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Meir-Keeler-type results on quasi-$b_v(s)$-metric spaces with new control functions

Leta Bekere Kumssa

Department of Mathematics, Madda Walabu University, Bale Robe, Ethiopia

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Abstract

A contraction mapping is generalized by defining an ambient space under consideration or by altering the contraction condition. In this study, we first define a new space called quasi-$b_v(s)$ metric space and verify that this space is a generalization of $b_v(s)$ metric spaces. We also define a new control function which is a generalization of the altering distance function. Finally, we prove the existence of a fixed point for ξ-generalized Meir-Keeler type contractions on quasi-$b_v(s)$-metric spaces. Many famous results in the field have been improved, generalized, and unified by the results presented here. The main result is used to drive several corollaries and an example is presented to back up the claim.

Keywords: fixed point, Meir-Keleer, quasi-$b_v(s)$-metric, $b_v(s)$-left-Cauchy, $b_v(s)$-right-Cauchy
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1 Introduction

Banach's assertion of 1922 [5] provides the foundation for the development of the fixed point theory. There are several variants of this assertion in the literature. Some instances include Ljubomir [6], Rhoades [20], and Taskovic [23].

One of the most important adaptations to Banach's theory is the Meir-Keeler contraction maps [16]. They asserted that a function S defined on a non-empty set Z with a metric δ is said to be Meir-Keeler contraction if there is a positive real number σ such that for any positive real number γ,

$$\gamma \leq \delta(z,w) < \gamma + \sigma(\gamma) \text{ implies } \delta(Sz,Sw) < \gamma \tag{1.1}$$

for all $z,\ w \in Z$. They also asserted and demonstrated that if a function S in a complete metric space (S,δ) meets (1.1), then such a mapping acquires exactly one fixed point. The Meir and Keeler's finding has been modified and changed in Kadelburg and Radenovic [12], Kadelburg et al. [13] and Karapinar et al. [14].

The Meir-Keeler technique has been also enlarged and extended in multiple ways by substituting the concept of metric space with more broader spaces or by altering the contraction criteria with different control functions (see [3, 4, 7, 18, 22]). During this study, $\mathbb{R}^+$ and $\mathbb{Z}^+$ stand for the set of all non-negative real numbers and the set of all positive integers respectively. In this study, we define the term quasi-$b_v(s)$ metric and demonstrate that this metric is a generalization of $b_v(s)$ metric. We also prove that fixed points of Meir-Keeler type contractions specified on the setting of quasi-$b_v(s)$-metric spaces exist.

Email address: first.letabekere@yahoo.com (Leta Bekere Kumssa)

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Definition 1.1. [21] A mapping $S : Z \to Z$ is denoted as $\alpha-$admissible if we have $\alpha(z, w) \geq 1$ implies $\alpha(Sz, Sw) \geq 1$ for any $z, w \in Z$, where $\alpha : Z \times Z \to \mathbb{R}^+$ is a given function.

Definition 1.2. [21] For any $z, w, v \in Z$, a mapping $S : Z \to Z$ is denoted as triangular $\alpha-$admissible if

a) S is α-admissible;
b) $\alpha(z, w) \geq 1$ and $\alpha(w, v) \geq 1 \implies \alpha(z, v) \geq 1$;
where $\alpha : Z \times Z \to \mathbb{R}^+$ is a given function.

Lemma 1.3. [14] Allow $S : Z \to Z$ to be a triangular α-admissible mapping. Assume that $z_0 \in Z$ exists and that $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0) \geq 1$ are true. If $z_k = S^k z_0$, then $\alpha(z_l, z_k) \geq 1$ for all $l, k \in \mathbb{Z}^+$.

Definition 1.4. [7] Assume Z is a non-empty set, and $s \geq 1$ denote a real number. A function $\delta : Z \times Z \to \mathbb{R}^+$ is a b-metric on Z if and only if the following requirements apply for any $z, w, v \in Z$:

a) $\delta(z, w) = 0 \iff z = w$;
b) $\delta(z, w) = \delta(w, z)$;
c) $\delta(z, w) \leq s[\delta(z, v) + \delta(v, w)]$.

The couple (Z, δ) is known as a b-metric space.

Advanced versions, like extended, dislocated, rectangular, partial, and so on, were introduced after b-metric spaces. In [8], Delfani et al. proved some fixed point results in b-metric spaces.

Definition 1.5. [19] Assume Z is a non-empty set, and $s \geq 1$ denote a real number. A function $\delta : Z \times Z \to \mathbb{R}^+$ is a rectangular b-metric on Z if and only if the following requirements apply for any $z, w, v, u \in Z$:

a) $\delta(z, w) = 0 \iff z = w$;
b) $\delta(z, w) = \delta(w, v)$;
c) $\delta(z, w) \leq s[\delta(z, v) + \delta(v, u) + \delta(u, w)]$.

The couple (Z, δ) is known as a rectangular b-metric space.

Definition 1.6. [11] Assume Z is a non-empty set, and $s \geq 1$ denote a real number. A function $\delta : Z \times Z \to \mathbb{R}^+$ is a quasi-b-metric on Z if and only if the following requirements apply for any $z, w, v \in Z$:

a) $\delta(z, w) = 0 \iff z = w$;
b) $\delta(z, w) \leq s[\delta(z, v) + \delta(v, w)]$.

Here the couple (Z, δ) is known as a quasi-b-metric space.

Definition 1.7. [11] Given that (Z, δ) is a quasi-b-metric space and (z_k) is a sequence in Z. We call the sequence (z_k) converges to $z \in Z$ if and only if

$$\lim_{k \to \infty} \delta(z_k, z) = 0 = \lim_{k \to \infty} \delta(z, z_k).$$

Definition 1.8. [11] Let (Z, δ) is a quasi-b-metric space, and (z_k) is a sequence in it. We say that the sequence (z_k) is

a) Cauchy on the left if and only if for all non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_l, z_k) \leq \gamma$ for all $l \geq k > n_0$;
b) Cauchy on the right if and only if for any non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_l, z_k) \leq \gamma$ for all $k \geq l > n_0$;
c) Cauchy provided that it is both Cauchy on the left and the right. This means, for any non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_l, z_k) \leq \gamma$ for all $l, k > n_0$.

Definition 1.9. [11] A quasi-b-metric space (Z, δ) is

a) complete on the left if and only if any sequence which is Cauchy on the left is convergent in Z ;
b) complete on the right if and only if any sequence which is Cauchy on the right is convergent in Z;

c) complete if and only if any Cauchy sequence in Z is convergent in Z.

In 2017, Mitrovic and Radenovic [17], introduced a more general version of b-metric.

Definition 1.10. [17] Assume Z is a non-empty set, and $s \geq 1$ denotes a real number. A function $\delta : Z \times Z \to \mathbb{R}^+$ is a $b_v(s)$-metric on Z if and only if the following requirements apply for any $z, w \in Z$ and all distinct points $z_1, z_2, ..., z_v \in Z$, each of them distinct from z and w:

a) $\delta(z, w) = 0 \iff z = w$;
b) $\delta(z, w) = \delta(w, z)$;
c) $\delta(z, w) \leq s[\delta(z, z_1) + \delta(z_1, z_2) + ... + \delta(z_v, w)]$.

The couple (Z, δ) is known as a $b_v(s)$-metric space.

In their work, Mitrovic and Radenovic [17] defined the terms convergence, Cauchyness of a sequence, mapping continuity, completeness, and other terms. Many writers contributed fixed point results to the $b_v(s)$-metric spaces in the following years [1, 2, 9, 10].

Inspired by the work of Mitrovic and Radenovic's [17], we now define the quasi-$b_v(s)$-metric spaces by omitting the symmetric property from Definition 1.10 and define the topological ideas of convergence, completeness, and continuity in this new space.

Definition 1.11. Assume Z is a non-empty set, and $s \geq 1$ denote a real number. A function $\delta : Z \times Z \to \mathbb{R}^+$ is a quasi-$b_v(s)$-metric on Z if and only if the following requirements apply for any $z, w \in Z$ and for all distinct points $z_1, z_2, ..., z_v \in Z$, each of them distinct from z and w:

a) $\delta(z, w) = 0 \iff z = w$;
b) $\delta(z, w) \leq s[\delta(z, z_1) + \delta(z_1, z_2) + ... + \delta(z_v, w)]$.

The couple (Z, δ) is known as a quasi-$b_v(s)$-metric space.

Remark 1.12. Any $b_v(s)$-metric is a quasi-$b_v(s)$-metric in general, but not the other way around. Hence quasi-$b_v(s)$-metric is a generalization of $b_v(s)$-metric. We will use the following example to demonstrate our point.

Example 1.13. Let $Z = \{\frac{1}{p} : p \in \{2, 3, 4, 5, 6, 7, 8\}\}$. We define $\delta : Z \times Z \to \mathbb{R}^+$ by

$$\delta(\frac{1}{p}, \frac{1}{q}) = \begin{cases} 0 & \text{if } p = q, \\ p - q & \text{if } p - q > 1, \\ \frac{1}{4} & \text{otherwise .} \end{cases}$$

Condition a) of Definition 1.11 is obvious. Since $\frac{1}{4} = \delta(\frac{1}{2}, \frac{1}{4}) \neq \delta(\frac{1}{4}, \frac{1}{2}) = 2$. i.e. symmetric property fails to hold. Hence (Z, δ) is not a $b_v(s)$-metric space.
To verify quasi-$b_v(s)$-metric inequality let us assume that $p = 8$ and $q = 2$. Hence

$$\begin{aligned} 4(\frac{6}{4}) &= 4[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}] \\ &= 4[\delta(\frac{1}{8}, \frac{1}{7}) + \delta(\frac{1}{7}, \frac{1}{6}) + \delta(\frac{1}{6}, \frac{1}{5}) + \delta(\frac{1}{5}, \frac{1}{4}) + \delta(\frac{1}{4}, \frac{1}{3}) + \delta(\frac{1}{3}, \frac{1}{2})] \\ &\geq [8 - 7 + 7 - 6 + 6 - 5 + 5 - 4 + 4 - 3 + 3 - 2] \\ &= 8 - 2 = \delta(\frac{1}{8}, \frac{1}{2}). \end{aligned}$$

Therefore, (Z, δ) is a quasi-$b_5(4)$-metric space. In a similar way as defined in [11], we can define some topological properties of quasi-$b_v(s)$-metric spaces.

Definition 1.14. Assume that (Z,δ) is a quasi-$b_v(s)$-metric space, $\{z_k\} \subseteq Z$, and that $z \in Z$. Then $\{z_k\}$ converges to z if and only if

$$\delta(z_k, z) \to 0 \text{ and } \delta(z, z_k) \to 0 \text{ as } k \to \infty.$$

Definition 1.15. Let (Z,δ) is a quasi-$b_v(s)$-metric space, and (z_k) is a sequence in it. We say that the sequence (z_k) is

a) $b_v(s)$ Cauchy on the left if and if for every non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_{k+p}, z_k) \leq \gamma$ for all $k \geq n_0,\ p \in \mathbb{Z}^+$;
b) $b_v(s)$ Cauchy on the right if and if for every non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_k, z_{k+p}) \leq \gamma$ for all $k \geq n_0,\ p \in \mathbb{Z}^+$;
c) $b_v(s)$ Cauchy provided that it is both $b_v(s)$ Cauchy on the left and on the right. This means, for any non-negative real number γ there is a positive integer $n_0 = n_0(\gamma)$ such that $\delta(z_l, z_k) \leq \gamma$ for all $l, k > n_0$.

Definition 1.16. A quasi-$b_v(s)$-metric on Z is

a) $b_v(s)$ complete on the left if and only if any sequence which is $b_v(s)$ Cauchy on the left is convergent in Z ;
b) $b_v(s)$ complete on the right if and only if any sequence which is $b_v(s)$ Cauchy on the right is convergent in Z;
c) $b_v(s)$ complete if and only if any $b_v(s)$ Cauchy sequence in Z is convergent in Z.

Definition 1.17. [15] If the following conditions are satisfied, a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is known as an altering distance function if

a) ξ is non-decreasing;
b) ξ is continuous;
c) $\xi(t) = 0$ if and only if $t = 0$.

We first define the following class of function that we use in the proof of our main result.

Definition 1.18. Assume that a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is referred to be a L-control function if

a) $\xi(t_1) \leq \xi(t_2)$ whenever $t_1 \leq t_2$ for every $t_1, t_2 \in \mathbb{R}^+$;
b) ξ is continuous;
c) $\xi(2st) \leq 2s\xi(t)$ for $s \geq 1$, for all $t > 0$;
d) $\xi(t) < t$ for all $t > 0$;
e) $\xi(t) = 0$ if and only if $t = 0$.

All over this paper, the class of all L-control function is represented by Ξ_L.

Example 1.19. Let us define a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\xi(t) = \dfrac{te^t}{1+te^t}$. Clearly ξ is non-decreasing and continuous and $e^t < 1 + te^t$ for all $t > 0$. Thus, $te^t < t + t^2e^t = t(1+te^t)$ and this implies that $\dfrac{te^t}{1+te^t} < t$. So, we have, $\xi(t) < t$. Again $te^t \leq 2ste^{2st}$ for all $t > 0$ and $s \geq 1$ implies that $1 + te^t \leq 1 + 2ste^{2st}$ and so $\dfrac{1}{1+2ste^{2st}} \leq \dfrac{1}{1+te^t}$. This means that $\dfrac{2st}{1+2ste^{2st}} \leq \dfrac{2st}{1+te^t}$ and this means that $\xi(2st) \leq 2s\xi(t)$. Hence $\xi \in \Xi_L$.

2 Main Result

Definition 2.1. Given that (Z,δ) is a quasi-$b_v(s)$-metric space. Allow $S : Z \to Z$ be a mapping and $\alpha : Z \times Z \to \mathbb{R}^+$ be a given function. Assume that for any non-negative real number γ there exists a non-negative real number σ such that for any $z, w \in Z$, $\gamma \leq M_\xi(z,w) < \gamma + \sigma$ implies that

$$\alpha(z,w)\delta(Sz,Sw) < \gamma \tag{2.1}$$

where

$$M_\xi(z,w) = \max\{\xi(\delta(z,w)), \xi(\delta(Sz,z)), \xi(\delta(Sw,w))\},$$

and $\xi \in \Xi_L$. Then S is called a ξ-generalized Meir-Keeler contraction of first kind.

Definition 2.2. Given that (Z,δ) is a quasi-$b_v(s)$-metric space. Allow $S : Z \to Z$ be a mapping and $\alpha : Z \times Z \to \mathbb{R}^+$ be a given function. Assume that for any non-negative real number γ there exists a non-negative real number σ such that for any $z, w \in Z$, $\gamma \le N_\xi(z,w) < \gamma + \sigma$ implies that

$$\alpha(z,w)\delta(Sz,Sw) < \gamma \tag{2.2}$$

where

$$N_\xi(z,w) = \max\{\xi(\delta(z,w)), \frac{1}{2s}[\xi(\delta(Sz,z)) + \xi(\delta(Sw,w))]\},$$

and $\xi \in \Xi_L$. Then S is called a ξ-generalized Meir-Keeler contraction of second kind.

Remark 2.3. If S satisfies conditions under Definition 2.1, then

$$\alpha(z,w)\delta(Sz,Sw) < M_\xi(z,w),$$

for all $z, w \in Z$ when $M_\xi(z,w) > 0$. Also, if $M_\xi(z,w) = 0$ then $z = w$, which implies $\delta(z,w) = 0$. In this case, we have

$$\alpha(z,w)\delta(Sz,Sw) \le M_\xi(z,w).$$

Remark 2.4. If S satisfies conditions under Definition 2.2, then

$$\alpha(z,w)\delta(Sz,Sw) < N_\xi(z,w),$$

for all $z, w \in Z$ when $N_\xi(z,w) > 0$.
Also, if $N_\xi(z,w) = 0$ then $z = w$, which implies $\delta(z,w) = 0$. In this case, we have

$$\alpha(z,w)\delta(Sz,Sw) \le N_\xi(z,w).$$

Remark 2.5. It is clear that

$$N_\xi(z,w) \le M_\xi(z,w)$$

for all $z, w \in Z$, where $M_\xi(z,w)$ and $N_\xi(z,w)$ are defined in (2.1) and (2.2) respectively.

Theorem 2.6. Assume that $\emptyset \ne Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S : Z \to Z$ a mapping such that the following requirements are satisfied:

a) S is ξ-generalized Meir-Keeler contraction of first kind;
b) S is triangular α-admissible;
c) S is continuous.

If there is a point $z_0 \in Z$ satisfying $\alpha(z_0, Sz_0) \ge 1$ and $\alpha(Sz_0, z_0) \ge 1$, then S has a fixed point in Z. Moreover, if $z, w \in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z,w) \ge 1$, then S has exactly one fixed point.

Proof . By hypothesis there is a point $z_0 \in S$ satisfying $\alpha(z_0, Sz_0) \ge 1$ and $\alpha(Sz_0, z_0) \ge 1$. Let's define $\{z_k\} \subseteq X$ as $z_{k+1} = Sz_k$ for all $k \in \mathbb{Z}^+$. Now we assume that there is some $k_0 \in \mathbb{Z}^+$ such that $z_{k_0} = z_{k_0+1}$, so z_{k_0} is a fixed point and we are through. As a result, we suppose that $z_k \ne z_{k+1}$, for all $k \in \mathbb{Z}^+ \cup \{0\}$. By the fact that S is α-admissible, we get $\alpha(z_0, Sz_0) = (z_0, z_1) \ge 1$. This implies that

$$\alpha(Sz_0, Sz_1) \ge 1. \tag{2.3}$$

If this procedure is continued, we obtain

$$\alpha(z_k, z_{k+1}) \ge 1, \text{ for all } k \in \mathbb{Z}^+. \tag{2.4}$$

Similarly, we consider that there is a point $z_0 \in S$ satisfying $\alpha(Sz_0, z_0) = (z_1, z_0) \ge 1$ and this implies that

$$\alpha(Sz_1, Sz_0) \ge 1. \tag{2.5}$$

If this procedure is continued, we have

$$\alpha(z_{k+1}, z_k) \geq 1 \text{ for any } k \in \mathbb{Z}^+. \tag{2.6}$$

On substituting $z = z_k$, $w = z_{k+1}$ in (2.1), for any $\gamma > 0$ there is $\sigma > 0$ satisfying $\gamma \leq M_\xi(z_k, z_{k+1}) < \gamma + \sigma$ and this implies that

$$\alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma, \tag{2.7}$$

where

$$\begin{aligned} M_\xi(z_k, z_{k+1}) &= \max\{\xi(\delta(z_k, z_{k+1})), \xi(\delta(Sz_k, z_k)), \xi(\delta(Sz_{k+1}, z_{k+1}))\} \\ &= \max\{\xi(\delta(z_k, z_{k+1})), \xi(\delta(z_{k+1}, z_k)), \xi(\delta(z_{k+2}, z_{k+1}))\}. \end{aligned} \tag{2.8}$$

In the following step we consider three cases.

Case 1. If $M_\xi(z_k, z_{k+1}) = \xi(\delta(z_k, z_{k+1}))$. Hence (2.7) becomes $\gamma \leq \xi(\delta(z_k, z_{k+1})) < \gamma + \delta$. Thus,

$$\alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma. \tag{2.9}$$

Now, from (2.9) and by property (d) of ξ, we obtain

$$\begin{aligned} \delta(z_{k+1}, z_{k+2}) = \delta(Sz_k, Sz_{k+1}) &\leq \alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma \\ &\leq \xi(\delta(z_k, z_{k+1})) < \delta(z_k, z_{k+1}). \end{aligned}$$

This implies that $\delta(z_{k+1}, z_{k+2}) \leq \delta(z_k, z_{k+1})$, for all $k \in \mathbb{Z}^+$. Hence, $\{\delta_k\} = \{\delta(z_k, z_{k+1})\}$ is a decreasing positive sequence in S and $\delta_k \to s \in \mathbb{R}^+$. We now establish that $s = 0$ by assuming the opposite, that is, $s > 0$. Then for all $k \in \mathbb{Z}^+$, we obtain

$$0 < s \leq \delta(z_k, z_{k+1}) \text{ for any } k \in \mathbb{Z}^+. \tag{2.10}$$

Now, choose $\gamma = s$ and from (2.9), it follows that $s \leq \xi(\delta(z_k, z_{k+1})) < s + \sigma$. This implies that

$$\alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < s.$$

But

$$\begin{aligned} \delta(z_{k+1}, z_{k+2}) = \delta(Sz_k, Sz_{k+1}) &\leq \alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < s \\ &\leq \xi(\delta(z_k, z_{k+1})) < \delta(z_k, z_{k+1}) \end{aligned}$$

Therefore, $\delta(z_{k+1}, z_{k+2}) < s < \delta(z_k, z_{k+1})$. This is a contradiction to (2.10), since $s > 0$ for any $k \in \mathbb{Z}^+$. As a result, we can deduce that $s = 0$. That is to say,

$$\lim_{k\to\infty} \delta(z_k, z_{k+1}) = 0.$$

Case 2. Assume that $M_\xi(z_k, z_{k+1}) = \xi(\delta(z_{k+1}, z_k))$. In this instance, (2.7) becomes $\gamma \leq \xi(\delta(z_{k+1}, z_k)) < \gamma + \sigma$. Hence,

$$\alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma. \tag{2.11}$$

Since,

$$\delta(z_{k+1}, z_{k+2}) = \delta(Sz_k, Sz_{k+1}) \leq \alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma$$

$$\leq \xi(\delta(z_{k+1}, z_k)) < \delta(z_{k+1}, z_k). \tag{2.12}$$

From (2.11) and (2.12), we have

$$\delta(z_{k+1}, z_{k+2}) < \delta(z_{k+1}, z_k) \tag{2.13}$$

for all $k \in \mathbb{Z}^+$. From Remark 2.3, it follows that

$$M_\xi(z_k, z_{k-1}) = \max\{\xi(\delta(z_k, z_{k-1})), \xi(\delta(z_{k+1}, z_k)), \xi(\delta(z_k, z_{k-1}))\} > 0.$$

Now,

$$\begin{aligned}\xi(\delta(z_{k+1}, z_k)) < \delta(z_{k+1}, z_k) = \delta(Sz_k, Sz_{k-1}) &\leq \alpha(z_k, z_{k-1})\delta(Sz_k, Sz_{k-1}) \\ &< M_\xi(z_k, z_{k-1}).\end{aligned} \tag{2.14}$$

This implies that $\delta(z_{k+1}, z_k) < M_\xi(z_k, z_{k-1})$. Hence $\xi(\delta(z_{k+1}, z_k))$ can not be the maximum for $M_\xi(z_k, z_{k-1})$. Now, assume that $M_\xi(z_k, z_{k-1}) = \xi(\delta(z_k, z_{k-1}))$. In this case, we have $\gamma \leq \xi(\delta(z_k, z_{k-1})) < \gamma + \sigma$ and so,

$$\alpha(z_k, z_{k-1})\delta(Sz_k, Sz_{k-1}) < \gamma. \tag{2.15}$$

But, from (2.15) and the property of ξ, we get

$$\begin{aligned}\delta(z_{k+1}, z_k) = \delta(Sz_k, Sz_{k-1}) &\leq \alpha(z_k, z_{k-1})\delta(Sz_k, Sz_{k-1}) < \gamma \\ &< M_\xi(z_k, z_{k-1}) = \xi(\delta(z_k, z_{k-1})) < \delta(z_k, z_{k-1}).\end{aligned}$$

This implies that $\delta(x_{k+1}, x_k) \leq \delta(x_n, x_{n-1})$ for all $k \geq 1$. Hence, $\{s_k\}$ $=\{\delta(z_{k+1}, z_k)\}$is a decreasing positive sequence in S and $s_k \to l \in \mathbb{R}^+$. We now establish that $l = 0$ by assuming the opposite, that is, $l > 0$. Then for all $k \in \mathbb{Z}^+$, we obtain

$$0 < l \leq \xi(\delta(z_{k+1}, z_k)) \text{ for all } k \in \mathbb{Z}^+. \tag{2.16}$$

Now, choose $\gamma = l$ and from (2.15), it follows that $l \leq \xi(\delta(z_k, z_{k-1})) < l + \sigma$. Thus,

$$\alpha(z_k, z_{k-1})\delta(Sz_k, Sz_{k-1}) < l. \tag{2.17}$$

Now, we obtain

$$\begin{aligned}\delta(z_{k+1}, z_k) = \delta(Sz_k, Sz_{k-1}) \leq \alpha(z_k, z_{k-1})\delta(Sz_k, Sz_{k-1}) &< l \\ &\leq \xi(\delta(z_k, z_{k-1})) < \delta(z_k, z_{k-1}).\end{aligned} \tag{2.18}$$

From (2.18), it follows that

$$\delta(z_{k+1}, z_k) < l < \delta(z_k, z_{k-1}).$$

This is a contradiction to (2.16), since $l > 0$ for any $k \in \mathbb{Z}^+$. As a result, we can deduce that $l = 0$. That is to say,

$$\lim_{k\to\infty} \delta(z_{k+1}, z_k) = 0$$

.

Case 3. Assume $M_\xi(z_k, z_{k+1}) = \xi(\delta(z_{k+2}, z_{k+1}))$. As a result, (2.7) becomes $\gamma \leq \xi(\delta(z_{k+2}, z_{k+1})) < \gamma + \sigma$, and so,

$$\alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma. \tag{2.19}$$

From (2.19), we get

$$\delta(z_{k+2}, z_{k+1})) = \delta(Sz_{k+1}, Sz_k) \leq \alpha(z_k, z_{k+1})\delta(Sz_k, Sz_{k+1}) < \gamma. \tag{2.20}$$

for every $k \in \mathbb{Z}^+$. Since $M_\xi(z_k, z_{k+1}) > 0$, by Remark 2.3, we obtain

$$\begin{aligned}\delta(z_{k+2}, z_{k+1}) = \delta(Sz_{k+1}, Sz_k) \leq \alpha(z_{k+1}, z_k))\delta(Sz_{k+1}, Sz_k) < M_\xi(z_k, z_{k+1}) \\ = \xi(\delta(z_{k+2}, z_{k+1})) < \delta(z_{k+2}, z_{k+1}),\end{aligned}$$

a contradiction. Hence, the case $M_\xi(z_k, z_{k+1}) = \xi(\delta(z_{k+1}, z_{+2}))$ is impossible. To demonstrate that the sequence $\{\delta(z_{k+1}, z_k)\}$ converges to 0, we repeat the procedure. Now, if we substitute $z = z_{k+1}$, $w = z_k$ in (2.1) for any $\gamma > 0$, there is $\sigma > 0$ such that $\gamma \leq M_\xi(z_{k+1}, z_k) < \gamma + \sigma$. This implies that

$$\alpha(z_{k+1}, z_k)\delta(Sz_{k+1}, Sz_k) < \gamma, \tag{2.21}$$

where

$$\begin{aligned}M_\xi(z_{k+1}, z_k) &= \max\{\xi(\delta(z_{k+1}, z_k)), \xi(\delta(Sz_{k+1}, z_{k+1})), \xi(\delta(Sz_k, z_k))\} \\ &= \max\{\xi(\delta(z_{k+1}, z_k)), \xi(\delta(z_{k+2}, z_{k+1})), \xi(\delta(z_{k+1}, z_k))\} \\ &= \max\{\xi(\delta(z_{k+1}, z_k)), \xi(\delta(z_{k+2}, z_{k+1}))\}.\end{aligned} \tag{2.22}$$

Here, suppose that $M_\xi(z_{k+1}, z_k) = \xi(\delta(z_{k+1}, z_k))$. Therefore (2.21), becomes $\gamma \leq \xi(\delta(z_{k+1}, z_k)) < \gamma + \sigma$, and so

$$\alpha(z_{k+1}, z_k)\delta(Sz_{k+1}, Sz_k) < \gamma. \tag{2.23}$$

We now consider

$$\begin{aligned}\delta(z_{k+2}, z_{k+1}) = \delta(Sz_{k+1}, Sz_k) \leq \alpha(z_{k+1}, z_k)\delta(Sz_{k+1}, Sz_k) < \gamma \\ \leq \xi(\delta(z_{k+1}, z_k)) < \delta(z_{k+1}, z_k).\end{aligned}$$

Hence

$$\delta(z_{k+2}, z_{k+1}) < \delta(z_{k+1}, z_k) \text{ for every } k \in \mathbb{Z}^+.$$

As a result, the sequence $\delta(z_{k+1}, z_k)$ is decreasing and converges to some non-negative real number h. It can be proven that $h = 0$ using a similar argument as previously. Again if $M_\xi(z_{k+1}, z_k) = \xi(\delta(z_{k+2}, z_{k+1}))$, then (2.21), becomes

$$\gamma \leq \xi(\delta(z_{k+2}, z_{k+1})) < \gamma + \sigma$$

and so $\alpha(z_{k+1}, z_k)\delta(Sz_{k+1}, Sz_k) < \gamma$. Now, we consider

$$\delta(z_{k+2}, z_{k+1}) = \delta(Sz_{k+1}, z_k) \leq \alpha(z_{k+1}, z_k)\delta(Sz_{k+1}, Sz_k) < \gamma.$$

$$\leq \xi(\delta(z_{k+2}, z_{k+1})) < \delta(z_{k+2}, z_{k+1}). \tag{2.24}$$

From (2.24), it follows that

$$\delta(z_{k+2}, z_{k+1}) < \delta(z_{k+2}, z_{k+1}) \text{ for every } k \in \mathbb{Z}^+.$$

This is a contradiction. Hence the only possibility is that $M_\xi(z_{k+1}, z_k) = \xi(\delta(z_{k+1}, z_k))$, in which case the sequence $\{s_k\} = \{\delta(z_{k+1}, z_k)\}$ converges to 0. The sequence (z_k) must then be shown to be both $b_v(s)$-Cauchy from the right and $b_v(s)$-Cauchy from the left. In this case, we first show that (z_k) is a $b_v(s)$-Cauchy sequence from the left in Z. We prove that for every $\gamma > 0$ and $s > 1$ there is $n_0 \in \mathbb{Z}^+$ $n_0 = n_0(\gamma)$ such that

$$\xi(\delta(z_{l+k}, z_l)) \leq \frac{\gamma}{s} < \gamma. \text{ for every } l \geq n_0 \text{ and } k \in \mathbb{Z}^+. \tag{2.25}$$

Since the sequences $\delta_k \to 0$ and $s_k \to 0$ as $k \to \infty$, for any $\sigma > 0$ there exist $N_1, N_2 \in \mathbb{Z}^+$ such that

$$\xi(\delta(z_k, z_{k+1})) \leq \frac{\sigma}{s} \text{ for every } k \geq N_1 \text{ and } \xi(\delta(z_{k+1}, z_k)) \leq \frac{\sigma}{s} \text{ for any } k \geq N_2. \tag{2.26}$$

We now choose σ in such a way that $\sigma = \gamma$. We proceed our proof by applying induction on k. Choose $n_0 = \max\{N_1, N_2\}$. For $k = 1$, (2.25) becomes

$$\xi(\delta(x_{l+1}, x_l)) \leq \frac{\gamma}{s} < \gamma \text{ for every } l \geq n_0$$

which holds due to (2.26) and the choice of σ. Now, assume (2.25) holds for $k = m$. We claim (2.25) holds for $k = m + 1$. By applying quasi-$b_v(s)$-metric inequality and the property of ξ, we have

$$\begin{aligned}\xi(\delta(z_{l+m}, z_{l-1})) &\leq \xi(s[\delta(z_{l+m}, z_l) + \delta(z_l, z_{l-1})]) \\ &\leq \xi(s[2\max\{\delta(z_{l+m}, z_l), \delta(z_l, z_{l-1})\}]) \\ &\leq 2s \max\ \{\xi(\delta(z_{l+m}, z_l)), \xi(\delta(z_l, z_{l-1}))\} \\ &< 2s \max\ \{\frac{\gamma}{s}, \frac{\epsilon}{s}\} = 2\gamma = \gamma + \sigma\end{aligned}$$

for all $l \geq n_0$. If we assume $\xi(\delta(z_{l+m}, z_{l-1})) \geq \gamma$, then we have

$$\begin{aligned}\gamma \leq \xi(\delta(z_{l+m}, z_{l-1})) &\leq M_\xi(z_{l+m}, z_{l-1}) \\ &= \max\{\xi(\delta(z_{l+m}, z_{l-1})), \xi(\delta(z_{l+m+1}, z_{l+m})), \xi(\delta(z_l, x_{l-1}))\} \\ &< \max\{\gamma + \sigma, \frac{\sigma}{s}, \frac{\sigma}{s}\} = \gamma + \sigma.\end{aligned}$$

By Remark 2.3, the contractive condition (2.1) holds for $z = z_{l+m}$ and $w = z_{l-1}$. i.e.

$$\gamma \leq M_\xi(z_{l+m}, z_{l-1}) < \gamma + \sigma \implies \delta(z_{l+m+1}, z_l) = \alpha(z_{l+m}, z_{l-1})\delta(Sz_{l+m}, Sz_{l-1}) < \gamma.$$

Again if we assume $\xi(d(z_{l+m}, z_{l-1})) < \gamma$, we have

$$\begin{aligned}M_\xi(z_{l+m}, z_{l-1}) &= \max\{\xi(\delta(z_{l+m}, z_{l-1})), \xi(\delta(Sz_{l+m}, z_{l+m})), \xi(\delta(Sz_{l-1}, z_{l-1}))\} \\ &= \max\{\xi(\delta(z_{l+m}, z_{l-1})), \xi(\delta(z_{l+m+1}, z_{l+m})), \xi(\delta(z_l, z_{l-1}))\} \\ &< \max\{\gamma, \frac{\gamma}{s}, \frac{\gamma}{s}\} = \gamma.\end{aligned}$$

By considering Remark 2.3, we can deduce that

$$\delta(z_{l+m+1}, z_l) = \alpha(z_{l+m}, z_{l-1})\delta(Sz_{l+m}, Sz_{l-1}) < M_\xi(z_{l+m}, z_{l-1}) < \gamma.$$

This implies that (2.25) holds for $k = l + m$. Hence $\xi(\delta(z_{l+k}, z_l)) \leq \gamma$ for every $l \geq n_0$ and $k \in \mathbb{Z}^+$. That is $\delta(z_n, z_m) < \gamma$ for every $k \geq m > n_0$. Thus, $\{z_k\}$ is a $b_v(s)$-Cauchy on the left. Similarly it can be proved that $\{z_k\}$ is a $b_v(s)$-Cauchy on the right. Consequently, $\{z_k\}$ is a $b_v(s)$-Cauchy sequence.

Because Z is $b_v(s)$-complete, $z \in Z$ exists such that

$$\lim_{k\to\infty} \delta(z, z_k) = 0 = \lim_{k\to\infty} \delta(z_k, z). \tag{2.27}$$

Since (Z, δ) is a quasi-$b_v(s)$-metric and S is continuous, from (2.27), we have

$$\lim_{k\to\infty} \delta(Sz, z_k) = 0 = \lim_{k\to\infty} \delta(Sz, Sz_{k-1}). \tag{2.28}$$

and

$$\lim_{k\to\infty} \delta(z_k, Sz) = 0 = \lim_{k\to\infty} \delta(Sz_{k-1}, Sz). \tag{2.29}$$

From (2.28) and (2.29), it follows that

$$\lim_{k\to\infty} \delta(Sz, z_k) = 0 = \lim_{k\to\infty} d(z_k, Sz). \tag{2.30}$$

Because of the uniqueness of limit and from (2.27) and (2.30), we get $Sz = z$. i.e. z is a fixed point of S. To show that S has exactly one fixed point, let $z, w \in$ F(S) with $z \neq w$.

$$\begin{aligned} M_\xi(z,w) &= \max\{\xi(\delta(z,w)), \xi(\delta(Sz,z)), \delta(Sw,w)\} \\ &= \max\{\delta(z,w), \delta(z,z), \delta(w,w)\} \\ &= \delta(z,w). \end{aligned}$$

By regarding Remark 2.3 and from the hypothesis $\alpha(z,w) \geq 1$, we have

$$\delta(z,w) \leq \alpha(z,w)\delta(Sz,Sw) < M_\xi(z,w) = \delta(z,w),$$

a contradiction. Hence $z = w$. □

Theorem 2.7. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S : Z \to Z$ a mapping such that the following requirements are satisfied:

a) S is ξ-generalized Meir-Keeler contraction of second kind;
b) S is triangular α-admissible;
c) S is continuous.

If there is a point $z_0 \in Z$ satisfying $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0) \geq 1$, then S has a fixed point in Z. Moreover, if $z, w \in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z,w) \geq 1$, then S has exactly one fixed point.

Proof . We use the same approach as the proof of Theorem 2.6 since $N_\xi(z,w) \leq M_\xi(z,w)$. □

The following is the effect of removing the continuity condition from Theorem 2.6.

Theorem 2.8. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S : Z \to Z$ a mapping such that the following requirements are satisfied:

a) S is a ξ generalized Meir-Keeler contraction of second type;
b) S is triangular α-admissible;
c) If $\{z_k\}$ is a sequence in Z which converges to z and satisfies $\alpha(z_{k+1}, z_k) \geq 1$ and $\alpha(z_k, z_{k+1}) \geq 1$ for all k, then there exists a sub-sequence $\{z_{k_n}\}$ of $\{z_k\}$ such that $\alpha(z, z_{k_n}) \geq 1$ and $\alpha(z_{k_n}, z) \geq 1$ for all n.

If there exists $z_0 \in Z$ such that $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0) \geq 1$, then S has a fixed point in Z. Moreover, let $z, w \in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z,w) \geq 1$, then S has exactly one fixed point.

Proof . Since $z_0 \in Z$ satisfies $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0)$, from the proof of Theorem 2.6 the sequence $\{z_k\}$ is defined by $z_{k+1} = Sz_k$ converges to some $z \in S$.
From (2.1) and condition (c) of the theorem , there exists a sequence $\{z_{k_n}\}$ of $\{z_k\}$ such that $\alpha(z, z_{k_n}) \geq 1$ and $\alpha(z_{k_n}, z) \geq 1$ for all n. By Remark 2.4, we get for any $n \in \mathbb{Z}^+$,

$$\delta(z, z_{k_{n+1}}) = \delta(Sz, Sz_{k_n}) \leq \alpha(z, z_{k_n})\delta(Sz, Sz_{k_n}) \leq N_\xi(z, x_{n_k}), \tag{2.31}$$

and

$$\delta(z_{k_{n+1}}, Sz) = \delta(Sz_{k_n}, Sz) \leq \alpha(z_{k_n}, z)\delta(Sz_{k_n}, Sz) \leq N_\xi(z_{k_n}, z), \tag{2.32}$$

where

$$N_\xi(z, z_{k_n}) = \max\{\xi(\delta(z, z_{k_n})), \frac{1}{2s}\big(\xi(\delta(Sz, z)) + \xi(\delta(Sz_{k_n}, x_{n_k}))\big)\} \tag{2.33}$$

and

$$N_\xi(z_{k_n}, z) = \max\{\xi(\delta(z_{k_n}, z)), \frac{1}{2s}\big(\xi(\delta(z, Sz)) + \xi(\delta(z_{k_n}, Sz_{k_n}))\big)\}. \tag{2.34}$$

On taking the limit in (2.33) and (2.34) as $k \to \infty$, we obtain

$$\lim_{n\to\infty} N_\xi(z, z_{k_n}) \leq \max\{\xi(0), \frac{1}{2s}\big(\xi(0) + \xi(\delta(Sz, z))\big)\} \leq \frac{1}{2s}\big(\xi(\delta(Sz, z)) < \frac{1}{2s}\delta(Sz, z) \tag{2.35}$$

and

$$\lim_{n\to\infty} N_\xi(x_{k_n}, z) \leq \max\{\xi(0), \frac{1}{2s}\big(\xi(\delta(Sz, z)) + \xi(0)\big)\} \leq \frac{1}{2s}\big(\xi(\delta(Sz, z)) < \frac{1}{2s}\delta(Sz, z). \tag{2.36}$$

On the other hand, taking the limit as $n \to \infty$ in (2.31), and (2.32) also by considering (2.35) and (2.36), we get

$$0 \leq \delta(Sz, z) \leq \frac{\delta(Sz, z)}{2s}. \tag{2.37}$$

$$0 \leq \delta(z, Sz) \leq \frac{\delta(z, Sz)}{2s}. \tag{2.38}$$

From (2.37), it follows that $d(Sz, z) = 0$. i.e., $Sz = z$. To show that S has exactly one fixed point, let $z, w \in$ F(S) with $z \neq w$.

$$\begin{aligned} N_\xi(z, w) &= \max\{\xi(\delta(z, w)), \frac{1}{2s}[\xi(\delta(Sz, z)) + \xi(\delta(Sw, w))]\} \\ &= \max\{\xi(\delta(z, w)), \frac{1}{2s}[\xi(\delta(z, z)) + \xi(\delta(w, w))]\} \\ &= \xi(\delta(z, w)) < \delta(z, w). \end{aligned}$$

By regarding Remark 2.4 and from the hypothesis $\alpha(z, w) \geq 1$, we have

$$\delta(z, w) = \delta(Sz, Sw) \leq \alpha(z, w)\delta(Sz, Sw) < N_\xi(z, w) < \delta(z, w),$$

a contradiction. Hence $z = w$. This completes the proof. □

Corollary 2.9. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S : Z \to Z$ be a continuous triangular α-admissible mapping. Assume that for every $\gamma > 0$ there exists $\sigma > 0$ satisfying $\gamma \leq L_\xi(z, w) < \gamma + \sigma$, then

$$\alpha(z, w)\delta(Sz, Sw) < \sigma, \tag{2.39}$$

where

$$L_\xi(z, w) = \max\{\delta(z, w), \frac{[1 + \delta(z, w)]\delta(Sz, z)}{1 + \delta(Sz, z)}, \frac{[1 + \delta(z, w)]\delta(Sw, w)}{1 + \delta(Sw, w)}\}$$

for all $z, w \in Z$. If there is a point $z_0 \in Z$ satisfying $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0) \geq 1$, then S has a fixed point in Z. Moreover, if $z, w \in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z, w) \geq 1$, then S has exactly one fixed point.

Proof . By defining $\xi(t) = \dfrac{t}{1+t} + \dfrac{t}{1+t}\delta(z, w)$ in $M_\xi(z, w)$, the proof follows from Theorem 2.6. □

Corollary 2.10. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S : Z \to Z$ be a continuous triangular α-admissible mapping. Assume that for every $\gamma > 0$ there exists $\sigma > 0$ satisfying $\gamma \leq \xi(\delta(z, w)) < \gamma + \sigma$, then

$$\alpha(z, w)\delta(Sz, Sw) < \gamma \tag{2.40}$$

for all $z, w \in Z$. If there is a point $z_0 \in Z$ satisfying $\alpha(z_0, Sz_0) \geq 1$ and $\alpha(Sz_0, z_0) \geq 1$, then S has a fixed point in Z. Moreover, if $z, w \in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z, w) \geq 1$, then S has exactly one fixed point.

Proof . This is a special case of Theorem 2.6. That is, if $M_\xi(z,w)=\xi(\delta(z,w))$ in Theorem 2.6, we have the result, and therefore the proof follows. □

Corollary 2.11. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S:Z\to Z$ be a continuous triangular α-admissible mapping. Assume that for every $\gamma>0$ there exists $\sigma>0$ satisfying

$$\gamma \le \delta(z,w) < \gamma+\sigma \implies \alpha(z,w)\delta(Sz,Sw) < \gamma \tag{2.41}$$

for all $z,w\in Z$. If there is a point $z_0\in Z$ satisfying $\alpha(z_0,Sz_0)\ge 1$ and $\alpha(Sz_0,z_0)\ge 1$, then S has a fixed point in Z. Moreover, if $z,w\in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z,w)\ge 1$, then S has exactly one fixed point.

Proof . This corollary's evidence is inferred from Corollary 2.9's proof. □

Corollary 2.12. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S:Z\to Z$ be a continuous mapping. Assume that for every $\gamma>0$ there exists $\sigma>0$ satisfying

$$\gamma \le \delta(z,w) < \gamma+\sigma \implies \delta(Sz,Sw) < \gamma \tag{2.42}$$

for all $z,w\in Z$. Then S has exactly one fixed point in Z.

Proof . The proof follows from Corollary 2.11 with $\alpha(z,w)=1$. □

Remark 2.13. Corollary 2.12 is the Meir-Keeler result [16] in quasi-$b_v(s)$-metric space.

We get the following result if we remove the function $S's$ continuation condition from Corollary 2.11.

Corollary 2.14. Assume that $\emptyset \neq Z$ and there exists a complete quasi-$b_v(s)$-metric on Z. Suppose that $S:Z\to Z$ a triangular α-admissible mapping. Suppose (Z,d) satisfies the following condition H: If $\{z_k\}$ is a sequence in Z which converges to z and satisfies $\alpha((z_{k+1},z_k)\ge 1$ and $\alpha(z_k,z_{k+1})\ge 1$ for all k then there exists a sub-sequence $\{z_{k_n}\}$ of $\{z_k\}$ such that $\alpha(z,z_{k_n})\ge 1$ and $\alpha(z_{k_n},z)\ge 1$ for all n. Assume that for every $\gamma>0$ there is $\sigma>0$ satisfying

$$\gamma \le \delta(z,w) < \gamma+\sigma \implies \alpha(z,w)\delta(Sz,Sw) < \gamma \tag{2.43}$$

for all $z,w\in Z$. If there is a point $z_0\in Z$ satisfying $\alpha(z_0,Sz_0)\ge 1$ and $\alpha(Sz_0,z_0)\ge 1$, then S has a fixed point in Z. Moreover, if $z,w\in F(S)$, where $F(S)$ is the set of all fixed points of S, such that $\alpha(z,w)\ge 1$, then S has exactly one fixed point.

Proof . The evidence flows from Corollary 2.11's proof. □

The following example demonstrates Theorem 2.6.

Example 2.15. Given that $Z=[0,\ \frac{3}{2}]$. Let $\alpha,\delta: Z\times Z\to\mathbb{R}^+$ be given functions such that

$$\delta(z,w)=\begin{cases} 0 & \text{if } z=w,\\ |z-w|^4+z & \text{if z not equals to w .}\end{cases} \quad \text{and}$$

$$\alpha(z,w)=\begin{cases} 1 & \text{provided that } z,w\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

$S:Z\to Z$ and $\xi:\mathbb{R}^+\to\mathbb{R}^+$ are defined as

$$S(z)=\begin{cases} \frac{1}{2}z & \text{if } z\in[0,1],\\ \\ 2z-\frac{3}{2} & \text{if } z\in[1,\frac{3}{2}],\end{cases} \qquad \xi(t)=\frac{t}{1+t}.$$

It is clear that $\delta(z,w)\neq\delta(w,z)$ for $z\neq w$. As a result, δ is a quasi-$b_1(s)$-metric space with a parameter $s=4$. S is clearly non-decreasing and continuous on Z. To demonstrate that S is a triangular α-admissible, if $\alpha(z,w)\ge 1$ is true, then $z,w\in[0,1]$. As a result $Sz,Sw\in[0,1]$ and thus, $\alpha(Sz,Sw)=1\ge 1$. If $\alpha(z,u)\ge 1$ and $\alpha(u,w)\ge 1$, then $z,u,w\in[0,1]$ and hence, $\alpha(z,w)\ge 1$. We then show that S is a ξ-generalized Meir-Keeler contraction of first kind. To that purpose, we will look at the following scenarios. Suppose that $z, w \in [0,1]$ with $z \neq w$. Without loss of generality assume that $z > w$. In this case, for every $\gamma > 0$, choose $\sigma = \gamma$ satisfying

$$\gamma \leq M_\xi(z,w) < \gamma + \sigma = 2\gamma, \tag{2.44}$$

where

$$\begin{aligned} M_\xi(z,w) &= \max\{\xi(\delta(z,w)), \xi(\delta(Sz,z)), \xi(\delta(Sw,w))\} \\ &= \max\{\xi(\delta(z,w)), \xi(\delta(\frac{1}{2}z,z)), \xi(\delta(\frac{1}{2}w,w))\} \\ &= \max\{\frac{|z-w|^4+z}{1+|z-w|^4+z}, \frac{|\frac{1}{2}z-z|^4+\frac{1}{2}z}{1+|\frac{1}{2}z-z|^4+\frac{1}{2}z}, \frac{|\frac{1}{2}w-w|^4+\frac{1}{2}y}{1+|\frac{1}{2}w-w|^4+\frac{1}{2}w}\} \\ &\leq \max\{|z-w|^4+z, (\frac{1}{2}z)^4+\frac{1}{2}z, (\frac{1}{2}w)^4+\frac{1}{2}w\} \\ &= \max\{|z-w|^4+z, (\frac{1}{2}z)^4+\frac{1}{2}z\}. \end{aligned}$$

Case i: $M_\xi(z,w) \leq \left(\frac{1}{2}z\right)^4 + z$, if $\frac{1}{2}z = w$.

Hence (2.44) becomes

$$\gamma \leq M_\xi(z,w) \leq \left(\frac{1}{2}z\right)^4 + z < \gamma + \sigma = 2\gamma. \tag{2.45}$$

Now,

$$\begin{aligned} \delta(Sz,Sw) \leq \alpha(z,w)\delta(Sz,Sw) &= \delta(\frac{1}{2}z, \frac{1}{2}w) \\ &= \delta(\frac{1}{2}z, \frac{1}{4}z) = \left(\frac{1}{4}z\right)^4 + \frac{1}{2}z < \frac{1}{2}[\left(\frac{1}{2}z\right)^4 + z] < \frac{1}{2}(2\gamma) = \gamma. \end{aligned}$$

Case ii: $M_\xi(z,w) \leq |z-w|^4 + z$, if $0 \leq \dfrac{1}{2}z < w < z \leq 1$. and hence from (2.44) it follows that

$$\gamma \leq M_\xi(z,w) \leq |z-w|^4 + z < \gamma + \delta = 2\gamma. \tag{2.46}$$

Thus,

$$\begin{aligned} \delta(Sz,Sw) \leq \alpha(z,w)\delta(Sz,Sw) &= \delta(\frac{1}{2}z, \frac{1}{2}w) \\ &= \frac{1}{16}|z-w|^4 + \frac{1}{2}z < \frac{1}{2}[|z-w|^4 + z] \\ &< \frac{1}{2}(2\gamma) = \gamma. \end{aligned}$$

Case iii: $M_\xi(z,w) \leq (\frac{1}{2}z)^4 + \frac{1}{2}z$, if $0 \leq w < \frac{1}{2}z < z \leq 1$ and hence from (2.44) it follows that

$$\epsilon \leq M_\xi(z,w) \leq (\frac{1}{2}z)^4 + \frac{1}{2}z < z^4 + \frac{1}{2}z < \gamma + \delta = 2\gamma. \tag{2.47}$$

Thus, by Remark 2.3, we have that

$$\begin{aligned} \delta(Sz,Sw) \leq \alpha(z,w)\delta(Sz,Sw) &= \delta(\frac{1}{2}z, \frac{1}{2}w) \\ &= \frac{1}{16}|z-w|^4 + \frac{1}{2}z < \frac{1}{16}z^4 + \frac{1}{2}z \\ &\leq M_\xi(z,w). \end{aligned}$$

Hence S is a ξ-generalized Meir-Keeler contraction of first kind. Also we have $z = 0$ such that $\alpha(0, S0) = 1 \geq 1$ and $\alpha(S0, 0) = 1 \geq 1$. Since all the hypotheses of Theorem 2.6 are satisfied, then S has fixed point. The fixed points are $z = 0$ and $z = \frac{3}{2}$.

Since quasi $b_v(s)$-metric space(an ambient space) is a generalization of metric and quasi metric spaces, the results presented in this paper generalize/extend the results on [3], [16] and [18]. In addition, if $\xi(t) = t$, then M_ξ of Definition 2.1 of this paper is equals to M of Definition 11 in [3]. Also if $\xi(t) = t$ and $M_\xi(z, w) = \delta(z, w)$ in Theorem 2.6, then we get the Theorem in [16]. Hence Theorem 2.6 is more general and broader than the existing results on [3], [16] and [18].

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