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Some existence and uniqueness results of a houseflies model with a delay depending on time and state

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Abstract

This work elucidates the sufficient conditions for establishing some existence and uniqueness results for a Musca Domestica model that is governed by a first-order nonlinear differential equation with iterative terms resulting from a time and state-dependent delay. The existence of at least one positive periodic solution is proved by using Schauder's fixed point theorem with the help of some properties of an obtained Green's function. Furthermore, under an additional condition, the Banach contraction principle is applied to guarantee the existence, uniqueness and stability of solutions. Finally, the validity of our main findings is demonstrated by two examples. Our findings are completely new and generalize previous ones to some degree.

Keywords: Delay differential equation, fixed point theorem, Green's function, periodic solution, positive solution
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1 Introduction

The common housefly, Musca Domestica is a cosmopolitan pest of both humans and animals that is recognized for their ability to spoil food, to cause irritations and also to harbor and to transmit many diseases as amoebic dysentery, cholera, yaws, Newcastle disease, typhoid fever, poliomyelitis and diarrhea, etc.

Among the first delay models that have been used to describe the growth of single-species insect populations, we can cite the work of Maynard Smith [15] who has proposed the following first order differential equation with a constant delay for modeling a single-species population with two stages: larva and adult:

$$M'(t) = -p\ M(t) + bM(t-\tau)\left[1 - M(t-\tau)\right],$$

with $M(t)$ is denoting the size of the population at time t, $p > 0$ is the adult mortality rate, b is the number of eggs laid per adult and the time delay τ stands for the time elapsing between oviposition and adult eclosion.

In 1976, motivated by the above equation, and in order to describe the oscillations in the dynamics of laboratory populations of houseflies, Musca Domestica, Taylor and Sokal [18] have set down the following delay differential equation:

$$M'(t) = -pM(t) + bM(t-\tau)\left[\beta - b\alpha M(t-\tau)\right],$$

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where β is the maximum egg-adult survival rate and α is the reduction in survival produced by each additional egg.

A growing number of experiments that have clearly manifested that the delays in many hereditary phenomena are generally depending in both time and state, have fascinated us and attracted our interests to consider a housefly model with iterative terms resulting from a time and state dependent delay. Indeed, several studies have revealed that due many factors such as the competition for food during the three larval stages, the duration of the fly life cycle depends in fact on the time and the population size. Simply said, adult females lay clusters of eggs in several batches by stacking them on top of each other which affects the duration of their life cycles and hence, the life cycles of the maggots at the bottom (near the moist and nutrient-rich place) are faster than those superimposed above them, which in turn are faster than those at the top which means that the mean duration of the life cycles τ varies depending not only on the time but also on the number of adult flies that lay eggs. By taking into account this information, we can revisit Taylor and Sokal model to the following one:

$$M'(t) = -p(t) M(t) + bM(t - \tau(t, M(t))) [\beta - b\alpha M(t - \tau(t, M(t)))],$$

and by assuming that $\tau(t, M(t)) = t - M(t)$, we arrive at the following first order iterative differential equation that involves implicitly the above time and state dependent delay:

$$M'(t) + p(t) M(t) = b\beta M^{[2]}(t) - b^2\alpha \left(M^{[2]}(t)\right)^2, \tag{1.1}$$

where $M^{[2]}(t) = M(M(t))$, $p \in \mathcal{C}(\mathbb{R},]0, \infty[)$ is a $w-$periodic function and the remainder parameters are positive.

Equation (1.1) is an iterative differential equation and equations of this kind have tremendous applications in an extremely wide range of areas, including biology, medicine, classical electrodynamics, physics, epidemiology, hematology, population dynamics and many other branches of science and technology. They have been of vital importance in modeling various natural phenomena over the last three centuries- and have been even more so in the last ten years- thanks to some papers (see [1]-[14], [16], [19] and references therein). Here, we would like to mention some recent works on iterative problems that arise in life sciences.

In [4] , the authors have used Schauder's fixed point theorem to study the following Nicholson's blowflies equation with an iterative harvesting effort:

$$N'(t) = -p(t) N(t) + a(t) N(t-\tau) e^{-\gamma(t)N(t-\tau)} - qN(t-\tau) E\left(t, N(t), N^{[2]}(t), ..., N^{[n]}(t)\right),$$

where $N(t)$ denotes the population density of the sheep blowfly, Lucilia Cuprina.

In [4], by virtue of the same aforementioned fixed point theorem, Bouakkaz has investigated the following class of first-order iterative differential equations with application to three iterative hematopoiesis models for humans and animals.

$$x'(t) = -p(t) x(t) + a(t) x^m(t - \tau(t)) f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),$$

where $x(t)$ is the density of mature cells in blood circulation at time t, $p(t)$ and $a(t)$ are, respectively the death and the production rates of blood cells.

In [16], Mezghiche et al. have applied Banach and Krasnoselskii's fixed point theorems together with the Green's functions method to establish the existence, uniqueness and stability results for the following class of first order neutral delay differential equations with an iterative harvesting term:

$$\frac{d}{dt}[x(t) - cx(t - \tau(t))] = -p(t) x(t) + f(t, x(t - \tau(t))) - H\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right).$$

Here $x(t)$ can represent for example, a size of a population, a number of individuals or a blood cell density and H is the harvesting function.

In [12], Khemis et al. have utilized the same approach adopted in [16] to discuss the existence, uniqueness and stability of positive periodic solutions for the following Lasota–Wazewska model with an iterative production term and a delayed harvesting one:

$$x'(t) = -p(t) x(t) + a(t) e^{-\gamma x^{[2]}(t)} - H(t, x(t-\tau)),$$

where $x(t)$ is the density of mature erythrocytes in an animal at time t.

So, we draw our motivation first and foremost from these works and as we said before, from the fact that delays in life sciences depend generally on the time and the state and also from our contribution in enriching and complementing some earlier publications whether on insect population dynamics or iterative problems. It is worth noting here that these latter are obviously burning topics and hence in many cases they are difficult to deal with. For this, the theory of such kind of equations which can be considered as a special type of the so-called functional differential equations with delays depending upon both the time and the state variables; has not yet been well established (see [1]-[14], [16], [19]). The chief problem lies in the iterative terms that generally impede the application of usual methods and could make the study somewhat difficult.

The main purpose of this work is to establish some sufficient criteria for ensuring the existence, uniqueness and stability of positive periodic solutions of the iterative differential equation (1.1). For achieving our goals, we use an attractive technique based on converting the problem at hand into an equivalent integral equation before constructing an integral operator with a Green's function type kernel. Next, through the fixed point theory, some functional analysis tools together with some properties of the obtained Green's function, we success in establishing some new existence, uniqueness and stability results for our problem.

The plan of the article is as follows. In Section 2, we introduce our notations, assumptions and some preliminaries. In Section 3, we state and prove our main results concerning the existence, uniqueness and continuous dependence on parameters of positive periodic solutions for the proposed model, whilst Section 4 is dedicated to give two examples to corroborate the effectiveness of the obtained findings.

2 Preliminaries

For $L \geq 0$ and $w, \lambda_1 > 0$, let

$$X = \{M \in \mathcal{C}(\mathbb{R}, \mathbb{R}),\ M(t+w) = M(t)\},$$

equipped with the supremum norm

$$\|M\| = \sup_{t \in \mathbb{R}} |M(t)| = \sup_{t \in [0,w]} |M(t)|,$$

and

$$\Omega = \{M \in X : 0 < M(t) \leq \lambda_1,\ \ |M(t_2) - M(t_1)| \leq L|t_2 - t_1|,\ \forall t_1, t_2 \in [0,w]\},$$

then $(X, \|\cdot\|)$ is a Banach space and Ω is a closed convex and bounded subset of X.

For convenience, we introduce the following notations:

$$p = \sup_{t \in [0,w]} p(t),\ \ \gamma_0 = \frac{\exp\left(-\int_0^w p(u)\,du\right)}{\exp\left(\int_0^w p(u)\,du\right) - 1},\ \ \gamma_1 = \frac{\exp\left(\int_0^w p(u)\,du\right)}{\exp\left(\int_0^w p(u)\,du\right) - 1}.$$

Furthermore, it will be assumed that

($\mathbf{H}_1$) For all $M \in \Omega$ and $s \in [0,w]$, we suppose that

$$\min_{s \in [0,w]} \left\{ b\beta M^{[2]}(s) - b^2 \alpha \left(M^{[2]}(s) \right)^2 \right\} > 0. \tag{2.1}$$

($\mathbf{H}_2$) The following estimates are satisfied:

$$\gamma_1 w b \beta \leq 1, \tag{2.2}$$

$$(\gamma_1 p w + 2\gamma_1)\left(b\beta\lambda_1 + b^2\alpha\lambda_1^2\right) \leq L. \tag{2.3}$$

Definition 2.1. (Green's function) [6] We will consider two − point n^{th}− order linear boundary value problems of the form

$$\begin{cases} L_n y(t) = \sigma(t),\ t \in I \equiv [c,d], \\ U_i(y) = \xi_i,\ i = \overline{1,m}, \end{cases} \tag{P_1}$$

where

$$L_n y(t) = a_0(t)\, y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + \dots + a_{n-1}(t)\, y'(t) + a_n(t)\, y(t),$$

and

$$U_i(y) = \sum_{j=0}^{n-1} \left(\alpha_j^i y^{(j)}(c) + \beta_j^i y^{(j)}(d) \right), \; i = \overline{1,m}, \; m \leq n,$$

being α_j^i, β_j^i and ξ_i real constants for all $i = \overline{1,m}$ and $j = \overline{0,n-1}$, σ and a_k continuous real functions for all $k = \overline{0,n}$, and $a_0(t) \neq 0$ for all $t \in I$.
We say that G is a Green's function for problem (P_1) if it satisfies the following properties:
(**G1**) G is defined on the square $I \times I$.
(**G2**) For $k = \overline{0,n-2}$, the partial derivatives $\dfrac{\partial^k G}{\partial t^k}$ exist and they are continuous on $I \times I$.
(**G3**) $\dfrac{\partial^{k-1} G}{\partial t^{k-1}}$ and $\dfrac{\partial^k G}{\partial t^k}$ exist and are continuous on the triangles $c \leq s < t \leq d$ and $c \leq t < s \leq d$.
(**G4**) For each $t \in (c,d)$ there exist the lateral limits

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t,t^+\right) \text{ and } \frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t,t^-\right),$$

(i.e., the limits when $(t,s) \to (t,t)$ with $s > t$ or with $s < t$) and, moreover

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t,t^+\right) - \frac{\partial^{n-1} G}{\partial t^{n-1}}\left(t,t^-\right) = -\frac{1}{a_0(t)}.$$

(**G5**) For each $s \in (c,d)$, the function $t \to G(t,s)$ is a solution of the differential equation $L_n y = 0$ on $t \in [c,s)$ and $t \in (s,d]$. That is,

$$a_0(t) \frac{\partial^n G}{\partial t^n}(t,s) + a_1(t) \frac{\partial^{n-1} G}{\partial t^{n-1}}(t,s) + \ldots + a_{n-1}(t) \frac{\partial G}{\partial t}(t,s) + a_n(t) G(t,s) = 0,$$

on both intervals.
(**G6**) For each $s \in (c,d)$, the function $t \to G(t,s)$ satisfies the boundary conditions

$$\sum_{j=0}^{n-1} \left(\alpha_j^i \frac{\partial^j G}{\partial t^j}(c,s) + \beta_j^i \frac{\partial^j G}{\partial t^j}(d,s) \right) = 0, \; i = \overline{1,m}.$$

Theorem 2.2. [6] Let us suppose that the homogeneous problem

$$\begin{cases} L_n y(t) = 0, \; t \in I \equiv [c,d], \\ U_i(y) = 0, \; i = \overline{1,n}, \end{cases} \tag{P_2}$$

has only the trivial solution. Then there exists a unique Green's function, $G(t,s)$, related to (P_2). Moreover, for each continuous function σ, the unique solution of the problem

$$\begin{cases} L_n y(t) = \sigma(t), \; t \in I \equiv [c,d], \\ U_i(y) = 0, \; i = \overline{1,n}, \end{cases}$$

is given by the expression

$$y((t)) = \int_c^d G(t,s)\,\sigma(s)\,ds.$$

Here, we state and prove a useful equivalence between equation (1.1) with the periodic properties and a certain integral equation.

Lemma 2.3. $M \in \Omega \cap \mathcal{C}^1(\mathbb{R},\mathbb{R})$ is a solution of equation (1.1) if and only if $M \in \Omega$ is a solution of the following integral equation:

$$M(t) = \int_t^{t+w} G(t,s) \left[b\beta M^{[2]}(s) - b^2 \alpha \left(M^{[2]}(s) \right)^2 \right] ds, \tag{2.4}$$

where G is a Green's function giving by

$$G(t,s) = \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^w p(u)\,du\right) - 1}. \tag{2.5}$$

Proof . Let $M \in \Omega$ be a solution of equation (1.1). Multiplying both sides of this equation by $\exp\left(\int_0^t p(u)\,du\right)$ we obtain

$$\frac{d}{ds}\left[M(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds = \left[b\beta M^{[2]}(t) - b^2\alpha\left(M^{[2]}(t)\right)^2\right]\exp\left(\int_0^t p(u)\,du\right).$$

The integration from t to $t+w$ gives

$$\int_t^{t+w}\frac{d}{ds}\left[M(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds = \int_t^{t+w}\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right]\exp\left(\int_0^s p(u)\,du\right)ds.$$

The fact that $M(t) = M(t+w)$ implies that

$$\begin{aligned}\int_t^{t+w}\frac{d}{ds}\left[M(s)\exp\left(\int_0^s p(u)\,du\right)\right]ds &= M(t)\left[\exp\left(\int_0^{t+w} p(u)\,du\right) - \exp\left(\int_0^t p(u)\,du\right)\right]\\ &= M(t)\left[\exp\left(\int_0^t p(u)\,du\right)\left(\exp\left(\int_t^{t+w} p(u)\,du\right) - 1\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}M(t) &= \int_t^{t+w}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_t^{t+w} p(u)\,du\right) - 1}\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right]ds\\ &= \int_t^{t+w} G(t,s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right]ds,\end{aligned}$$

which completes the first step of the proof.

Conversely, if we assume that M satisfies (2.4) and by the derivation of this integral equation, we infer that M satisfies equation (1.1). □

Remark 2.4. The Green's function G satisfies the following properties:

$$G(t+w, s+w) = G(t,s),\ \forall t,s \in \mathbb{R},$$

and

$$\gamma_0 \le G(t,s) \le \gamma_1. \tag{2.6}$$

Lemma 2.5. [19]If $M, N \in \Omega$, then

$$\left\|M^{[2]} - N^{[2]}\right\| \le (1+L)\|M - N\|. \tag{2.7}$$

Remark 2.6. We can establish inequality (2.7) as follows. If $M, N \in \mathbb{D}$, then

$$\begin{aligned}\left|M^{[2]}(t) - N^{[2]}(t)\right| &\le |M(M(t)) - M(N(t))| + |M(N(t)) - N(N(t))|\\ &\le L|M(t) - N(t)| + \|M - N\|\\ &\le L_2\|M - N\| + \|M - N\|.\end{aligned}$$

So

$$\left\|M^{[2]} - N^{[2]}\right\| = \sup_{t\in[0,w]}\left|M^{[2]}(t) - N^{[2]}(t)\right| \le (1+L)\|M - N\|.$$

Theorem 2.7. (Schauder's fixed point theorem) [17] Let Ω be a closed, bounded, convex, and nonempty subset of a Banach space $(\mathbb{X}, \|.\|)$. Then any continuous compact mapping $\mathcal{A} : \Omega \longrightarrow \Omega$ has at least one fixed point in Ω.

Theorem 2.8. (Arzelà-Ascoli theorem) [5] Let $\mathbb{X}$ be a compact metric space and let $C(\mathbb{X})$ be the space of all bounded and continuous real-valued functions on $\mathbb{X}$. If Ω is an equicontinuous and bounded subset of $C(\mathbb{X})$, then Ω is relatively compact.

3 Main results

3.1 Existence result

In the first part of this section, we will use the Schauder's fixed point theorem combined with the obtained Green's function properties to prove the existence of at least one positive periodic solution for equation (1.1). For this and by virtue of Lemma 2.3, we define an operator $\mathcal{K} : \Omega \to X$ as follows:

$$(\mathcal{K}M)(t) = \int_t^{t+w} G(t,s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds. \tag{3.1}$$

So, fixed points of $\mathcal{K}$ are solutions of equation (1.1) and vice versa. This means that our main task is clearly to show that the operator $\mathcal{K}$ has fixed points which are solutions to equation (1.1).

Let us begin by pointing out that by virtue of the periodic properties, the operator $\mathcal{K}$ is well-defined. Next, we state and prove the following lemma which establishes the continuity and compactness of $\mathcal{K}$.

Lemma 3.1. The operator $\mathcal{K}$: $\Omega \to X$ given by (3.1) is continuous and compact.

Proof . Thanks to Arzelà-Ascoli theorem, Ω is a compact subset of X. So, to show that $\mathcal{K}$ is a compact operator it suffices to show that it is continuous. For $M, N \in \Omega$, we have

$$\begin{aligned}|(\mathcal{K}M)(t) - (\mathcal{K}N)(t)| &\le b\beta\int_t^{t+w} G(t,s)\left|M^{[2]}(s) - N^{[2]}(s)\right| ds \\ &+ b^2\alpha\int_t^{t+w} G(t,s)\left|M^{[2]}(s) + N^{[2]}(s)\right|\left|M^{[2]}(s) - N^{[2]}(s)\right| ds.\end{aligned}$$

Taking into account (2.6) and (2.7), we obtain

$$\begin{aligned}|(\mathcal{K}M)(t) - (\mathcal{K}N)(t)| &\le \left(\gamma_1 wb\beta(1+L) + 2\gamma_1 wb^2\alpha\lambda_1(1+L)\right)\|M-N\| \\ &= \mu\|M-N\|,\end{aligned}$$

where,

$$\mu = \gamma_1 wb(\beta + 2b\alpha\lambda_1)(1+L),$$

which shows that the operator $\mathcal{K}$ is Lipschitz continuous and hence continuous. Therefore, $\mathcal{K}$ is compact. □

Lemma 3.2. If conditions (2.1) and (2.2) hold, then

$$0 < (\mathcal{K}M)(t) \le \lambda_1,$$

for all $M \in \Omega$.

Proof . Let $M \in \Omega$. In view of (2.1) we have

$$\begin{aligned}(\mathcal{K}M)(t) &= \int_t^{t+w} G(t,s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &> \gamma_0 w\min_{s\in[0,w]}\left\{b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right\} \\ &> 0,\end{aligned}$$

and by using (2.2) we have

$$\begin{aligned}(\mathcal{K}M)(t) &= \int_t^{t+w} G(t,s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &\le b\beta\int_t^{t+w} G(t,s)M^{[2]}(s)\,ds \\ &\le \gamma_1 wb\beta\lambda_1 \\ &\le \lambda_1.\end{aligned}$$

Consequently, $0 < (\mathcal{K}M)(t) \le \lambda_1$. □

Remark 3.3. Let $p \in X$. For $t_1, t_2 \in [0, w]$ and $t_1 < t_2$ we have

$$\int_{t_1}^{t_1+w} \left|\exp\left(\int_{t_2}^{s} p(u)\,du\right) - \exp\left(\int_{t_1}^{s} p(u)\,du\right)\right| ds \le wp \exp\left(\int_0^w p(u)\,du\right)|t_2 - t_1|. \tag{3.2}$$

Lemma 3.4. If condition (2.3) holds, then

$$|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| \le L|t_2 - t_1|,$$

for all $t_1, t_2 \in \mathbb{R}$.

Proof . Let $t_1, t_2 \in [0, w]$ with $t_1 < t_2$. For $M \in \Omega$, we have

$$\begin{aligned}|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| = \Bigg|&\int_{t_2}^{t_2+w} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &- \int_{t_1}^{t_1+w} G(t_1, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds\Bigg|.\end{aligned}$$

Thereby

$$\begin{aligned}|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| = \Bigg|&\int_{t_2}^{t_1} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &+ \int_{t_1}^{t_1+w} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &+ \int_{t_1+w}^{t_2+w} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &- \int_{t_1}^{t_1+w} G(t_1, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds\Bigg|,\end{aligned}$$

from which we infer that

$$\begin{aligned}|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| \le &\int_{t_2}^{t_1} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &+ \int_{t_1+w}^{t_2+w} G(t_2, s)\left[b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right] ds \\ &+ \int_{t_1}^{t_1+w} |G(t_2, s) - G(t_1, s)|\left|b\beta M^{[2]}(s) - b^2\alpha\left(M^{[2]}(s)\right)^2\right| ds.\end{aligned}$$

It follows from (2.3), (2.6) and (3.2) that

$$\begin{aligned}|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| &\le 2\gamma_1\left(b\beta\lambda_1 + b^2\alpha\lambda_1^2\right)|t_2 - t_1| \\ &\quad + \gamma_1 pw\left(b\beta\lambda_1 + b^2\alpha\lambda_1^2\right)|t_2 - t_1| \\ &= (\gamma_1 pw + 2\gamma_1)\left(b\beta\lambda_1 + b^2\alpha\lambda_1^2\right)|t_2 - t_1| \\ &\le L|t_2 - t_1|.\end{aligned}$$

So, $|(\mathcal{K}M)(t_2) - (\mathcal{K}M)(t_1)| \le L|t_2 - t_1|$ for all $t_1, t_2 \in \mathbb{R}$. □

Our first main result is the following theorem:

Theorem 3.5. Suppose that conditions (2.1) – (2.3) hold, then equation (1.1) has at least one positive periodic solution $M \in \Omega$.

Proof . As a result of Lemmas 3.2 and 3.4, $\mathcal{K}$ maps Ω into itself, i.e. $\mathcal{K}(\Omega) \subset \Omega$ and from Lemma 3.1, $\mathcal{K}$ is a compact and continuous operator, so all requirements of the Schauder's fixed point theorem are satisfied. This shows that $\mathcal{K}$ has at least one fixed point in Ω, which means that equation (1.1) admits at least one positive periodic solution. □

3.2 Uniqueness result

The second part of this section will be devoted to establishing the existence and uniqueness of solutions by using the Banach contraction principle.

Theorem 3.6. Besides of the assumptions (2.1) – (2.3) if we further assume that $\mu < 1$, then equation (1.1) has one and only one solution $M \in \Omega$.

Proof . From Lemma 3.1, for all $M, N \in \Omega$ we arrived at

$$|(\mathcal{K}M)(t) - (\mathcal{K}N)(t)| \leq \mu \|M - N\|.$$

Thanks to condition $\mu < 1$ and the Banach fixed point theorem, operator $\mathcal{K}$ has a unique fixed point in Ω. From Lemma 2.3, this unique fixed point is the unique positive periodic solution of equation (1.1). □

3.3 Continuous dependence on parameters

Now, we establish the continuous dependence of the solution upon the adult mortality rate.

Theorem 3.7. The solution obtained in Theorem 3.6 depends continuously on the function p.

Proof . Let

$$M_1(t) = \int_t^{t+w} G_1(t,s)\left[b\beta M_1^{[2]}(s) - b^2\alpha\left(M_1^{[2]}(s)\right)^2\right] ds,$$

be the unique solution of equation (1.1) and let

$$M_2(t) = \int_t^{t+w} G_2(t,s)\left[b\beta M_2^{[2]}(s) - b^2\alpha\left(M_2^{[2]}(s)\right)^2\right] ds,$$

be a solution of the perturbed equation with a small perturbation in the adult mortality rate $p_1(t)$ where

$$G_1(t,s) = \frac{\exp\left(\int_t^s p_1(u)\,du\right)}{\exp\left(\int_0^w p_1(u)\,du\right) - 1},\ G_2(t,s) = \frac{\exp\left(\int_t^s p_2(u)\,du\right)}{\exp\left(\int_0^w p_2(u)\,du\right) - 1}.$$

Estimating the difference between $M_1(t)$ and $M_2(t)$, we get

$$\begin{aligned} |M_1(t) - M_2(t)| \leq & \int_t^{t+w} |G_1(t,s) - G_2(t,s)| \left|b\beta M_1^{[2]}(s) - b^2\alpha\left(M_1^{[2]}(s)\right)^2\right| ds \\ & + \int_t^{t+w} G_2(t,s)\left[\left|b\beta M_1^{[2]}(s) - b\beta M_2^{[2]}(s)\right|\right. \\ & \left. + \left|b^2\alpha\left(M_1^{[2]}(s)\right)^2 - b^2\alpha\left(M_2^{[2]}(s)\right)^2\right|\right] ds. \end{aligned}$$

Thanks to the mean value theorem, we obtain

$$\int_t^{t+w} |G_1(t,s) - G_2(t,s)|\, ds \leq \eta \|p_1 - p_2\|, \tag{3.3}$$

where

$$\eta = \frac{w^2 e^{w(\|p_2\| + \max(\|p_1\|,\|p_2\|))}}{\left(\exp\left(\int_0^w p_1(u)\,du\right) - 1\right)\left(\exp\left(\int_0^w p_2(u)\,du\right) - 1\right)} + \frac{w^2 e^{w\max(\|p_1\|,\|p_2\|)}}{\exp\left(\int_0^w p_1(u)\,du\right) - 1}.$$

It follows from (2.6), (3.3) and Lemma 2.5 that

$$\begin{aligned} |M_1(t) - M_2(t)| \leq & \, wb\lambda_1(\beta - b\alpha\lambda_1)\eta\|p_1 - p_2\| \\ & + \gamma_1 wb(1+L)(\beta + 2b\alpha\lambda_1)|M_1(t) - M_2(t)|. \end{aligned}$$

So

$$\|M_1 - M_2\| \leq \frac{wb\lambda_1\eta(\beta - b\alpha\lambda_1)}{1-\mu}\|p_1 - p_2\|.$$

This completes the proof. □

4 Examples

Here are two concrete examples illustrating Theorems 3.5, 3.6 and 3.7.

Example 4.1. Consider the following iterative houseflies model:

$$M'(t)+\left(0.025+0.024\sin^2\frac{2\pi t}{35}\right)M(t)=(0.05)\,(0.4)\,M^{[2]}\,(t)-(0.05)^2\,(0.000226)\left(M^{[2]}\,(t)\right)^2, \tag{4.1}$$

where

$$p\,(t)=0.025+0.024\sin^2\frac{2\pi t}{35},\ b=0.05,\ \beta=0.4 \text{ and } \alpha=0.000226.$$

Let

$$\Omega_1=\{M\in X:0<\lambda_0\leq M\,(t)\leq\lambda_1,\ \ |M\,(t_2)-M\,(t_1)|\leq L\,|t_2-t_1|\,,\ \forall t_1,t_2\in[0,w]\}\,,$$

where $w=35$, $\lambda_0=0.1$, $\lambda_1=0.8$ and $L=0.2$.
We define an integral operator $\mathcal{K}_1:\Omega_1\to X$ as follows:

$$(\mathcal{K}_1M)\,(t)=\int_t^{t+35}G\,(t,s)\left[(0.02)\,M^{[2]}\,(s)-\left(565\times10^{-9}\right)\left(M^{[2]}\,(s)\right)^2\right]ds,$$

where its kernel is the following Green's function:

$$G\,(t,s)=\frac{\exp\left(\int_t^{t+35}\left(0.025+0.024\sin^2\frac{2\pi u}{35}\right)du\right)}{\exp\left(\int_t^{t+35}\left(0.025+0.024\sin^2\frac{2\pi u}{35}\right)du\right)-1}.$$

Thanks to the periodic properties, the operator $\mathcal{K}_1$ is well-defined.

We have

$$p=0.049,\ \gamma_1\approx1.3772 \text{ and } \mu=\gamma_1wb\,(\beta+2b\alpha\lambda_1)\,(1+L)\approx1.1569>1.$$

Moreover, we have

$$\min_{s\in[0,35]}\left\{b\beta M^{[2]}\,(s)-b^2\alpha\left(M^{[2]}\,(s)\right)^2\right\}\approx1.9996\times10^{-3}>0,$$

which means that condition (2.1) is satisfied. And

$$\gamma_1wb\beta=0.96404<1,$$

which implies that condition (2.2) is fulfilled. We have also

$$(\gamma_1pw+2\gamma_1)\left(b\beta\lambda_1+b^2\alpha\lambda_1^2\right)\approx8.1863\times10^{-2}\leq L=0.2.$$

Then condition (2.1) is also satisfied.

Finally, we conclude that all conditions of Theorem 3.5 hold and hence equation (4.1) has at least one positive periodic solution in Ω_1. Indeed since conditions $(2.1)-(2.3)$ are satisfied, then Lemmas 3.2 and 3.4 show that $\mathcal{K}_1$ maps Ω_1 into itself. Furthermore, we get

$$|(\mathcal{K}_1M)\,(t)-(\mathcal{K}_1N)\,(t)|\leq1.1569\ \ \|M-N\|\,,$$

and therefore the continuity of the operator $\mathcal{K}_1$ results immediately afterwards. In addition, Arzelà-Ascoli theorem ensures the compactness of the departure set Ω_1 which, in turn, proves the compactness of the continuous operator $\mathcal{K}_1$. Therefore, we conclude by the Schauder's fixed point theorem that the operator $\mathcal{K}_1$ has at least one fixed point in Ω_1 which is a positive periodic solution of equation (4.1).
But

$$\mu=\gamma_1wb\,(\beta+2b\alpha\lambda_1)\,(1+L)\approx1.1569>1.$$

Therefore, Theorems 3.6 and 3.7 cannot be applied here. Indeed, the additional criterion (3.3) is not fulfilled and hence the solution of equation (4.1) is not necessarily unique and we cannot get any information about the stability of solutions.

Example 4.2. We consider the same previous iterative houseflies model (4.1) with

$$\Omega_2 = \{M \in X : 0 < \lambda_0 \leq M(t) \leq \lambda_1, \ |M(t_2) - M(t_1)| \leq L |t_2 - t_1|, \ \forall t_1, t_2 \in [0, w]\},$$

where $\lambda_0 = 0.01$, $\lambda_1 = 0.02$ and $L = 0.003$.
We define an integral operator $\mathcal{K}_2 : \Omega_2 \to X$ where $\mathcal{K}_1$ and $\mathcal{K}_2$ have the same expression and the same Green's kernel. We have

$$\begin{aligned} &\min_{s\in[0,35]} \left\{ b\beta M^{[2]}(s) - b^2\alpha \left(M^{[2]}(s)\right)^2 \right\} \approx 2 \times 10^{-4} > 0, \\ &\gamma_1 w b\beta = 0.96404 < 1, \\ &(\gamma_1 p w + 2\gamma_1)\left(b\beta\lambda_1 + b^2\alpha\lambda_1^2\right) \approx 2.0465 \times 10^{-3} \leq L = 0.003, \end{aligned}$$

and

$$\mu = \gamma_1 w b (\beta + 2b\alpha\lambda_1)(1 + L) = 0.96693 < 1.$$

Here conditions (2.1) − (2.3) and (3.3) are satisfied. Therefore, all requirements of Theorem 3.6 are fulfilled which means that equation (4.1) has a unique positive periodic solution in Ω. Indeed, this is due to the application of the Banach contraction principle which guarantees the existence of a unique fixed point of operator $\mathcal{K}_2$. Moreover, Theorem 3.7 ensures the continuous dependence on the adult mortality rate p of this unique solution. Indeed, if M is the unique solution of equation (4.1) and if M_2 is a solution of the following perturbed equation:

$$M_2'(t) + p_2(t) M_2(t) = b\beta M_2^{[2]}(t) - b^2\alpha \left(M_2^{[2]}(t)\right)^2,$$

with the perturbed parameter p_2. Then we get

$$\|M - M_2\| \leq 195.62 e^{(35)\max(0.049, \|p_2\|)} \left(e^{\|p_2\|} + 1\right) \|p - p_2\|,$$

which implies that the unique solution depends on the function p.

5 Conclusion and remarks

This article has considered a first order iterative differential equation describing the dynamics of a houseflies population where the iterative terms have resulted from a varying delay depending on the time and the population size of adult houseflies. By virtue of a powerful approach that combines the fixed point theory and the Green's functions method, some sufficient conditions ensuring the existence, uniqueness and stability of positive periodic solutions of a Musca Domestica model have been set up. One of our fundamental goals has been first and foremost to construct a Banach space and a subset of it for paving the way to the application of the chosen fixed point theorems in the one hand and for guaranteeing some mathematical and biological requirements in the other. Indeed, this choice has ensured the continuity, positivity, boundedness and periodicity of the solutions and also has allowed us to control the iterative terms and hence to avoid any expected hitches in our study. Next, we have converted our problem into an equivalent integral equation with a Green's kernel for applying the Schauder's fixed point theorem that has guaranteed the existence of at least one positive periodic solution. Then, by the help of Banach's fixed point theorem, an additional criterion has been found under which the solution has became unique and depended continuously on the adult mortality rate. In the end we have given two examples justifying the validity of the acquired theoretical findings.

The highlights of this paper are listed as follows:

(*i*) Some sufficient conditions have been derived to establish the existence, uniqueness and stability of positive periodic solutions. These findings are completely new and have extremely vital significance in studying the dynamics of insect populations.

(*ii*) Although there are some authors that have dealt with such problems (see for example [4] and [16]), as far as we know, houseflies model with iterative terms has not been studied till now. Thus, it was worthwhile to investigate in this direction.

(*iii*) The technique adopted here can also be used to study numerous delay models in plentiful fields and especially it is of considerable significance in handling many iterative models that appear frequently in life sciences such as disease transmission models, models for blood cell production, model for two-body problems of classical electrodynamics, population models, and so on and so forth.

The topic is vast and important, since such equations appear in different scientific disciplines (population dynamics, biology, medicine, classical electrodynamics, physics, epidemiology, hematology) and although the proofs of the existence, uniqueness and stability of solutions have been achieved, there are still some interesting problems relating to the following points:

(i) Generalize the findings of this article which could be generalized to an iterative houseflies model involving multiple delays, harvesting strategy, dispersal or competition.

(ii) Obtain existence, uniqueness and stability results for a neutral houseflies model with iterative terms.

(iii) Consider problem (1.1) with a nonlinear density-dependent adult mortality rate.

(iv) Extend the study to other insect populations.

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