, U(s_i), U''(s_i)\Big) - \omega\Big(x_i, \widehat{U}(s_i), \widehat{U}''(s_i)\Big)\right|,$$

by following theorem (2.1) we obtain

$$\begin{aligned}\|D - \widehat{D}\|_\infty &\leq M\Big(|U(s) - \widehat{U}(s)| + |U''(s) - \widehat{U}''(s)|\Big)\\ &\leq M(\lambda_0 h^4 + \lambda_2 h^2),\end{aligned} \tag{2.20}$$

where $\|\omega''(z)\|_\infty \leq M$. Thus we can rewrite (2.20) as follows

$$\|D - \widehat{D}\|_\infty \leq M_1 h^2, \tag{2.21}$$

where $M_1 = ML(\lambda_0 h^2 + \lambda_2)$. The matrix A in (2.18) is an ill-conditioned matrix, thus by Tikhonov regularization solution [21], we have

$$(C - \widehat{C}) = [A^T A + \alpha (R^{(2)})^T R^2]^{-1} A^T (D - \widehat{D}), \tag{2.22}$$

taking the infinity norm and then by using (2.21) we find

$$\|C - \widehat{C}\|_\infty \leq \|[A^T A + \alpha (R^{(2)})^T R^2]^{-1} A^T\|_\infty \|D - \widehat{D}\|_\infty \leq M_2 h^2, \tag{2.23}$$

where $M_2 = M_1 \|[A^T A + \alpha (R^{(2)})^T R^2]^{-1} A^T\|_\infty$. Now, we will be able to prove the convergence of our present method. Therefore, we recall a following lemma first

Lemma 2.2. The extended B-splines $\{B_{-1}, B_0, ..., B_{N+1}\}$ defined in relation (2.2), satisfy the following inequality

$$\left|\sum_{i=-1}^{N+1} B_i(s, \eta)\right| \leq 1.75, \quad (0 \leq s \leq 1). \tag{2.24}$$

Proof . For proof see[22] □ Now, observe that we have

$$U(s) - \widehat{U}(s) = \sum_{i=-1}^{N+1} (c_i - \hat{c}_i) B_i(s),$$

thus taking the infinity norm and using (2.23) and (2.24) we get

$$\|U(s) - \widehat{U}(s)\|_\infty = \left\|\sum_{i=-1}^{N+1} (c_i - \hat{c}_i) B_i(s, \eta)\right\|_\infty \leq \|(c_i - \hat{c}_i)\|_\infty \left|\sum_{i=-1}^{N+1} B_i(s, \eta)\right| \leq \frac{7}{4} M_2 h^2 \tag{2.25}$$

Theorem 2.3. Let $u(s)$ be the exact solution of the equation (1.1) with the over-specified condition (2.1) and initial conditions (1.2)-(1.3) and also $U(s)$ be the extended B-spline collocation approximation to $u(s)$ then the method has second order convergence

$$\|u(s) - U(s)\| \leq \Omega h^2,$$

where $\Omega = \lambda_0 L h^2 + \dfrac{7}{4} M_2$ is some finite constant.

Proof . From theorem (2.1) we have

$$\|u(s)-\widehat{U}(s)\| \leq \lambda_0 L h^4, \tag{2.26}$$

thus substituting from (2.25) and (2.26) we have

$$\|u(s)-U(s)\| \leq \|u(s)-\widehat{U}(s)\|+\|\widehat{U}(s)-U(s)\| \leq \lambda_0 L h^4+\frac{7}{4} M_2 h^2=\Omega h^2, \tag{2.27}$$

where $\Omega=\lambda_0 L h^2+\frac{7}{4} M_2$. □

Theorem 2.4. The time discretization process (2.7) that we use to discretize equation (1.1) in time variable is of the first order convergence.

Proof . See [22]. □

We suppose that $u(s,t)$ be the solution of equation (1.1) and $U(s,t)$ be the approximate solution by our present method then we have

$$\|u(s,t_j)-U(s,t_j)\| \leq \varrho(k+h^2),$$

(ϱ is some finite constant), thus the order of convergence of our process is $O(k+h^2)$.

3 Numerical illustrations

In this Section, the extended cubic B-spline method is employed to obtain the numerical solutions for unknown boundary conditions in the problem (1.1)-(1.5). Numerical example is discussed in this Section to demonstrate the accuracy of the presented methods described in Sections 2 and these numerical results are compared with cubic B-spline method[26].

In an inverse problem, there are two sources of error in the estimation. The first source is the unavoidable bias deviation (deterministic error) and the second one is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors is considered in the mean squared error or the total error [6]. Therefore, we compare the exact and the approximate solutions by considering the total error S defined by

$$S_{g_2}=\left[\frac{1}{N-1}\sum_{s=1}^{N}\Big(g_2(t_{s+1})-g_2^*(t_{s+1})\Big)^2\right]^{\frac{1}{2}},$$

where N is the number of estimated values, g_2 is the exact value, g_2^* is the estimated value.

In order to illustrate the performance of the methods and justify the accuracy and efficiency of the proposed methods, we also offer the infinity-norm of absolute error for $0 \leq s \leq 1$ and $0 \leq t \leq t_f$.

$$L_\infty^{g_2}=||g_2(t)-g_2^*(t)||_\infty=\max|g_2(t)-g_2^*(t)|,$$

also,

$$L_\infty^{u}=||u(s,t)-u^*(s,t)||_\infty=\max|u(s,t)-u^*(s,t)|,$$

where, $u^*(s,t)$ is the estimated value of $u(s,t)$. We consider two examples of linear telegraph equation in [20], we take $a=0.8$, $t_f=1$, $k=0.001$, $\alpha=\beta=10$, $\eta=7.5\times10^{-5}$ and the noisy data (input data+0.0001×rand(1)).

Example 3.1. In this example, we solve the Telegraph equation given in,

$$u_{tt}+2\alpha u_t-u_{ss}+\beta^2 u=(2-2\alpha+\beta^2)\exp^{-t}\sin(s), \qquad (s,t)\in[0,1]\times[0,t_f],$$

with given data

$$\begin{aligned}
&u(s,0)=\sin(s), && \\
&u(0,t)=g_1(t)=\exp(-t), && 0\leq t\leq 1,\\
&u(a,t)=h_1(t)=\exp(-t)\sin(a), && 0\leq t\leq 1.
\end{aligned}$$

The exact solutions of this problem are,

$$u(s,t) = \exp(-t)\sin(s), \qquad 0 \le s \le 1, \quad 0 \le t \le 1,$$
$$u(1,t) = g_2(t) = \exp(-t)\sin(1), \qquad 0 \le t \le 1.$$

The numerical results of the unknown boundary condition $u(1,t)$ and the obtained numerical solutions for $u(s,t)$ at point $s = 0.2$ is reported in Tables 2 and 3. In order to clarify the accuracy of the present method, the corresponding graphical illustration are presented in Figure 1.

Table 2: The comparison between exact and numerical solutions for $g_2(t)$ by using the ExCuBs collocation and cubic B-spline collocation method in Example 3.1, when $h = 1/100$ and $k = 1/1000$.

t	Exact	ExCuBs	cubic B-spline
0.1	0.762156	0.764418	0.841064
0.2	0.689627	0.693961	0.750809
0.3	0.624000	0.629308	0.661028
0.4	0.564619	0.570130	0.585855
0.5	0.510888	0.516203	0.524208
0.6	0.462271	0.467225	0.472057
0.7	0.418280	0.422826	0.425221
0.8	0.378475	0.382615	0.384071
0.9	0.342458	0.346215	0.347276
1	0.309869	0.313273	0.314218
S_{g_2}	-	0.0044	0.0356
$L_\infty^{g_2}$	-	0.0055	0.0790
Execution time (second)	-	367.324243	507.359855
Condition Number of Matrix A	-	$3.7719e+17$	257201357416804960.0

Table 3: The comparison between exact and numerical solutions for $u(0.2,t)$ by using the extended cubic B-spline collocation and cubic B-spline method in Example 3.1, when $h = 1/100$ and $k = 1/1000$.

t	Exact	ExCuBs	cubic B-spline
0.1	0.179943	0.180369	-0.227643
0.2	0.162819	0.163067	0.518497
0.3	0.147325	0.147329	0.224198
0.4	0.133305	0.133220	0.230347
0.5	0.120619	0.120509	0.206538
0.6	0.109141	0.109027	0.186207
0.7	0.098754	0.098646	0.168547
0.8	0.089357	0.089258	0.152532
0.9	0.080853	0.080764	0.138153
1	0.073159	0.073078	0.124933
S_{g_2}	-	$1.8023e-04$	0.1493
$L_\infty^{g_2}$	-	$1.8454e-07$	0.4086

Example 3.2. In this example, we solve the Telegraph equation given in,

$$u_{tt} + 2\alpha u_t - u_{ss} + \beta^2 u = \beta^2 \cos(t)\sin(s) - 2\alpha\sin(t)\sin(s), \qquad (s,t) \in [0,1] \times [0,t_f],$$

with given data

$$u(s,0) = \sin(s),$$
$$u(0,t) = g_1(t) = 0, \qquad 0 \le t \le 1,$$
$$u(a,t) = h_1(t) = \cos(t)\sin(a), \qquad 0 \le t \le 1.$$

The exact solutions of this problem are,

$$u(s,t) = \cos(t)\sin(s), \qquad 0 \le s \le 1, \quad 0 \le t \le 1,$$
$$u(1,t) = g_2(t) = \cos(t)\sin(1), \qquad 0 \le t \le 1.$$

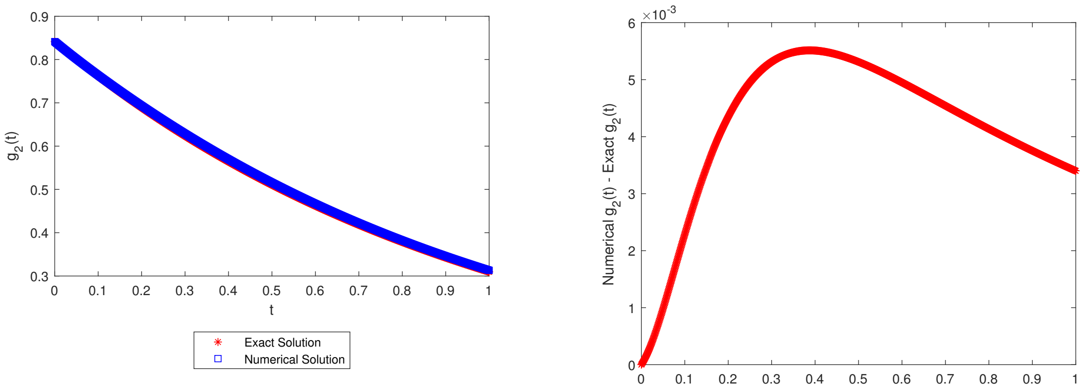

Figure 1: The comparison between the exact and numerical solutions (using extended cubic B-spline method) $u(1,t)$ for Example 3.1.

The numerical results of the unknown boundary condition $u(1,t)$ and the obtained numerical solutions for $u(s,t)$ at point $s = 0.2$ is reported in Tables 4 and 5. In order to clarify the accuracy of the present method, the corresponding graphical illustration are presented in Figure 2.

Table 4: The comparison between exact and numerical solutions for $g_2(t)$ by using the ExCuBs collocation and cubic B-spline collocation method in Example 3.2, when $h = 1/100$ and $k = 1/1000$.

t	Exact	ExCuBs	cubic B-spline
0.1	0.837350	0.839880	0.883721
0.2	0.824864	0.829926	0.862334
0.3	0.804136	0.810640	0.828925
0.4	0.775373	0.782477	0.792018
0.5	0.738863	0.746060	0.751575
0.6	0.694970	0.701980	0.705613
0.7	0.644134	0.650790	0.653436
0.8	0.586861	0.593056	0.594852
0.9	0.523725	0.529378	0.530833
1	0.455356	0.460404	0.461726
S_{g_2}	-	0.0060	0.0226
$L_{\infty}^{g_2}$	-	0.0072	0.0465
Execution time (second)	-	376.554380	603.750734
Condition Number of Matrix A	-	$2.4771e+17$	257201357416804960.0

Table 5: The comparison between exact and numerical solutions for $u(0.2, t)$ by using the extended cubic B-spline collocation and cubic B-spline method in Example 3.2, when $h = 1/100$ and $k = 1/1000$.

t	Exact	ExCuBs	cubic B-spline
0.1	0.197696	0.198168	-0.218570
0.2	0.194748	0.195078	0.545804
0.3	0.189854	0.189947	0.275556
0.4	0.183063	0.183053	0.296238
0.5	0.174443	0.174384	0.283766
0.6	0.164080	0.163992	0.269843
0.7	0.152078	0.151968	0.253204
0.8	0.138556	0.138430	0.233710
0.9	0.123650	0.123509	0.212050
1	0.107508	0.107355	0.188062
S_{g_2}	-	$2.0628e-04$	0.1600
$L_\infty^{g_2}$	-	$2.2301e-07$	0.4172

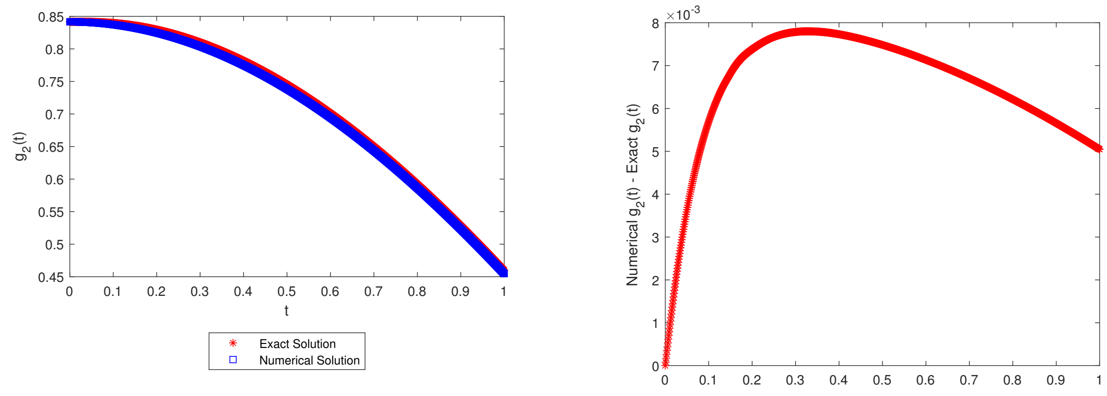

Figure 2: The comparison between the exact and numerical solutions (using extended cubic B-spline method) $u(1, t)$ for Example 3.2.

4 Conclusion and Future Work

The extended cubic B-spline method has been employed to estimate unknown boundary conditions were proposed for the inverse generalized telegraph equation (1.1)-(1.5). Numerical comparisons have been made between the extended cubic B-spline and cubic B-spline method. The numerical results showed that the extended cubic B-spline has the best performance. More precisely, it is the most accurate and fastest in comparison with cubic B-spline method. These results are obtained in the MATLAB 7.10 (R2010a) and is tested on a personal computer with intel(R) core(TM)2 Duo CPU and 4GB RAM. Our computational technique in particular is useful for a number of interesting open problems. For instance, it can be employed for alternative variant of the inverse dynamical system. These method should help to solve inverse analysis of coupled nonlinear thermo-elastic problem. We plan to pursue these issue in our future research. Finally, it would be helpful for engineers and scientists to apply the ExCuBs basis functions for solving a fractional telegraph equation in high dimensional and nonlinear telegraph equation.

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