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An application of the power series distribution for univalent function classes with positive coefficients

Gangadharan Murugusundaramoorthy[a], Sibel Yalçın[b,*], Elif Yaşar[b], Serkan Çakmak[c]

[a]School of Advanced Sciences, Vellore Institute of Technology, Vellore - 632014, India
[b]Department of Mathematics, Bursa Uludag University, 16059, Görükle, Bursa, Turkey
[c]Istanbul Gelisim University, Istanbul, Turkey

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Abstract

The primary motivation of the paper is to give necessary and sufficient condition for the power series distribution (Pascal model) to be in the subclasses $\mathcal{VS}_p(\vartheta,\gamma,\kappa)$ and $\mathcal{VC}_p(\vartheta,\gamma,\kappa)$ of analytic functions. Further, to obtain certain connections between the Pascal distribution series and subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, \tag{1.1}$$

which are analytic in the open unit disk $\mathbf{U}=\{z\in\mathbb{C}: \ |z|<1\}$. Also, let $\mathcal{V}$ be the subclasses of $\mathcal{A}$ consisting of functions which are univalent in $\mathbf{U}$, and with positive coefficients given by (see [11])

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, a_j>0 \tag{1.2}$$

respectively. We denote by $\mathcal{SV}(\gamma)$ $(1\leq\gamma<\frac{4}{3})$ a subset of $\mathcal{V}$ consisting of all functions starlike of order γ, i.e. such that $\Re\left(zf'(z)/f(z)\right)<\gamma$ $(z\in\mathbf{U})$, subclasses studied in [11]. By $\mathcal{S}_p(\gamma,\kappa)$ we denote the class of $\kappa-$ starlike functions of order γ, $0\leq\gamma<1$, that is a class of function f, which satisfy the condition

$$\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right)>\kappa\left|\frac{zf'(z)}{f(z)}-1\right| \quad (\kappa\geq0),$$

*Corresponding author
Email addresses: gmsmoorthy@yahoo.com (Gangadharan Murugusundaramoorthy), syalcin@uludag.edu.tr (Sibel Yalçın), elifyasar@uludag.edu.tr (Elif Yaşar), secakmak@gelisim.edu.tr (Serkan Çakmak)

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for details see [5]. Motivated by the above definitions and earlier work of Murugusundaramoorthy [6] we define the following new subclasses of $\mathcal{A}$.

Definition 1.1. For $1 \leq \gamma < \frac{4+\kappa}{3}, 0 \leq \vartheta < 1, \kappa \geq 0$ we let

1. $\mathcal{S}_p(\vartheta,\gamma,\kappa) = \left\{ f \in \mathcal{A} : \Re\left((1+\kappa e^{i\theta}) \frac{zf'(z)}{(1-\vartheta) f(z) + \vartheta z f'(z)} - \kappa e^{i\theta} \right) < \gamma, z \in \mathbf{U} \right\}$
   and
2. $\mathcal{C}_p(\vartheta,\gamma,\kappa) = \left\{ f \in \mathcal{A} : \Re\left((1+\kappa e^{i\theta}) \frac{f'(z) + zf''(z)}{f'(z) + \vartheta z f''(z)} - \kappa e^{i\theta} \right) < \gamma, z \in \mathbf{U} \right\}.$

Also let $\mathcal{VS}_p(\vartheta,\gamma,\kappa) = \mathcal{S}_p(\vartheta,\gamma,\kappa) \cap \mathcal{V}$ and $\mathcal{VC}_p(\vartheta,\gamma,\kappa) = \mathcal{C}_p(\vartheta,\gamma,\kappa) \cap \mathcal{V}$.

By fixing $\vartheta = 0$ we get the following subclasses studied in [9]

Example 1.2. For $1 \leq \gamma < \frac{4+\kappa}{3}, \kappa \geq 0$, we let
(i) $\mathcal{S}_p(\gamma,\kappa) = \left\{ f \in \mathcal{A} : \Re\left((1+\kappa e^{i\theta}) \frac{zf'(z)}{f(z)} - \kappa e^{i\theta} \right) < \gamma, z \in \mathbf{U} \right\}$
and
(ii) $\mathcal{C}_p(\gamma,\kappa) = \left\{ f \in \mathcal{A} : \Re\left(1 + (1+\kappa e^{i\theta}) \frac{zf''(z)}{f'(z)} \right) < \gamma, z \in \mathbf{U} \right\}.$
Also by fixing $\kappa = 0$ we get the subclasses studied in [11].

Further from Definition 1.1, by assuming $\kappa = 0$ we get the subclasses studied by Murugusundaramoorthy [6].

Example 1.3. Let $1 \leq \gamma < \frac{4}{3}, 0 \leq \vartheta < 1$ we let

$$\begin{aligned} \mathcal{VM}_p(\vartheta,\gamma) &= \left\{ f \in \mathcal{V} : \Re\left(\frac{zf'(z)}{(1-\vartheta) f(z) + \vartheta z f'(z)} \right) < \gamma, z \in \mathbf{U} \right\} \\ \mathcal{VG}_p(\vartheta,\gamma) &= \left\{ f \in \mathcal{V} : \Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \vartheta z f''(z)} \right) < \gamma, z \in \mathbf{U} \right\}. \end{aligned}$$

Now, we determine the following necessary and sufficient conditions for $f \in \mathcal{VS}_p(\vartheta,\gamma,\kappa)$ and $f \in \mathcal{VC}_p(\vartheta,\gamma,\kappa)$.

Theorem 1.4. A function $f \in \mathcal{V}$ given by (1.2) is in the class $\mathcal{VS}_p(\vartheta,\gamma,\kappa)$ if and only if

$$\sum_{j=2}^{\infty} \left[j(1+\kappa) - (\gamma+\kappa)(1+j\vartheta-\vartheta) \right] a_j \leq \gamma - 1. \tag{1.3}$$

Proof . Let f be of the form (1.2). To show that $f \in \mathcal{VS}_p(\vartheta,\gamma,\kappa)$, it suffices to prove that

$$\left| \frac{(1+\kappa e^{i\theta}) \left[\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta z f'(z)} - 1 \right]}{2(\gamma-1) - (1+\kappa e^{i\theta}) \left[\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta z f'(z)} - 1 \right]} \right| \leq 1.$$

We have

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$$

and

$$(1-\vartheta)f(z) + \vartheta z f'(z) = z + \sum_{j=2}^{\infty} (1+j\vartheta-\vartheta) a_j z^j.$$

Substituting these values we get

$$\left| \frac{(1+\kappa e^{i\theta}) \sum_{j=2}^{\infty} [j - (1+j\vartheta-\vartheta)] a_j z^{j-1}}{2(\gamma-1)[1 + \sum_{j=2}^{\infty} (1+j\vartheta-\vartheta) a_j z^{j-1}] - (1+\kappa e^{i\theta}) \sum_{j=2}^{\infty} [j - (1+j\vartheta-\vartheta)] a_j z^{j-1}} \right| \leq 1$$

and simple computation yields the desired result.
Conversely, we need only to prove the if $f \in \mathcal{VS}_p(\vartheta, \gamma, \kappa)$ and z is real then

$$\Re\left(\left(1+\kappa e^{i\theta}\right)\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta zf'(z)}-\kappa e^{i\theta}\right)<\gamma$$

that is,

$$\Re\left(\left(1+\kappa e^{i\theta}\right)\left[\frac{zf'(z)}{(1-\vartheta)f(z)+\vartheta zf'(z)}-1\right]\right)<\gamma-1.$$

Thus, for proper $\theta \in \mathbb{R}$, $z = r < 1$ and $a_j > 0$, $j \in \mathbb{N}$, we get

$$\frac{\sum\limits_{j=2}^{\infty}(1+\kappa)\{j-(j\vartheta-\vartheta+1)\}a_j}{1+\sum\limits_{j=2}^{\infty}(j\vartheta-\vartheta+1)a_j}<\gamma-1.$$

This completes the proof of case(i). □

Theorem 1.5. Let $f \in \mathcal{V}$ be of the form (1.2), then $f \in \mathcal{VC}_p(\vartheta, \gamma, \kappa)$ if and only if

$$\sum_{j=2}^{\infty} j\left[j\left(1+\kappa\right)-\left(\gamma+\kappa\right)\left(1+j\vartheta-\vartheta\right)\right]a_j \leq \gamma-1. \tag{1.4}$$

Let $f \in \mathcal{VC}_p(\vartheta, \gamma, \kappa)$ be of the form (1.2). Then by definition we have

$$f \in \mathcal{VC}_p(\vartheta, \gamma, \kappa) \Longleftrightarrow zf' \in \mathcal{VS}_p(\vartheta, \gamma, \kappa)$$

thus we have $f(z) = \left(z + \sum\limits_{j=2}^{\infty}(ja_j)z^j\right) \in \mathcal{VS}_p(\vartheta, \gamma, \kappa)$. Hence by proceeding on lines similar to Theorem 1.4, we easily get (1.4). The main purpose of this paper is to survey the power series distribution for the analytic function classes $\mathcal{VS}_p(\vartheta, \gamma, \kappa)$ and $\mathcal{VC}_p(\vartheta, \gamma, \kappa)$. We note that $\mathcal{VS}_p(\vartheta, \gamma, \kappa) \equiv \mathcal{M}\left(\vartheta, \frac{\gamma+\kappa}{1+\kappa}\right)$[6] and $\mathcal{VC}_p(\vartheta, \gamma, \kappa) \equiv \mathcal{N}\left(\vartheta, \frac{\gamma+\kappa}{1+\kappa}\right)$[6]. In the following sections, we obtain certain connections between the Pascal distribution series and subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution.

2 Necessary and sufficient conditions

The power series distribution is very useful in multivariate data research fields. This family of distributions, particularly is used in survival and reliability studies. However, nowadays, the elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the Burr-Weibull have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1], [2], [3], [4], [7], [8]). In this paper, we focus on the Pascal power series distribution.

In this study we consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathbf{X}=j)=\binom{j+\mathbf{t}-1}{\mathbf{t}-1}p^j\left(1-p\right)^{\mathbf{t}}, \quad j \in \{0,1,2,3,...\},$$

where p, $\mathbf{t}$ are called the parameters.

Now, based upon the Pascal distribution, consider the following power series:

$$\mathcal{P}(\mathbf{t},p,z)=z+\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}\left(1-p\right)^{\mathbf{t}}z^j. \tag{2.1}$$

where $\mathbf{t} \geq 1, 0 \leq p \leq 1, z \in \mathbf{U}$. Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity.

We considered the linear operator

$$\mathcal{I}_p^{\mathbf{t}} : \mathcal{A} \to \mathcal{A}$$

defined by Hadamard product

$$\mathcal{I}_p^{\mathbf{t}} f(z) = \mathcal{P}(\mathbf{t}, p, z) * f(z) = z + \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} p^{j-1} (1-p)^{\mathbf{t}} a_j z^j, \; z \in \mathbf{U}.$$

By considering above definitions and lemmas, we have the following necessary and sufficient conditions for the function $\mathcal{P}$.

Theorem 2.1. For $p \neq 1$, a necessary and sufficient condition for the function $\mathcal{P}$ given by (2.1) to be in the class $\mathcal{VS}_p(\vartheta, \gamma, \kappa)$ is

$$\frac{(1-\vartheta\gamma+\kappa(1-\vartheta))\,\mathbf{t}p}{1-p} - (1-\gamma)(1-p)^{\mathbf{t}} \leq 2(\gamma-1). \tag{2.2}$$

Proof . According to Theorem 1.4, we must show that

$$\sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} [j(1+\kappa) - (\gamma+\kappa)(1+j\vartheta-\vartheta)]\, p^{j-1} (1-p)^{\mathbf{t}} \leq \gamma - 1.$$

Therefore, by combining the relation (1.3) and the implication (2.1), we let the equality

$$\begin{aligned}
\Theta_1(j,\vartheta,\gamma,\kappa) &= \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} [j(1+\kappa) - (\gamma+\kappa)(1+j\vartheta-\vartheta)]\, p^{j-1} (1-p)^{\mathbf{t}} \\
&= \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} [(1-\vartheta\gamma+\kappa(1-\vartheta))(j-1) + 1 - \gamma]\, p^{j-1} (1-p)^{\mathbf{t}},
\end{aligned}$$

$$\begin{aligned}
\Theta_1(j,\vartheta,\gamma,\kappa) &= (1-\vartheta\gamma+\kappa(1-\vartheta)) \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} (j-1)\, p^{j-1} (1-p)^{\mathbf{t}} \\
&+ (1-\gamma) \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} p^{j-1} (1-p)^{\mathbf{t}} \\
&= (1-\vartheta\gamma+\kappa(1-\vartheta))\,\mathbf{t}p\,(1-p)^{\mathbf{t}} \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}} p^{j-2} \\
&+ (1-\gamma)(1-p)^{\mathbf{t}} \sum_{j=2}^{\infty} \binom{j+\mathbf{t}-2}{\mathbf{t}-1} p^{j-1} \\
&= (1-\vartheta\gamma+\kappa(1-\vartheta))\,\mathbf{t}p\,(1-p)^{\mathbf{t}} \sum_{j=0}^{\infty} \binom{j+\mathbf{t}}{\mathbf{t}} p^{j} \\
&+ (1-\gamma)(1-p)^{\mathbf{t}} \sum_{j=0}^{\infty} \binom{j+\mathbf{t}-1}{\mathbf{t}-1} p^{j} - (1-\gamma)(1-p)^{\mathbf{t}} \\
&= \frac{(1-\vartheta\gamma+\kappa(1-\vartheta))\,\mathbf{t}p}{1-p} + (1-\gamma) - (1-\gamma)(1-p)^{\mathbf{t}}.
\end{aligned}$$

But $\Theta_1(j,\vartheta,\gamma,\kappa)$ is bounded above by $\gamma-1$ if and only if (2.2) holds. Thus the proof of Theorem 2.1 is now completed. □

Theorem 2.2. For $p \neq 1$, a necessary and sufficient condition for the function $\mathcal{P}$ given by (2.1) to be in the class $\mathcal{VC}_p(\vartheta, \gamma, \kappa)$ is

$$\frac{[1-\gamma\vartheta+\kappa(1-\vartheta)]\,\mathbf{t}(\mathbf{t}+1)\,p^2}{(1-p)^2}+\frac{[3-\gamma+2\kappa-2\vartheta(\gamma+\kappa)]\,\mathbf{t}p}{1-p} - (1-\gamma)(1-p)^{\mathbf{t}} \leq 2(\gamma-1). \tag{2.3}$$

Proof . To prove that $\mathcal{P} \in \mathcal{VC}_p(\vartheta, \gamma, \kappa)$, we must show that

$$\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1} j\left[j(1+\kappa)-(\gamma+\kappa)(1+j\vartheta-\vartheta)\right]p^{j-1}(1-p)^{\mathbf{t}} \leq \gamma-1$$

From the relation (1.4) and the implication (2.1) , we let

$$\begin{aligned}
\Theta_2(j,\vartheta,\gamma,\kappa) &= \sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1} j\left[j(1+\kappa)-(\gamma+\kappa)(1+j\vartheta-\vartheta)\right]p^{j-1}(1-p)^{\mathbf{t}} \\
&= [1-\gamma\vartheta+\kappa(1-\vartheta)](1-p)^{\mathbf{t}}\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}(j-1)(j-2)\,p^{j-1} \\
&+ [3-\gamma+2\kappa-2\vartheta(\gamma+\kappa)](1-p)^{\mathbf{t}}\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}(j-1)\,p^{j-1} \\
&+ (1-\gamma)(1-p)^{\mathbf{t}}\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1} \\
&= [1-\gamma\vartheta+\kappa(1-\vartheta)](1-p)^{\mathbf{t}}\sum_{j=3}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}+1}\mathbf{t}(\mathbf{t}+1)\,p^{j-3}p^2 \\
&+ [3-\gamma+2\kappa-2\vartheta(\gamma+\kappa)](1-p)^{\mathbf{t}}\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}}\mathbf{t}p^{j-2}p \\
&+ (1-\gamma)(1-p)^{\mathbf{t}}\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1} \\
&= [1-\gamma\vartheta+\kappa(1-\vartheta)]\,\mathbf{t}(\mathbf{t}+1)\,p^2(1-p)^{\mathbf{t}}\sum_{j=0}^{\infty}\binom{j+\mathbf{t}+1}{\mathbf{t}+1}p^{j} \\
&+ [3-\gamma+2\kappa-2\vartheta(\gamma+\kappa)]\,\mathbf{t}p\,(1-p)^{\mathbf{t}}\sum_{j=0}^{\infty}\binom{j+\mathbf{t}}{\mathbf{t}}p^{j} \\
&+ (1-\gamma)(1-p)^{\mathbf{t}}\sum_{j=0}^{\infty}\binom{j+\mathbf{t}-1}{\mathbf{t}-1}p^{j}-(1-\gamma)(1-p)^{\mathbf{t}} \\
&= \frac{[1-\gamma\vartheta+\kappa(1-\vartheta)]\,\mathbf{t}(\mathbf{t}+1)\,p^2}{(1-p)^2}+\frac{[3-\gamma+2\kappa-2\vartheta(\gamma+\kappa)]\,\mathbf{t}p}{1-p} \\
&+ (1-\gamma)-(1-\gamma)(1-p)^{\mathbf{t}} \leq \gamma-1
\end{aligned}$$

Thus, according to Theorem 1.5, we conclude that $\mathcal{P} \in \mathcal{VC}_p(\vartheta, \gamma, \kappa)$. □

3 Inclusion Properties

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(\mu, \varrho)$, $(\tau \in \mathbb{C}\backslash\{0\},\ 0 < \mu \leq 1; \varrho < 1)$, if it satisfies the inequality

$$\left|\frac{(1-\mu)\frac{f(z)}{z}+\mu f'(z)-1}{2\tau(1-\varrho)+(1-\mu)\frac{f(z)}{z}+\mu f'(z)-1}\right| < 1, \quad (z \in \mathbf{U}).$$

The class $\mathcal{R}^{\tau}(\mu,\varrho)$ was introduced earlier by Swaminathan [10](for special cases see the references cited there in) and obtained the following estimate.

Lemma 3.1. [10] If $f \in \mathcal{R}^{\tau}(\mu,\varrho)$ is of form (1.2), then

$$|a_j| \leq \frac{2\,|\tau|\,(1-\varrho)}{1+\mu(j-1)}, \quad j \in \mathbb{N}\setminus\{1\}. \tag{3.1}$$

The bounds given in (3.1) is sharp.

Making use of the Lemma 3.1, we will study the action of the Pascal distribution series on the class $\mathcal{VC}_p(\vartheta,\gamma,\kappa)$ in the following theorem.

Theorem 3.2. If $p \neq 1$ and $f \in \mathcal{R}^{\tau}(\mu,\varrho)$, if the inequality

$$\left[\frac{(1-\vartheta\gamma+\kappa\,(1-\vartheta))\,\mathbf{t}p}{1-p}+(1-\gamma)-(1-\gamma)\,(1-p)^{\mathbf{t}}\right] \leq \frac{\mu(\gamma-1)}{2\,|\tau|\,(1-\varrho)} \tag{3.2}$$

is satisfied, then $\mathcal{I}_p^{\mathbf{t}}f(z) \in \mathcal{VC}_p(\vartheta,\gamma,\kappa)$.

Proof . Let f be of the form (1.2) belong to the class $\mathcal{R}^{\tau}(\mu,\varrho)$. By virtue of Theorem 1.5 it suffices to show that

$$\sum_{j=2}^{\infty} j\left[j\,(1+\kappa)-(\gamma+\kappa)\,(1+j\vartheta-\vartheta)\right]p^{j-1}\,(1-p)^{\mathbf{t}}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}a_j \leq \gamma-1$$

Since $f \in \mathcal{R}^{\tau}(\mu,\varrho)$ then by Lemma 3.1 we have

$$|a_j| \leq \frac{2\,|\tau|\,(1-\varrho)}{1+\mu(j-1)}, \quad j \in \mathbb{N}\setminus\{1\}.$$

Let $\Theta_3(\vartheta,\gamma,\kappa)=\sum_{j=2}^{\infty} j\left[j\,(1+\kappa)-(\gamma+\kappa)\,(1+j\vartheta-\vartheta)\right]\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}\,(1-p)^{\mathbf{t}}\,a_j$

$$\Theta_3(\vartheta,\gamma,\kappa) \leq 2\,|\tau|\,(1-\varrho)\sum_{j=2}^{\infty}\frac{j\left[j\,(1+\kappa)-(\gamma+\kappa)\,(1+j\vartheta-\vartheta)\right]}{1+\mu(j-1)}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}\,(1-p)^{\mathbf{t}}.$$

Since $1+\mu(j-1) \geq j\mu$, we get

$$\Theta_3(\vartheta,\gamma,\kappa) \leq \frac{2\,|\tau|\,(1-\varrho)}{\mu}\sum_{j=2}^{\infty}\left[j\,(1+\kappa)-(\gamma+\kappa)\,(1+j\vartheta-\vartheta)\right]\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}\,(1-p)^{\mathbf{t}}.$$

Proceeding as in Theorem 1.5, we get

$$\Theta_3(\vartheta,\gamma,\kappa) \leq \frac{2\,|\tau|\,(1-\varrho)}{\mu}\left[\frac{(1-\vartheta\gamma+\kappa\,(1-\vartheta))\,\mathbf{t}p}{1-p}+(1-\gamma)-(1-\gamma)\,(1-p)^{\mathbf{t}}\right].$$

But $\Theta_3(\vartheta,\gamma,\kappa)$ bounded above by $\gamma-1$ if and only if (3.2) holds. Thus the proof is complete. □

Theorem 3.3. Let $p \neq 1$, then $\mathcal{L}(\mathbf{t},z)=\int_0^z \frac{\mathcal{I}_p^{\mathbf{t}}(\xi)}{\xi}d\xi$ is belong to the class $\mathcal{VC}_p(\vartheta,\gamma,\kappa)$ if and only if

$$\frac{(1-\vartheta\gamma+\kappa\,(1-\vartheta))\,\mathbf{t}p}{1-p}-(1-\gamma)\,(1-p)^{\mathbf{t}} \leq 2(\gamma-1). \tag{3.3}$$

Proof . Since

$$\mathcal{L}(\mathbf{t},z)=z+\sum_{j=2}^{\infty}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}(1-p)^{\mathbf{t}}\frac{z^j}{j}$$

then by Theorem 1.5 we need only to show that

$$\sum_{j=2}^{\infty}j\left[j\left(1+\kappa\right)-\left(\gamma+\kappa\right)\left(1+j\vartheta-\vartheta\right)\right]\frac{1}{j}\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}(1-p)^{\mathbf{t}}\leq\gamma-1.$$

That is, let

$$\begin{aligned}\Theta_4(\vartheta,\gamma,\kappa) &= \sum_{j=2}^{\infty}\left[j\left(1+\kappa\right)-\left(\gamma+\kappa\right)\left(1+j\vartheta-\vartheta\right)\right]\binom{j+\mathbf{t}-2}{\mathbf{t}-1}p^{j-1}(1-p)^{\mathbf{t}}\\ &\leq \gamma-1.\end{aligned}$$

Now by writing $j=(j-1)+1$ and Proceeding as in Theorem 1.5, we get

$$\Theta_4(\vartheta,\gamma,\kappa)=\left[\frac{\left(1-\vartheta\gamma+\kappa\left(1-\vartheta\right)\right)\mathbf{t}p}{1-p}+\left(1-\gamma\right)-\left(1-\gamma\right)\left(1-p\right)^{\mathbf{t}}\right]$$

which is bounded above by $\gamma-1$ if and only if (3.3) holds. □

Concluding Remark: By taking $\kappa=0$ and specializing $\vartheta=0$ one can deduce above analogues results for various subclasses given in Examples 1.2 and 1.3 and studied in [6, 9, 11] with positive coefficients.

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