Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1385–1398
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27475.3642

Controlling chaos in a discretized prey-predator system

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(Communicated by Haydar Akca)

Abstract

In this work, we have considered a discrete-time discrete prey-predator. The discrete-time model is formulated in terms of difference equations and is obtained by applying a nonstandard finite difference scheme of Mickens type. We have discussed the existence and the local dynamics of the fixed points. Analytically, we demonstrated that the system undergoes a Neimark-Sacker bifurcation. Under a parametric condition, all chaotic features are justified numerically. Finally, we use two chaos control strategies to control the Neimark-Sacker bifurcation.

Keywords: Discrete model, Asymptotic stability, Bifurcation analysis, Chaos control
2020 MSC: 65Q10, 34H10, 34G20, 39A28, 39A30, 39A33, 92D25

1 Introduction

Since Lotka and Volterra's [1, 2] formulation of the mathematical model of predator-prey interactions, many researchers have devoted themselves to modeling and investigating the relationship between predators and prey. In comparison to the continuous dynamical system represented by differential equations, discrete dynamical systems may provide a more efficient computational model for numerical simulations and more complex dynamics. Furthermore, these models seem to be more realistic than continuous ones when populations have non-overlapping generations [3, 4, 5]. Therefore, in recent time, more attention has been given to a discrete dynamical system. Din [6] provided a discrete-time prey-predator model through Euler method. Neimark-Sacker bifurcation and periodic doubling bifurcation are studied and three chaos control methods are implemented to avoid chaotic orbits. In [7], a discrete-time system derived from the continuous-time Rosenzweig–MacArthur model using the forward Euler scheme with unit integral step size is investigated. The derived discrete model reveals multi-stability and rich dynamics compared to its related continuous model. Ackleh et al.[8] developed a continuous-time model for Alzheimer's disease with two corresponding discrete-time approximations. They show numerically that the continuous-time model produces sigmoidal growth, while the discrete-time approximations may exhibit oscillatory dynamics. The reader interested in other discrete time models is addressed to, among many others, [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].
Motivated by the previously cited works, we discretize the following prey-predator model, subject to the effect of predator population harvesting:

$$x' = ax(1-x) - cx\,y, \quad y' = bx\,y - ey - Hy. \tag{1.1}$$

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Received: June 2022 *Accepted:* October 2022

In the model (1.1), the prey grows logistically, where a is the growth rate in the absence of the predator. On the contrary, the predator decays exponentially in the absence of the prey, where $e > 0$ stands for the decay rate of the predator. The nonnegative coefficients c and b stand for the consumption rate and consumption-energy rate, respectively. The coefficient H corresponds to the harvesting effect. The initial conditions are $x_0 > 0$ and $y_0 > 0$.
Different strategies for discretizing differential equation systems have been mentioned above. To do this, standard difference schemes (e.g., Euler approximations, Runge-Kutta techniques) are often utilized. In [23], the forward Euler technique is used to conduct a discrete version of the system (1.1). In this paper, we apply a nonstandard difference scheme proposed by Liu and Elaydi in [24].The discretization is based on modified Mickens discretization scheme [25]. The Mickens framework aims at constructing finite-difference schemes that are dynamically consistent with the underlying continuous models.
The system's discretization (1.1) yields

$$\frac{x_{t+1} - x_t}{\phi_1(h)} = ax_t - ax_t x_{t+1} - c\, x_t y_t, \tag{1.2}$$

$$\frac{y_{t+1} - y_t}{\phi_2(h)} = bx_t y_t - ey_t - Hy_t, \tag{1.3}$$

where

$$\phi_1(h) = \frac{\exp(ah) - 1}{a}, \phi_2(h) = \frac{1 - \exp(-(e+H)h)}{e+H}.$$

Thus, from system (1.2)-(1.3) one obtains the following nonlinear discrete model:

$$x_{t+1} = \frac{\exp(ah)x_t - c\phi_1(h)x_t y_t}{1 + (\exp(ah) - 1)x_t}, \tag{1.4}$$

$$y_{t+1} = (\exp(-(e+H)h) + b\phi_2(h)x_t)y_t. \tag{1.5}$$

The goal of this research is to find the system's fixed points (1.4)-(1.5) and analyze their asymptotic stability. We use bifurcation theory along with numerical simulation to show the existence of Neimark–Sacker bifurcation in the system. Furthermore, we employ two methods to completely eliminate or delay the chaotic behaviors.
To sum up, the paper is organized as follows: In Section 2, the existence and the asymptotic stability of the fixed points are investigated. The existence of the Neimark-Sacker bifurcation is proved analytically in Section 3. The chaos control system is developed in Section 4. Detailed numerical simulations and computation analysis are developed to support the analytical findings in Section 5. Finally, Section 6 draws the conclusion to this paper.

2 Local dynamics

The system (1.4)-(1.5) has the following fixed points, the trivial fixed prey-predator point $O = (0, 0)$, the boundary fixed point A=(1,0) and the coexistence fixed point $C = (x^*, y^*)$ where

$$x^* = \frac{e+H}{b},\ y^* = \frac{a(b - (e+H))}{bc}.$$

For all parametric values, the fixed points O and A exist, and the coexistence fixed point C exists if $b > (e + H)$.
The Jacobian matrix $J_{(x,y)}$ of (1.4)-(1.5) evaluated at any fixed point (x, y) is given by

$$J_{(x,y)} = \begin{pmatrix} \dfrac{\exp(ah) - c\phi_1(h)y}{(1 + (\exp(ah) - 1)x)^2} & \dfrac{-c\phi_1(h)x}{1 + (\exp(ah) - 1)x} \\ & \\ b\phi_2(h)y & \exp(-(e+H)h) + b\phi_2(h)x \end{pmatrix}.$$

The characteristic equation of J is

$$\rho^2 - tr J(x, y)\rho + \det J(x, y) = 0, \tag{2.1}$$

where

$$T := tr J(x, y) = \frac{\exp(ah) - c\phi_1(h)y}{(1 + (\exp(ah) - 1)x)^2} + \exp(-(e+H)h) + b\phi_2(h)x,$$

and

$$D := \det J(x,y) = \frac{\exp(ah) - c\phi_1(h)y}{(1+(\exp(ah)-1)x)^2}[\exp(-(e+H)h) + b\phi_2(h)x] + \frac{-cb\phi_1(h)\,\phi_2(h)\,x\,y}{1+(\exp(ah)-1)x}.$$

The following lemma describes the various conditions associated to local stability analysis of feasible fixed points

Lemma 2.1. [26] Let $\psi(\rho) = \rho^2 - T\rho + D$. Suppose that $\psi(1) > 0$, ρ_1, ρ_2 are two roots of $\psi(\rho) = 0$. Then

- $\mid \rho_1 \mid < 1$ and $\mid \rho_2 \mid < 1$ if and only if $\psi(-1) > 0$ and $D < 1$;
- ($\mid \rho_1 \mid > 1$ and $\mid \rho_2 \mid < 1$) or ($\mid \rho_1 \mid < 1$ and $\mid \rho_2 \mid > 1$) if and only if $\psi(-1) < 0$;
- $\mid \rho_1 \mid > 1$ and $\mid \rho_2 \mid > 1$ if and only if $\psi(-1) > 0$ and $D > 1$;
- $\rho_1 = -1$ and $\mid \rho_2 \mid \neq 1$ if and only if $\psi(-1) = 0$ and $D \neq 1$;
- ρ_1 and ρ_2 are complex and $\mid \rho_1 \mid = 1$ and $\mid \rho_2 \mid = 1$ if and only if $T^2 - 4D < 0$ and $D = 1$.

Let ρ_1 and ρ_2 be two roots of (2.1), which called eigenvalues of the fixed point (x,y). The following typological classifications are considered:

1. (x,y) is locally asymptotic stable if $\mid \rho_1 \mid < 1$ and $\mid \rho_2 \mid < 1$.
2. (x,y) is called a source if $\mid \rho_1 \mid > 1$ and $\mid \rho_2 \mid > 1$. A source is locally unstable.
3. (x,y) is called a saddle if $\mid \rho_1 \mid < 1$ and $\mid \rho_2 \mid > 1$ or ($\mid \rho_1 \mid > 1$ and $\mid \rho_2 \mid < 1$).
4. (x,y) is called non-hyperbolic if either $\mid \rho_1 \mid = 1$ or $\mid \rho_2 \mid = 1$.

• For the fixed point $O(0,0)$ we have

$$J_{(O)} = \begin{pmatrix} \exp(ah) & 0 \\ 0 & \exp(-(e+H)h) \end{pmatrix}.$$

The eigenvalues of $J_{(O)}$ are

$$\lambda_1 = \exp(ah) \quad \text{and} \quad \lambda_2 = \exp(-(e+H)h). \tag{2.2}$$

From (2.2), it is easy to see that $O(0,0)$ is saddle because $\lambda_1 > 1$ and $\lambda_2 < 1$.
• For the fixed point $A(1,0)$ we have

$$J_{(A)} = \begin{pmatrix} \exp(ah) & 0 \\ 0 & \exp(-(e+H)h) + \frac{b(1-\exp(-(e+H)h))}{e+H} \end{pmatrix}.$$

The eigenvalues of $J_{(A)}$ are

$$\lambda_1 = \exp(-ah) \quad \text{and} \quad \lambda_2 = \exp(-(e+H)h) + \frac{b(1-\exp(-(e+H)h))}{e+H}. \tag{2.3}$$

From (2.3), if $b < e+H$. Then $A(1,0)$ is locally asymptotically stable. Otherwise $A(1,0)$ is saddle.

• The Jacobian matrix about the coexistence fixed point C is given by:

$$J_C = \begin{pmatrix} \dfrac{1}{1+(\exp(ah)-1)\frac{e+H}{b}} & -\dfrac{c(\exp(ah)-1)(e+H)}{a(b+(\exp(ah)-1))(e+H)} \\ \dfrac{a(b-(e+H))(1-\exp(-(e+H)h))}{c(e+H)} & 1 \end{pmatrix}. \tag{2.4}$$

The characteristic equation of Jacobian matrix (2.4) can be written as

$$\omega^2 - tr\Big(J_C\Big)\omega + \det\Big(J_C\Big) = 0, \tag{2.5}$$

where

$$tr\Big(J_C\Big) = 1 + \frac{1}{1 + (\exp(ah) - 1)\frac{e+H}{b}}, \tag{2.6}$$

and

$$\det\Big(J_C\Big) = \frac{1}{1 + (\exp(ah) - 1)\frac{e+H}{b}} + \frac{(\exp(ah) - 1)(b - (e + H))(1 - \exp(-(e + H)h))}{(e + H)(b + (\exp(ah) - 1))}. \tag{2.7}$$

The discriminant of (2.5) is

$$\Delta = tr\Big(J_C\Big)^2 - 4\det\Big(J_C\Big). \tag{2.8}$$

Lemma 2.2.
- The coexistence fixed point C is locally asymptotically stable if

$$e + H < b < (e + H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right).$$

- The coexistence fixed point C is source if

$$b > \max\left\{e + H, (e + H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right)\right\}.$$

- The coexistence fixed point C is non-hyperbolic if

$$b = (e + H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right).$$

Proof. The characteristic equation of the fixed point can be represented as

$$F(\omega) = \omega^2 - tr\Big(J_C\Big)\omega + \det\Big(J_C\Big) = 0. \tag{2.9}$$

By using Jury criteria [26], clearly $F(1) > 0$ implies

$$\frac{(\exp(ah) - 1)(b - (e + H))(1 - \exp(-(e + H)h))}{(e + H)(b + (\exp(ah) - 1))} > 0,$$

Moreover, we have

$$F(-1) = 2 + \frac{2}{1 + (\exp(ah) - 1)\frac{e+H}{b}} + \frac{(\exp(ah) - 1)(b - (e + H))(1 - \exp(-(e + H)h))}{(e + H)(b + (\exp(ah) - 1))} > 0.$$

Now

$$\det\Big(J_C\Big) < 1$$

implies that

$$\frac{(\exp(ah) - 1)(b - (e + H))(1 - \exp(-(e + H)h))}{(e + H)(b + (\exp(ah) - 1))} < 1. \tag{2.10}$$

Simplifying the inequality (2.10), one gets

$$b < (e + H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right).$$

If the discriminant Δ defined in (2.8) is negative and $b = (e+H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right)$ holds, then the Jacobian matrix (2.4) has two complex conjugate eigenvalues with modulus 1. Thus, this conditions can be written as

$$N_B = \left\{(a, c, e, h, H) > 0,\ \Delta < 0,\ b = (e + H)\left(1 + \frac{1}{1 - \exp(-(e + H)h)}\right)\right\}. \tag{2.11}$$

3 Neimark-Sacker bifurcation

The roots of the characteristic equation (2.9) at $C(x^*, y^*)$ are a pair of complex conjugate numbers ω_1, ω_2 given by

$$\omega_{1,2} = \frac{tr\Big(J_C\Big) \pm i\sqrt{4\det\Big(J_C\Big) - \Big(tr\Big(J_C\Big)\Big)^2}}{2}, \tag{3.1}$$

with $tr J_C$ and $\det J_C$ are given in (2.6) and (2.7) respectively. Let Neimark-Sacker bifurcation occur in the neighborhood of $b = \overline{b}$, we construct then the following set

$$\mathcal{N}_B = \left\{(a, c, e, h, H) > 0,\ \Delta\ < 0,\ \ b = \overline{b}\right\}. \tag{3.2}$$

Setting the values of all parameters in (3.2), and if we vary b in a small neighborhood of $b = \overline{b}$, then the coexistence fixed point C will experiences Neimark-Sacker bifurcation.
Taking a small perturbation b^* where ($b^* \ll 1$) of the parameter b in the neighborhood of $b = \overline{b}$ in the system (1.4)-(1.5), we obtain

$$x_{t+1} = \frac{\exp(ah)x_t - c\phi_1(h)x_t y_t}{1 + (\exp(ah) - 1)x_t} = f(x_t, u_t, b^*), \tag{3.3a}$$

$$y_{t+1} = \Bigg(\exp(-(e + H)h) + (\overline{b} + b^*)\phi_2(h)x_t\Bigg)y_t = g(x_t, u_t, b^*). \tag{3.3b}$$

Let $v_t = x_t - x^*$, $w_t = y_t - y^*$. Then from (3.3) we set

$$v_{t+1} = \frac{\exp(ah)(v_t + x^*) - c\,\phi_1(h)(v_t + x^*)(w_t + y^*)}{1 + (\exp(ah) - 1)(v_t + x^*)} - x^*, \tag{3.4a}$$

$$w_{t+1} = \Bigg(\exp(-(e + H)h) + (\overline{b} + b^*)\phi_2(h)(v_t + x^*)\Bigg)(w_t + y^*) - y^*. \tag{3.4b}$$

Expanding (3.4) in Taylor series at $(v_t, w_t) = (0, 0)$ up to second order, we obtain

$$v_{t+1} = \alpha_1 v_t + \alpha_2 w_t + \alpha_{12} v_t w_t + \alpha_{11} v_t^2 + O\Big((\mid v_t \mid + \mid w_t \mid)^2\Big), \tag{3.5a}$$

$$w_{t+1} = \beta_1 v_t + \beta_2 w_t + \beta_{12} v_t w_t + O\Big((\mid v_t \mid + \mid w_t \mid)^2\Big), \tag{3.5b}$$

where

$$\alpha_1 = f_x(x^*, y^*, 0) = \frac{1}{1 + (\exp(ah) - 1)x^*},\ \ \alpha_2 = f_y(x^*, y^*, 0) = -\frac{c\phi_1(h)x^*}{1 + (\exp(ah) - 1)x^*},$$

$$\alpha_{12} = f_{xy}(x^*, y^*, 0) = -\frac{c\phi_1(h)}{(1 + (\exp(ah) - 1)x^*)^2},\ \alpha_{11} = f_{xx}(x^*, y^*, 0) = \frac{(\exp(ah) - 1)(\exp(ah) - c\phi_1(h)y^*)}{(1 + (\exp(ah) - 1)x^*)^2}$$

$$\left(\frac{x^*(\exp(ah) - 1)}{1 + (\exp(ah) - 1)x^*} - 1\right),\ \ \beta_1 = g_x(x^*, y^*, 0) = b\phi_2(h)y^*,\ \ \beta_2 = g_y(x^*, y^*, 0) = 1,\ \ \beta_{12} = g_{xy}(x^*, y^*, 0) = b\phi_2(h).$$

The linearized system (3.5) at $(v(t), w(t)) = (0, 0)$ admits the characteristic equation with the roots

$$\omega_{1,2}(b^*) = \frac{trJ(b^*) \pm i\sqrt{4\det J(b^*) - (tr(b^*))^2}}{2}, \tag{3.6}$$

and

$$|\omega_{1,2}(b^*)| = \sqrt{\det J(b^*)}.$$

Moreover, when b^* tends to zero, yields

$$\det(J(0)) = 1\ \ and\ \ \frac{d|\omega_{1,2}|}{db^*}\ |_{b^*=0} \neq 0. \tag{3.7}$$

Additionally, we required that when $b^* = 0$, $\omega_{1,2}^m \neq 1$, $m = 1, 2, 3, 4$. This is equivalent to $trJ(0) \neq -2, -1, 1, 2$.

Let $\eta = Re(\omega_{1,2})$, and $\xi = Im(\omega_{1,2})$. The model (3.5) is written as

$$\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} \alpha_{12} v_t w_t + \alpha_{11} v_t^2 \\ \beta_{12} v_t w_t \end{pmatrix}. \tag{3.8}$$

Let us consider the invertible matrix P associated to the eigenvalue $\lambda_{1,2} = \eta \pm i\xi$

$$P = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \begin{pmatrix} \alpha_2 & 0 \\ \eta - \alpha_1 & -\xi \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix}.$$

The system (3.8) can be written as

$$P \begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} P \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} \Big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\Big) X_t^2 - \Big(\xi\alpha_{12}\alpha_2\Big) X_t Y_t \\ \Big(\beta_{12}\alpha_2(\eta - \alpha_1)\Big) X_t^2 - \Big(\xi\beta_{12}\alpha_2\Big) X_t Y_t \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} \eta & -\xi \\ -\xi & \eta \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + P^{-1} \begin{pmatrix} \Big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\Big) X_t^2 - \Big(\xi\alpha_{12}\alpha_2\Big) X_t Y_t \\ \Big(\beta_{12}\alpha_2(\eta - \alpha_1)\Big) X_t^2 - \Big(\xi\beta_{12}\alpha_2\Big) X_t Y_t \end{pmatrix},$$

where

$$P^{-1} = \begin{pmatrix} \frac{1}{\alpha_2} & 0 \\ \frac{\eta - \alpha_1}{\xi\alpha_2} & \frac{1}{\xi} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} \eta & -\xi \\ -\xi & \eta \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} F(X_t, Y_t) \\ G(X_t, Y_t) \end{pmatrix}, \tag{3.9}$$

where

$$F(X_t, Y_t) = \frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\Big) X_t^2 - \frac{1}{\alpha_2}(\xi\alpha_{12}\alpha_2) X_t Y_t,$$

and

$$G(X_t, Y_t) = \Big((\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2)\frac{\eta - \alpha_1}{\xi\alpha_2} + \frac{\beta_{12}\alpha_2(\eta - \alpha_1)}{\xi}\Big) X_t^2 - \Big((\xi\alpha_{12}\alpha_2)\frac{\eta - \alpha_1}{\xi\alpha_2} + \frac{\xi\beta_{12}\alpha_2}{\xi}\Big) X_t Y_t.$$

In order for (5.5) to undergo a Neimark-Sacker bifurcation, it is required that the following Lyapunov coefficient is nonzero [27]:

$$\chi = -\Re\Big[\frac{(1 - 2\overline{\omega})\overline{\omega}^2}{1 - \omega}\tau_{11}\tau_{20}\Big] - \frac{1}{2} \mid \tau_{11} \mid^2 - \mid \tau_{02} \mid^2 + \Re(\overline{\omega}\tau_{21}), \tag{3.10}$$

$$\begin{aligned}
\tau_{02} &= \frac{1}{8}[F_{X_t X_t} - F_{Y_t Y_t} - 2G_{X_t Y_t} + i(G_{X_t X_t} - G_{Y_t Y_t} + 2F_{X_t Y_t})]_{(0,0)}, \\
&= \frac{1}{4}\Big[\frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\Big) + \Big((\xi\alpha_{12}\alpha_2)\frac{\eta - \alpha_1}{\xi\alpha_2} + \frac{\xi\beta_{12}\alpha_2}{\xi}\Big) + \\
&\quad i\Big(\Big((\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2)\frac{\eta - \alpha_1}{\xi\alpha_2} + \frac{\beta_{12}\alpha_2(\eta - \alpha_1)}{\xi}\Big) - \frac{1}{\alpha_2}(\xi\alpha_{12}\alpha_2)\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
\tau_{11} &= \frac{1}{4}[F_{X_t X(t)} + F_{Y_t Y_t} + i(G_{X_t X_t} + G_{Y_t Y_t})]_{(0,0)}, \\
&= \frac{1}{4}\Big[\frac{1}{\alpha_2}\Big(\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2\Big) + i\Big(((\alpha_{12}\alpha_2(\eta - \alpha_1) + \alpha_{11}\alpha_2^2)\frac{\eta - \alpha_1}{\xi\alpha_2} + \frac{\beta_{12}\alpha_2(\eta - \alpha_1)}{\xi}\Big)\Big],
\end{aligned}$$

$$\tau_{20} = \frac{1}{8}[F_{X_tX_t} - F_{Y_tY_t} + 2G_{X_tY_t} + i(G_{X_tX_t} - G_{Y_tY_t} - 2F_{X_tY_t})]_{(0,0)},$$
$$= \frac{1}{4}\Bigg[\Bigg(\frac{1}{\alpha_2}\Bigg(\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2\Bigg) - \Bigg((\xi\alpha_{12}\alpha_2)\frac{\eta-\alpha_1}{\xi\alpha_2} + \frac{\xi\beta_{12}\alpha_2}{\xi}\Bigg) +$$
$$i\Bigg(\Bigg((\alpha_{12}\alpha_2(\eta-\alpha_1) + \alpha_{11}\alpha_2^2)\frac{\eta-\alpha_1}{\xi\alpha_2} + \frac{\beta_{12}\alpha_2(\eta-\alpha_1)}{\xi}\Bigg) + \frac{1}{\alpha_2}(\xi\alpha_{12}\alpha_2)\Bigg)\Bigg],$$

and

$$\tau_{21} = \tfrac{1}{16}[F_{X_tX_tX_t} + F_{X_tY_tY_t} + G_{X_tX_tY_t} + G_{Y_tY_tY_t} + i(G_{X_tX_tX_t} + G_{X_tY_tY_t} - F_{X_tX_tY_t} - F_{Y_tY_tY_t})]_{(0,0)} = 0.$$

Based on the above analysis, we state the following result on Neimark-Sacker bifurcation.

Theorem 3.1. ([27]). If the condition (3.7) holds and the Lyapunov coefficient χ defined in (3.10) is nonzero. Then, the system (1.4)-(1.5) experiences a Neimark-Sacker bifurcation at its coexistence fixed point $C(x^*, y^*)$, whenever b deviates in the neighborhood of $\bar{b} = (e+H)\left(1 + \frac{1}{1-\exp(-(e+H)h)}\right)$. Moreover, if $\chi < 0$ ($\chi > 0$) then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point $C(x^*, y^*)$ for $b > b^*$ (respectively, $b < b^*$).

4 Chaos Control

Controlling chaos attempts to stabilize an unstable orbit in a given system. To do this, we apply small perturbations to the values of certain parameters known as bifurcation parameters. In this paper, we employ two methods to stabilize the chaos produced by Neimark-Sacker bifurcation: The state feedback method [6] developed in Subsection (4.1) and the hybrid method [28] developed in Subsection (4.2).

4.1 Feedback Control method

We write the system (1.4)-(1.5) in the following form to apply the state feedback method

$$x_{t+1} = \frac{\exp(ah)x_t - c\phi_1(h)x_ty_t}{1 + (\exp(ah)-1)x_t}, \tag{4.1}$$

$$y_{t+1} = \Bigg(\exp(-(e+H)h) + b\phi_2(h)x_t\Bigg)y_t - S_t. \tag{4.2}$$

Here, $S_t = \alpha(x_t - x^*) + \beta(y_t - y^*)$ is the feedback control force applied to the fixed point $C(x^*, y^*) = (\frac{e+H}{b}, \frac{ab-a(e+H)}{bc})$, where α, β are the feedback gains.
The Jacobian matrix at (x^*, y^*) is

$$J(x^*, y^*) = \begin{pmatrix} \frac{1}{1+(\exp(ah)-1)x^*} & -\frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*} \\ b\phi_2(h)y^* - \alpha & \exp(-(e+H)h) + b\phi_2(h)x^* - \beta \end{pmatrix}. \tag{4.3}$$

The corresponding characteristic equation is

$$\lambda^2 - \left[\frac{1}{1+(\exp(ah)-1)x^*} + e^{-(e+H)h} + b\phi_2(h)x^* - \beta\right]\lambda + \frac{\exp(-(e+H)h) + b\phi_2(h)x^* - \beta}{1+(\exp(ah)-1)x^*} + \tag{4.4}$$
$$\frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*}\Bigg(b\phi_2(h)y^* - \alpha\Bigg) = 0.$$

Let λ_1 , λ_2 are the eigenvalues of the characteristic (4.4) then sum and product of the roots is given by

$$\lambda_1 + \lambda_2 = \frac{1}{1+(\exp(ah)-1)x^*} + \exp(-(e+H)h) + b\phi_2(h)x^* - \beta, \tag{4.5}$$

$$\lambda_1\lambda_2 = \frac{1}{1+(\exp(ah)-1)x^*}\Bigg(\exp(-(e+H)h) + b\phi_2(h)x^* - \beta\Bigg) + \frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*}\Bigg(b\phi_2(h)y^* - \alpha\Bigg). \tag{4.6}$$

Lemma 4.1. The system (4.1)-(4.2) is asymptotically stable if the eigenvalues of (4.4) have absolute value less than 1.

Proof. The marginal stability lines can be obtained from the conditions:
$\lambda_1 = \pm 1$, $\lambda_1\lambda_2 = 1$.
For the condition $\lambda_1\lambda_2 = 1$, the equation 4.6 gives

$$L_1 : \frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*}\alpha + \frac{1}{1+(\exp(ah)-1)x^*}\beta = -1 + \frac{\exp(-(e+H)h) + b\phi_2(h)x^* + c\phi_1(h)x^* b\phi_2(h)y^*}{1+(\exp(ah)-1)x^*}, \quad (4.7)$$

Eq. (4.7) expresses the first condition for marginal stability.
For $\lambda_1 = 1$, the equation (4.5) yields

$$L_2 : \frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*}\alpha - \frac{(\exp(ah)-1)x^*}{1+(\exp(ah)-1)x^*}\beta = 1 - \frac{1 + bc\phi_1(h)\phi_2(h)x^*y^* - (\exp(-(e+H)h) + b\phi_2(h)x^*)}{1+(\exp(ah)-1)x^*} - (\exp(-(e+H)h) + b\phi_2(h)x^*). \quad (4.8)$$

Similarly for $\lambda_1 = -1$, one obtains

$$L_3 : \frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*}\alpha + \frac{2+(\exp(ah)-1)x^*}{1+(\exp(ah)-1)x^*}\beta = 1 + \frac{1 + bc\phi_1(h)\phi_2(h)x^*y^* + \exp(-(e+H)h) + b\phi_2(h)x^*}{1+(\exp(ah)-1)x^*} + (\exp(-(e+H)h) + b\phi_2(h)x^*). \quad (4.9)$$

The lines L_1 , L_2 , L_3 give the conditions for the eigenvalues to have absolute value less than 1. The triangular region bounded by these lines accommodates stable eigenvalues.

4.2 Hybrid control

The modified controlled system can be written for the hybrid method's implementation as follows:

$$x_{t+1} = \mu\left(\frac{\exp(ah)x_t - c\phi_1(h)x_t y_t}{1+(\exp(ah)-1)x_t}\right) + (1-\mu)x_t, \quad (4.10)$$

$$y_{t+1} = \mu\left(\left(\exp(-(e+H)h) + b\phi_2(h)x_t\right)y_t\right) + (1-\mu)y_t, \quad (4.11)$$

where $\mu \in (0,1)$, is the controlled parameter. The Jacobian of system (4.10)-(4.11) evaluated at $C(x^*, y^*)$ is

$$J_\mu(C) = \begin{pmatrix} 1+\mu\left(\frac{\exp(ah)-c\phi_1(h)y^*}{(1+(\exp(ah)-1)x^*)^2} - 1\right) & -\mu\frac{c\phi_1(h)x^*}{1+(\exp(ah)-1)x^*} \\ \mu b\phi_2(h)y^* & 1+\mu\left(\exp(-(e+H)h) + b\phi_2(h)x^* - 1\right) \end{pmatrix}, \quad (4.12)$$

and the associated characteristic equation takes the form

$$\zeta^2 - T\zeta + D = 0, \quad (4.13)$$

where

$$T = \left(1+\mu\left(\frac{\exp(ah)-c\phi_1(h)y^*}{(1+(\exp(ah)-1)x^*)^2} - 1\right) + 1+\mu\left(\exp(-(e+H)h) + b\phi_2(h)x^* - 1\right)\right), \quad (4.14)$$

and

$$D = \left(1+\mu\left(\frac{\exp(ah)-c\phi_1(h)y^*}{(1+(\exp(ah)-1)x^*)^2} - 1\right)\right)\left(1+\mu\left(\exp(-(e+H)h) + b\phi_2(h)x^* - 1\right)\right) + \mu^2\frac{bc\phi_1(h)\phi_2(h)x^*y^*}{1+(\exp(ah)-1)x^*}. \quad (4.15)$$

Lemma 4.2. The coexistence fixed point $C(x^*, y^*)$ of controlled system (4.10)-(4.11) is locally asymptotically stable, whenever the following condition is satisfied:

$$| T | < 1 + D < 2,$$

where T and D are defined in (4.14) and (4.15) respectively.

5 Numerical simulations

In this section, we give some numerical simulations to confirm our theoretical analysis and to show more complex dynamical behaviors for the system (1.4)-(1.5). We set $(h, a, H, c, e) = (0.9, 2.5, 1, 1, 0.2)$. The coexistence fixed point $C(0.41, 1.47)$ is locally asymptotically stable, which means that all orbits are eventually attracted to it, as shown in Fig. (1). However, the system (1.4)-(1.5) starts to lose its stability and a closed curve related to Neimark-Sacker bifurcation is appeared. To see this, if $b > 2.95$ the model (1.4)-(1.5) becomes

$$x_{t+1} = \frac{e^{2.5*0.9}x_t - \frac{e^{2.5*0.9}-1}{2.5}x_t y_t}{1 + (e^{2.5*0.9} - 1)x_t} = f(x_t, y_t, b^*), \tag{5.1}$$

$$y_{t+1} = \left(e^{-(0.2+1)*0.9} + (2.95 + b^*)\frac{1 - e^{-(0.2+1)*0.9}}{1.2}x_t\right)y_t = g(x_t, y_t, b^*). \tag{5.2}$$

We transform $(0.41, 1.47)$ into $(0, 0)$ by $v_t = x_t - 0.41$ and $w_t = y_t - 1.47$. Therefore, the system (5.1)-(5.2) becomes

$$v_{t+1} = 0.2232156961v_t - 0.3107137215w_t - 0.1691614138v_t w_t - 0.424505997v_t^2, \tag{5.3a}$$

$$w(t + 1) = 2.3865366694v_t + w_t + 1.6234943329v_t w_t. \tag{5.3b}$$

The eigenvalues of the linear part of (5.3) is

$$\omega_{1,2} = 0.6116078481 \pm\ 0.7685578874\,i. \tag{5.4}$$

Now, (5.3) takes the following form

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.6116078481 & -0.7685578874 \\ -0.7685578874 & 0.6116078481 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \begin{pmatrix} F(X_t, Y_t) \\ G(X_t, Y_t) \end{pmatrix}, \tag{5.5}$$

where

$$F(X_t, Y_t) = 0.0661988724X_t^2 + 0.1300103388X_t Y_t,$$

and

$$G(X_t, Y_t) = -0.221466959X_t^2 + 0.5701429315X_t Y_t.$$

Computation yields

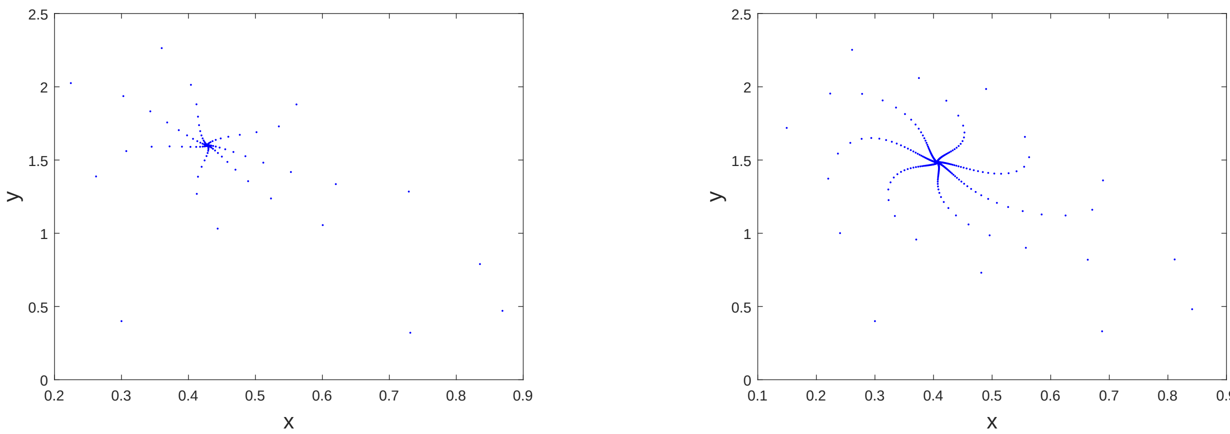
$$\tau_{02} = -0.125986015\ - 0.022864155\,i, \tag{5.6}$$

$$\tau_{11} = 0.0330994362\ - 0.1107334795\,i, \tag{5.7}$$

$$\tau_{20} = 0.159085451\ - 0.087869325\,i, \tag{5.8}$$

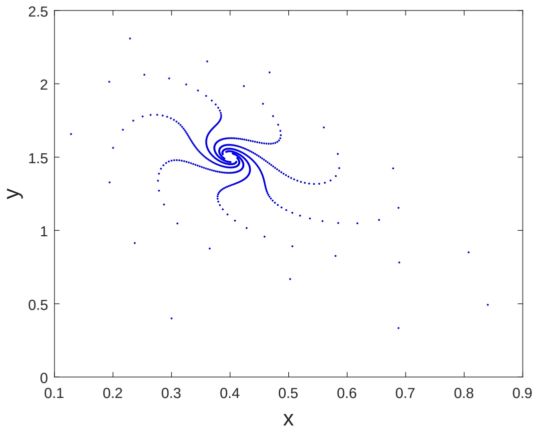
$$\tau_{21} = 0. \tag{5.9}$$

Using (5.4)-(5.6)-(5.7)-(5.8) and (5.9) in (3.10), one gets $\chi = 0.049457454 > 0$. Hence, the model (1.4)-(1.5) undergoes a repelling Neimark-Sacker bifurcation if $b > 2.95$, and meanwhile, a stable curve appears, which is depicted in Fig. (2)

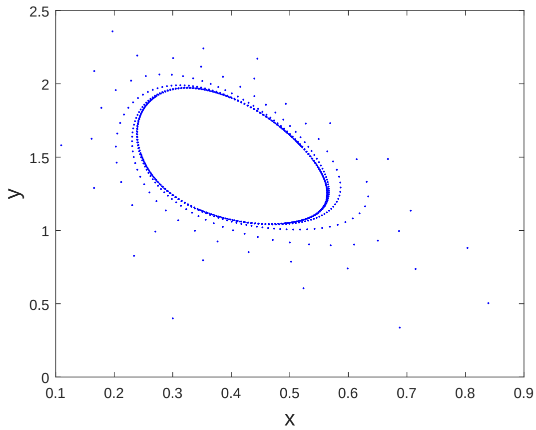


(a): $b = 2.8$ (b): $b = 2.95$

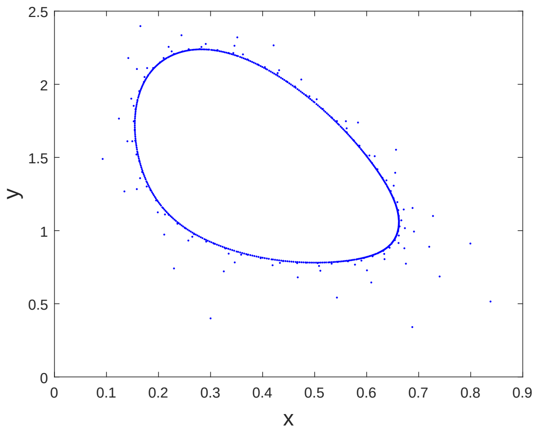
Figure 1: Phase portraits of model for discrete model (1.4)-(1.5) for different value of b.



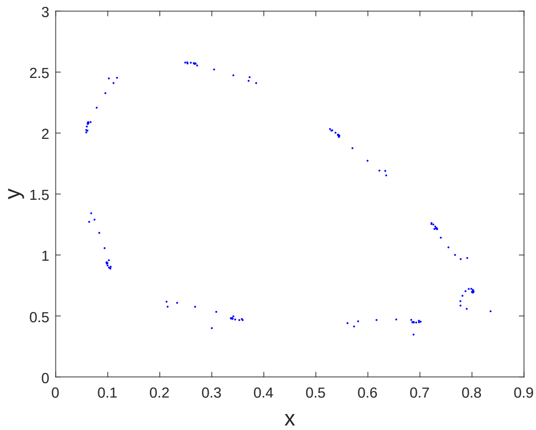
(a): $b = 3$



(b): $b = 3.05$



(c): $b = 3.1$



(d): $b = 3.2$

Figure 2: The appearance of the Neimark-Sacker bifurcation to full chaos in the discrete model(1.4)-(1.5).

Now, in order to stabilize the chaos in the system (1.4)-(1.5) about the fixed point $C(x^*, y^*) = (0.41, 1.47)$. For the first method *state feedback control*, we chose $b = 3.05$. The corresponding chaotic behavior is produced in Fig(2)(b). For these parameters, Lemma (4.1) gives the following lines of marginal stability for the system (4.1)-(4.2):

$$L_1 : -0.3107137215\alpha - 0.2232156961\beta = 0.0039143987, \tag{5.10}$$

$$L_2 : 0.3107137215\alpha + 0.776784304\beta = -0.7450778836, \tag{5.11}$$

$$L_3 : 0.3107137215\alpha - 1.2232156961\beta = -3.247093319. \tag{5.12}$$

The system (4.1)-(4.2) is stable for the triangular region bounded by the marginal lines L_1 , L_2 and L_3. Now, in order to make the fixed point locally asymptotically stable, consider the feedback controlling force $S_t = \alpha(x_t - x^*) + \beta(y_t - y^*)$ with feedback gains $\alpha = 0.45$, $\beta = 0.55$, chosen, from the triangular region from Fig.(3). For these values, a time series is drawn in Fig. (refD), which demonstrates that the system (4.1)-(4.2) converges to the fixed point $C(0.41, 1.47)$, and the stability is finally rebuilt.

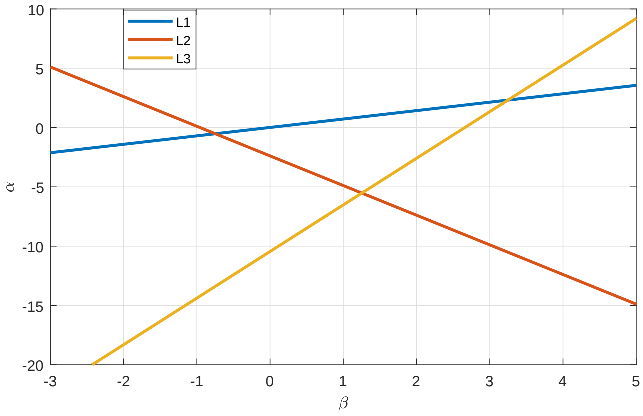
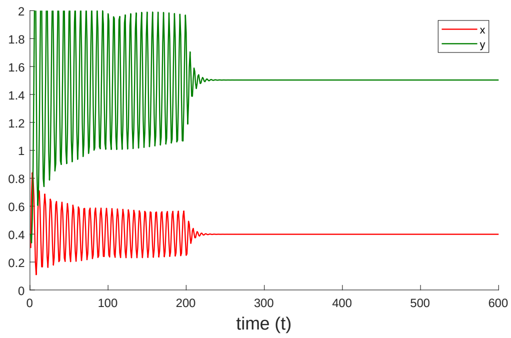
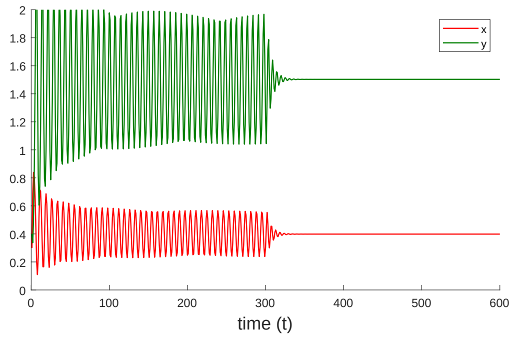


Figure 3: Stability Region for the controlled system (4.1)-(4.2).



(a) The chaos is controlled at time $t = 200$,



(b) The chaos is controlled at time $t = 300$.

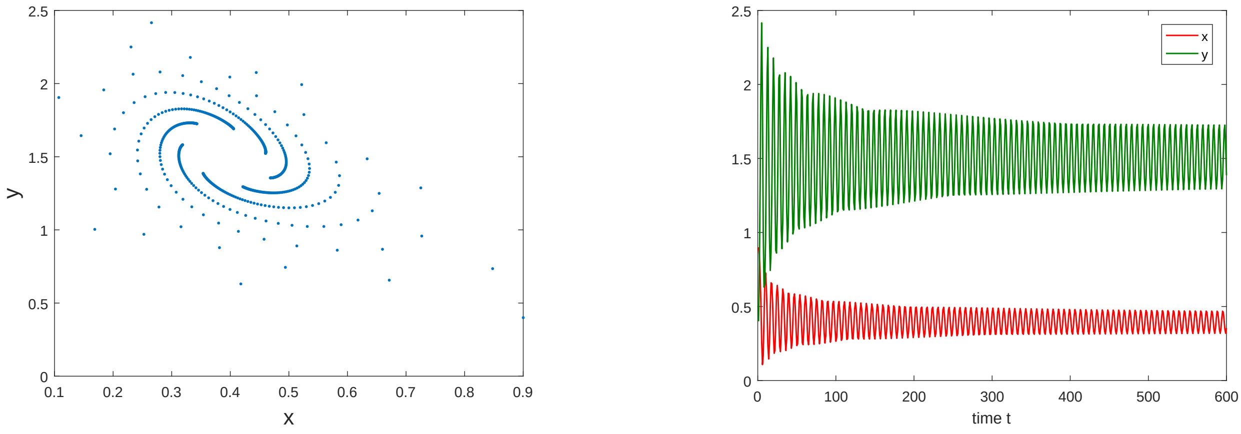
Figure 4: Stable time series for x and y for the controlled system (4.1)-(4.2) for $b = 3.05$ with initial conditions $(x_0, y_0) = (0.3, 0.4)$.

Now, in order to use *the hybrid control method*, the controlled system (4.10)-(4.11) related to (1.4)-(1.5) can be written as

$$x_{t+1} = \mu\left(\frac{e^{2.5*0.9}x_t - \frac{e^{2.5*0.9}-1}{2.5}x_t y_t}{1+(e^{2.5*0.9}-1)x_t}\right) + (1-\mu)x_t, \tag{5.13}$$

$$y_{t+1} = \mu\left(\left(e^{-(1.2)*0.9} + 1.6*3.05*\frac{1-e^{-1.2}}{1.2}x_t\right)y_t\right) + (1-\mu)y_t. \tag{5.14}$$

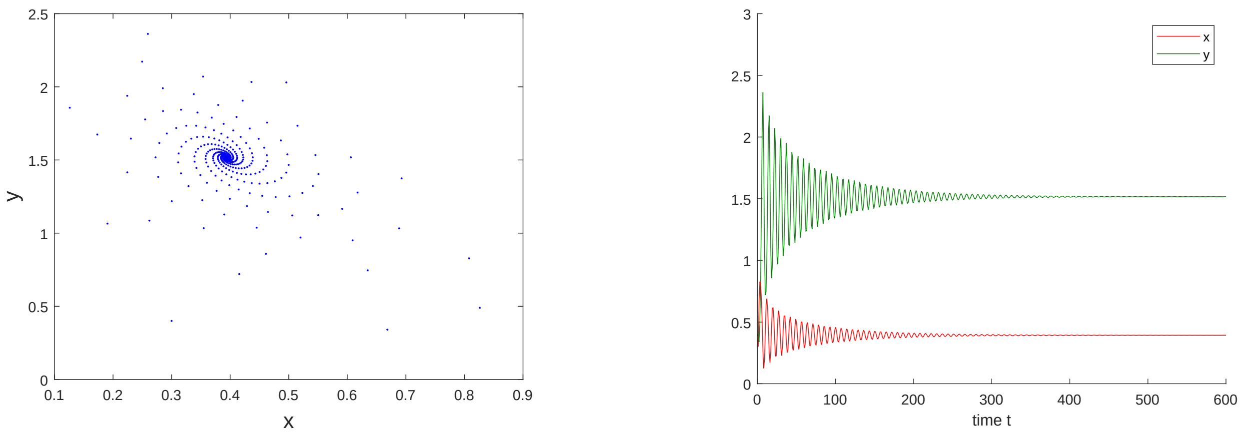
For the value $\mu = 0.98$, time series and phase portraits are plotted in Fig. (5) which reproduce the instability for the system.



(a) (b)

Figure 5: Phase portrait and time series of discrete model (4.10)-(4.11) respectively for $b = 3.05$ and for $\rho = 0.98$

Now, for the value $\mu = 0.96$, time series and phase portrait are plotted in Fig. (6) which show the stability for the system.



(a) (b)

Figure 6: Phase portrait and time series of discrete model (4.10)-(4.11) respectively for $b = 3.05$ and for $\rho = 0.96$.

6 Conclusion

In this work, we investigated the dynamical properties of a discrete-time prey-predator system. The model is developed by discretizing a differential Lotka-Voltera model and using a nonstandard finite difference scheme, which preserves the essential properties of the continuous system. The existence and local asymptotic stability of the fixed points are investigated. To support the complexity of (1.4)-(1.5), the presence of Neimark-Sacker bifurcation for the coexistence fixed point $(\frac{e+H}{b}, \frac{a(b-(e+H))}{bc})$ is proved analytically using bifurcation theory. Detailed numerical simulations have been performed with the purpose of validating our theoretical findings and exhibiting more complex features. Two control strategies, the state feedbach method and the hybrid method, are employed to control the choas produced by the Neimark-Sacker bifurcation. Numerical simulations give evidence of the successful implementation of these methods to stabilize the chaotic behavior produced by Neimark-Sacker bifurcation. In our perspective, introducing the evolution of phenotypic traits in a discrete prey-predator model utilizing evolutionary game theory is a fascinating topic [10, 22, 29]. Another challenging open question is the establishment of the global stability for the current developed system. As a result, new methods for determining the global stability of fixed points, such as the method of Lyapunov functions, as discussed in [17].

Acknowledgment

The authors would like to express their thanks to the main editor and anonymous referees for constructive comments and suggestions that improved the quality of this manuscript.

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