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Additive and quadratic set-valued k-functional inequalities and their stability

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Abstract

The aims of this work are first solving set-valued functional inequalities involving additive and quadratic set-valued mappings, and second investigations of their stability by using the fixed point method.

Keywords: set-valued functional equation, set-valued functional inequality, stability
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1 Introduction

One of the well-known and elementary functional equations are equations of the following forms:

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \tag{1.2}$$

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.3}$$

These equations are known ([5, Chapter 1, 4]), respectively, as *additive, Jensen's* and *quadratic functional equations*. The mappings satisfying the functional equations (1.1), (1.2) and (1.3) are called *additive, Jensen*, and *quadratic mappings*, respectively.

In the article [9], Park treated the functional inequalities of the following forms:

$$\|f(x+y) - f(x) - f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\|, \tag{1.4}$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|\rho\left(f(x+y) - f(x) - f(y)\right)\|, \tag{1.5}$$

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where ρ is a fixed non–Archimedean number with $|\rho| < 1$. These above inequalities were generalized to

$$\left\| f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \right\| \leq \left\| \rho \left(kf\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\|, \tag{1.6}$$

$$\left\| kf\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j) \right\| \leq \left\| \rho \left(f\left(\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f(x_j)\right)\right\|, \tag{1.7}$$

where $k \geq 2$ is a fixed integer and ρ is a fixed nonzero complex number with $|\rho| < 1$, in 2015 [10]. The stabilities of functional inequalities (1.4), (1.5), (1.6), (1.7) are also established.

In another direction, there are many mathematicians have studied the stability theory of set–valued functional equations in various types. For all closed convex subsets A, B in a Banach space, we here denoted by

$$A \oplus B := \overline{A + B} \tag{1.8}$$

a closure of the set $A + B := \{a + b : a \in A, b \in B\}$. In 2019, Park et al. [11] introduced and solved the functional inequalities, which respective are the set–valued equation forms of (1.4) and (1.5), as follows:

$$h(F(x+y), F(x) \oplus F(y)) \leq \rho \cdot h\left(F\left(\frac{x+y}{2}\right) \oplus F(x) \oplus F(y)\right), \tag{1.9}$$

$$h\left(F\left(\frac{x+y}{2}\right) \oplus F(x) \oplus F(y)\right) \leq \rho \cdot h(F(x+y), F(x) \oplus F(y)), \tag{1.10}$$

where $\rho < 1$ is a positive real number and $h(A, B)$ is a Hausdorff distance between A and B. An analysis of their stability was also investigated. In 2017, Lee et al. [7] introduced the set–valued functional equation of the following form

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y). \tag{1.11}$$

They also proved the stability of this set–valued functional equation (1.11).

In the present work, we treat the following set–valued functional inequalities:

$$h(F(x+y), F(x) \oplus F(y)) \leq k_1 \cdot h(F(x+y) \oplus F(x-y), 2F(x)), \tag{1.12}$$

$$h(G(x+y) \oplus G(x-y), 2G(x)) \leq k_2 \cdot h(G(x+y), G(x) \oplus G(y)), \tag{1.13}$$

$$\begin{aligned} &h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) \\ &\leq k_3 \cdot h(H(x+y+z) \oplus H(x+y-z) \oplus H(y+z-x) \oplus H(z+x-y), \\ &\quad 4[H(x) \oplus H(y) \oplus H(z)]), \end{aligned} \tag{1.14}$$

$$\begin{aligned} &h(L(x+y+z) \oplus L(x+y-z) \oplus L(y+z-x) \oplus L(z+x-y), 4[L(x) \oplus_4 L(y) \oplus L(z)]) \\ &\leq k_4 \cdot h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)), \end{aligned} \tag{1.15}$$

where k_1, k_2, k_3, and k_4 are fixed positive real numbers with $k_1 < \frac{\sqrt{2}}{2}, k_2 < 1, k_3 < \frac{1}{2}$, and $k_4 < 2$. The first two functional inequalities (1.12), (1.13) are related to addtive mappings, while the remaining inequalities (1.14), (1.15) are involved to quadratic mappings. By using fixed point method, moreover, an analysis of their stability is also demonstrated.

As witnessed above inequalities (1.12), (1.13), (1.14), (1.15) are associated with the fixed constants k_1, k_2, k_3 and k_4, we shall refer collectively to functional inequalities like (1.12) or (1.13) as an *additive set–valued k–functional inequality*, and also call the functional inequalities like (1.14) or (1.15) as a *quadratic set–valued k–functional inequality*.

2 Preliminaries

In this section, we gather some elementary properties involving the set–valued functional equations as well as the fundamental theorem in fixed point theory, which are needed in our main results.

Throughout this paper, let X be a real vector space, and Y a real Banach space. Let k_1, k_2, k_3, and k_4 be fixed positive real numbers subjecting to $k_1 < \frac{\sqrt{2}}{2}, k_2 < 1, k_3 < \frac{1}{2}$, and $k_4 < 2$. Denote generically, unless otherwise specified, by $2^Y, C_b(Y), C_c(Y)$, and $C_{cb}(Y)$ the set of all subsets of Y, the set of all closed bounded subsets of Y, the set of all closed convex subsets of Y, and the set of all closed convex bounded subsets of Y, respectively. Define the *addition* and the *scalar multiplication* on 2^Y as follows:

$$A + B := \{x + y : \ x \in A, y \in B\} \quad \text{and} \quad \lambda A := \{\lambda x : \ x \in A\}, \tag{2.1}$$

where $A, B \in 2^Y$ and $\lambda \in \mathbb{R}$. Using the same notation as (1.8), it is easily checks that

$$\lambda A + \lambda B = \lambda(A + B) \quad \text{and} \quad (\lambda + \mu)A \subseteq \lambda A + \mu A \tag{2.2}$$

for all $A, B \in 2^Y$ and all $\lambda, \mu \in \mathbb{R}$. Furthermore, we have

(1) If $C \in 2^Y$ is convex, then

$$(\lambda + \mu)C = \lambda C + \mu C \tag{2.3}$$

for all $\lambda, \mu \in \mathbb{R}^+$.

(2) If $U, V, W \in C_{cb}(Y)$, then

$$U \oplus W = V \oplus W \quad \text{implies} \quad U = V. \tag{2.4}$$

For any pair $F, G \in C_b(Y)$, define the *Hausdorff distance* between F and G by

$$h(F, G) := \inf\{ \ \lambda > 0 \ : \ F \subseteq G + \lambda \mathbb{B}_Y, \ G \subseteq F + \lambda \mathbb{B}_Y\}, \tag{2.5}$$

where $\mathbb{B}_Y$ is the unit closed ball in Y.

Since Y is a Banach space, it is proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup ([1, Chapter II]). Moreover, $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space ([2]). Some properties, which are directly obtained from the Hausdorff distance, are revealed as follows.

Proposition 2.1 ([1]). For every $A, B, C, D, E \in C_{cb}(Y)$ and $\lambda > 0$, we have

(i) $h(A \oplus B, C \oplus D) \le h(A, C) + h(B, D)$;
(ii) $h(\lambda A, \lambda B) = \lambda h(A, B)$;
(iii) $h(A, B) = h(A \oplus E, B \oplus E)$.

We next give the definition of a generalized metric space as well as the fundamental theorem in the fixed point theory which is related to the present work.

Definition 2.2 ([8]). Let X be a set. A function $d : X \times X \to [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions:

(D1) $d(x, y) = 0$ if and only if $x = y$;
(D2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(D3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The ordered pair (X, d) is also called a *generalized metric space*.

Note that the distinction between the generalized metric and the metric is that the range of the former is permitted to include the infinity. Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a *Lipschitz condition* with a *Lipschitz constant* L if there exists a constant $L \ge 0$ such that

$$d(Tx, Ty) \le L \ d(x, y) \tag{2.6}$$

for all $x, y \in X$. If the Lipschitz constant $L < 1$, then the operator T is called *a strictly contractive mapping*.

The following result is the well–known fundamental theorem in the fixed point theory.

Theorem 2.3 ([3]). Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that the following assertions hold:

(1) $d(J^n x, J^{n+1}x) < \infty \ \ \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \ : \ d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq \left(\frac{1}{1-\alpha}\right) d(y, Jy)$ for all $y \in Y$.

3 Additive and Quadratic Set–Valued k–Functional Inequalities

In this section, the function solutions satisfying the functional inequalities (1.12), (1.13), (1.14), and (1.15) are determined. Our results are involved the use of additive and quadratic set–valued mappings which are defined as follows.

Definition 3.1. Let $f, g : X \to (C_{cb}(Y), \oplus)$ be set–valued mappings.

(i) ([6], [11]) The *additive set–valued functional equation* is defined by

$$f(x+y) = f(x) \oplus f(y) \tag{3.1}$$

for all $x, y \in X$. Every solution of (3.1) is called a *additive set–valued mapping.*

(ii) ([7]) The *quadratic set–valued functional equation* is defined by

$$g(x+y) \oplus g(x-y) = 2g(x) \oplus 2g(y) \tag{3.2}$$

for all $x, y \in X$. Every solution of (3.2) is called a *quadratic set–valued mapping.*

Theorem 3.2. Let $F, G : X \to (C_{cb}(Y), \oplus)$ be set–valued mappings with $G(0) = \{0\}$. Then F and G satisfy the following functional inequalities

$$h(F(x+y), F(x) \oplus F(y)) \ \leq \ k_1 h(F(x+y) \oplus F(x-y), 2F(x)), \tag{3.3}$$
$$h(G(x+y) \oplus G(x-y), 2G(x)) \ \leq \ k_2 h(G(x+y), G(x) \oplus G(y)) \tag{3.4}$$

for all $x, y \in X$, where $k_1 < \frac{\sqrt{2}}{2}$ and $k_2 < 1$ are fixed positive real numbers, if and only if F and G are additive set–valued mappings.

Proof . To solve (3.3), putting $x = y = 0$ into (3.3), we have

$$h(F(0), 2F(0)) \leq k_1 h(2F(0), 2F(0)) = 0 \tag{3.5}$$

since $F(0)$ is convex, yielding that $F(0) = \{0\}$. Next, setting $x = y$ in (3.3), we get

$$(1 - k_1) h(F(2x), 2F(x)) \leq 0. \tag{3.6}$$

Since $k_1 < \frac{\sqrt{2}}{2}$, we must have $h(F(2x), 2F(x)) = 0$, and so

$$F\left(\frac{x}{2}\right) = \frac{1}{2} F(x) \tag{3.7}$$

for all $x \in X$.

Letting $t = x + y$ and $s = x - y$, the inequality (3.3) becomes

$$h\left(F(t), \frac{1}{2}F(t+s) \oplus \frac{1}{2}F(t-s)\right) \leq k_1 h\left(F(t) \oplus F(s), F(t+s)\right) \tag{3.8}$$

by using (3.7), so that

$$h\left(2F(t), F(t+s) \oplus F(t-s)\right) \leq 2k_1 h\left(F(t) \oplus F(s), F(t+s)\right) \tag{3.9}$$

for all $t, s \in X$. Using (3.3) and (3.9), we see that

$$h(F(x+y), F(x) \oplus F(y)) \leq 2k_1^2 h(F(x) \oplus F(y), F(x+y)), \tag{3.10}$$

and so

$$(1-2k_1^2)h(F(x+y), F(x) \oplus F(y)) \leq 0 \tag{3.11}$$

for all $x, y \in X$. Since $k_1 < \frac{\sqrt{2}}{2}$, we get

$$h(F(x+y), F(x) \oplus F(y)) = 0, \tag{3.12}$$

yielding that

$$F(x+y) = F(x) \oplus F(y) \tag{3.13}$$

for all $x, y \in X$, as desired.

To solve (3.4), setting $t = x+y$ and $s = x-y$ in (3.4), we obtain

$$h\left(G(t) \oplus G(s), 2G\left(\frac{t+s}{2}\right)\right) \leq k_2 h\left(G(t), G\left(\frac{t+s}{2}\right) \oplus G\left(\frac{t-s}{2}\right)\right) \tag{3.14}$$

for all $t, s \in X$. Putting $s = 0$ in (3.14), by using $G(0) = \{0\}$ and the convexity of range of G, we have

$$h\left(G(t), 2G\left(\frac{t}{2}\right)\right) \leq k_2 h\left(G(t), 2G\left(\frac{t}{2}\right)\right) \tag{3.15}$$

and so

$$(1-k_2)h\left(G(t), 2G\left(\frac{t}{2}\right)\right) \leq 0 \tag{3.16}$$

giving that

$$G(t) = 2G\left(\frac{t}{2}\right) \tag{3.17}$$

for all $t \in X$, since $k_2 < 1$. Taking (3.17) back into (3.14) and using (3.4), we see that

$$\begin{aligned} h(G(t) \oplus G(s), G(t+s)) &\leq k_2 h\left(G(t), \frac{1}{2}[G(t+s) \oplus G(t-s)]\right) \\ &= \frac{k_2}{2} h\left(2G(t), G(t+s) \oplus G(t-s)\right) \\ &\leq \frac{k_2^2}{2} h\left(G(t+s), G(t) \oplus G(s)\right), \end{aligned}$$

so that

$$\left(1 - \frac{k_2^2}{2}\right) h(G(t) \oplus G(s), G(t+s)) \leq 0 \tag{3.18}$$

for all $t, s \in X$. Since $k_2 < 1$, the inequality (3.18) yields

$$h(G(t) \oplus G(s), G(t+s)) = 0, \tag{3.19}$$

and the additivity of G henceforth follows immediately. The converse is obviously holds. □

Theorem 3.3. Let $H, L : X \to (C_{cb}(Y), \oplus)$ be set–valued mappings with $H(0) = \{0\}$ and $L(0) = \{0\}$. Then H and L satisfy the following functional inequalities

$$\begin{aligned} & h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) \\ \leq \; & k_3 h(H(x+y+z) \oplus H(x+y-z) \oplus H(y+z-x) \oplus H(z+x-y), \\ & 4[H(x) \oplus H(y) \oplus H(z)]), \end{aligned} \tag{3.20}$$

$$\begin{aligned} & h(L(x+y+z) \oplus L(x+y-z) \oplus L(y+z-x) \oplus L(z+x-y), 4[L(x) \oplus L(y) \oplus L(z)]) \\ \leq \; & k_4 h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)) \end{aligned} \tag{3.21}$$

for all $x, y, z \in X$, where $k_3 < \frac{1}{2}$ and $k_4 < 2$ are fixed positive real numbers, if and only if H and L are quadratic set–valued mappings.

Proof . If H and L are quadratic mappings, then the inequalities (3.20) and (3.21) clearly hold.

Conversely, to solve (3.20), taking $x = z = 0$ in (3.20) and using the convexity of range of H, we have

$$h(H(y) \oplus H(-y), 2H(y)) \leq k_3 h(3H(y) \oplus H(-y), 4Hy)) \tag{3.22}$$

which equivalent to

$$h(H(y) \oplus H(-y), H(y) \oplus H(y)) \leq k_3 h(3H(y) \oplus H(-y), 3H(y) \oplus H(y)) \tag{3.23}$$

for all $y \in X$. By virtue of Proposition 2.1 (iii), the inequality (3.23) goes over into

$$h(H(-y), H(y)) \leq k_3 h(H(-y), H(y)), \tag{3.24}$$

and so $(1 - k_3)h(H(-y), H(y)) \leq 0$. Since $k_3 < \frac{1}{2}$, we must have

$$H(y) = H(-y) \tag{3.25}$$

for all $y \in X$. Putting $z = 0$ in (3.20) and using (3.25), one obtains

$$\begin{aligned} h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) &\leq k_3 h(2H(x+y) \oplus H(y-x) \oplus H(x-y), 4[H(x) \oplus H(y)]) \\ &= k_3 h(2H(x+y) \oplus 2H(x-y), 4[H(x) \oplus H(y)]) \\ &= 2k_3 h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) \end{aligned}$$

so that

$$(1 - 2k_3)h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) \leq 0 \tag{3.26}$$

for all $x, y \in X$. Since $k_3 < \frac{1}{2}$, we get

$$h(H(x+y) \oplus H(x-y), 2H(x) \oplus 2H(y)) = 0 \tag{3.27}$$

and the result follows.

To solve (3.21), taking $y = z = 0$ in (3.21), we get

$$h(2L(x) \oplus L(-x) \oplus L(x), 4L(x)) \leq k_4 h(2Lx), 2L(x)) = 0 \tag{3.28}$$

giving that $h(L(-x), L(x)) = 0$. Thus,

$$L(x) = L(-x) \tag{3.29}$$

for all $x \in X$. Letting $z = 0$ in (3.21) and using (3.29), we see that

$$2h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)) \leq k_4 h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)) \tag{3.30}$$

so that

$$(2 - k_4)h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)) \leq 0 \tag{3.31}$$

for all $x, y \in X$. Since $k_4 < 2$, we have

$$h(L(x+y) \oplus L(x-y), 2L(x) \oplus 2L(y)) = 0 \tag{3.32}$$

and hence

$$L(x+y) \oplus L(x-y) = 2L(x) \oplus 2L(y) \tag{3.33}$$

for all $x, y \in X$, proving that L is a quadratic. $\square$

4 Stability

In this section, we investigate first the stability of the additive set–valued k–functional inequalities (1.12) and (1.13), and second of the quadratic set–valued k–functional inequalities (1.14) and (1.15).

4.1 Stability of Additive Set–Valued k–Functional Inequalities

Theorem 4.1. Let $\varphi : X^2 \to [0, \infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{4.1}$$

for all $x, y \in X$. If $f : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h\left(f(x+y), f(x) \oplus f(y)\right) \leq k_1 h(f(x+y) \oplus f(x-y), 2f(x)) + \varphi(x, y) \tag{4.2}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $F : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(f(x), F(x)\right) \leq \left(\frac{1}{2(1-k_1)(1-L)}\right)\varphi(x, x) \tag{4.3}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 1$ and M such that

$$\operatorname{diam}\, f(x) \leq M \left\|x\right\|_X^r$$

for all $x \in X$, then $F(x)$ is a singleton set.

Proof . Putting $y = x$ into (4.2) and using the convexity of range of f with $f(0) = \{0\}$, we have

$$h\left(f(2x), 2f(x)\right) \leq k_1 h(f(2x), 2f(x)) + \varphi(x, x) \tag{4.4}$$

for all $x \in X$, and so

$$h\left(f(x), \frac{1}{2}f(2x)\right) \leq \left(\frac{1}{2(1-k_1)}\right)\varphi(x, x) \tag{4.5}$$

for all $x \in X$. Constructing the set

$$S := \{g : X \to C_{cb}(Y) \mid g(0) = \{0\}\}$$

with the generalized metric

$$d(g, h) := \inf\{\mu \in (0, \infty) \mid h(g(x), f(x)) \leq \mu \cdot \varphi(x, x) \ \ \forall x \in X\},$$

where, as usual, $\inf \emptyset = +\infty$. Then it is easily shows that the pair (S, d) is a complete generalized metric space; see also [4]. Define a linear mapping $J : S \to S$ by

$$Jg(x) = \frac{1}{2}g\left(2x\right) \tag{4.6}$$

for all $x \in X$ and for each $g \in S$. Then for given $g, f \in G$ with $d(g, f) = \varepsilon$, we have

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, x)$$

and so

$$h(Jg(x), Jf(x)) = h\left(\frac{1}{2}g\left(2x\right), \frac{1}{2}f(2x)\right) = \frac{1}{2}h(g(2x), f(2x)) \leq \frac{\varepsilon}{2}\varphi(2x, 2x) \leq \varepsilon L\varphi(x, x)$$

yielding that $d(Jg, Jf) \leq \varepsilon L$, and we arrived at

$$d(Jg, Jf) \leq Ld(g, f) \tag{4.7}$$

for all $g, f \in S$. This means that J is a contractive mapping with a contractive constant $L < 1$. Moreover, the inequalities (4.5) and (4.6) give

$$h(f(x), Jf(x)) \leq \left(\frac{1}{2(1-k_1)}\right)\varphi(x, x), \tag{4.8}$$

and so

$$d(f, Jf) \leq \frac{1}{2(1-k_1)}. \tag{4.9}$$

Theorem 2.3 henceforth now applicable and implies that there exists a mapping $F : X \to (C_{cb}(Y), \oplus, h)$ satisfying the following assertions:

(i) F is a fixed point of J, i.e., $F(x) = JF(x) = \frac{1}{2}F(2x)$ and so

$$F(2x) = 2F(x) \tag{4.10}$$

for all $x \in X$. Moreover, the mapping F is unique in the set

$$K := \{g \in S \ : \ d(f,g) < \infty\}. \tag{4.11}$$

This means that F is a unique mapping satisfying (4.10) and

$$h(f(x), F(x)) \leq C\varphi(x,x)$$

for some $C \in (0,\infty)$ and all $x \in X$;

(ii) $d(J^n f, F) \to 0$ as $n \to \infty$. This gives

$$F(x) = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n f\left(2^n x\right) \tag{4.12}$$

for all $x \in X$;

(iii) $d(f,F) \leq \left(\frac{1}{1-L}\right) d(f, Jf)$. This implies that

$$d(f(x), F(x)) \leq \frac{1}{2(1-k_1)(1-L)} \tag{4.13}$$

for all $x \in X$, and the desired (4.3) follows.

To show that a mapping F is an additive set–valued mapping, using (4.1), (4.2), (4.12), and Proposition 2.1 we obtain

$$\begin{aligned} h(F(x+y), F(x) \oplus F(y)) &= h\left(\lim_{n\to\infty} \frac{1}{2^n} f(2^n(x+y)), \lim_{n\to\infty} \frac{1}{2^n} f(2^n y) \oplus \lim_{n\to\infty} \frac{1}{2^n} f(2^n y)\right) \\ &= \lim_{n\to\infty} \frac{1}{2^n} h(f(2^n x + 2^n y), f(2^n x) \oplus f(2^n y)) \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \left[k_1 h(f(2^n x + 2^n y) \oplus f(2^n x - 2^n y), 2f(2^n x)) + \varphi(2^n x + 2^n y)\right] \\ &\leq k_1 h\left(\lim_{n\to\infty} \frac{1}{2^n} f(2^n x + 2^n y) \oplus \lim_{n\to\infty} \frac{1}{2^n} f(2^n x - 2^n y), 2\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)\right) \\ &\quad + \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) \\ &\leq k_1 h(F(x+y) \oplus F(x-y), 2F(x)) + \lim_{n\to\infty} \frac{1}{2^n} \cdot 2^n \cdot L^n \cdot \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \end{aligned}$$

and so

$$h(F(x+y), F(x) \oplus F(y)) \leq k_1 h(F(x+y) \oplus F(x-y), 2F(x))$$

for all $x \in X$. Theorem 3.2 hence now implies that F is an additive.

Moreover, let $r, M \in \mathbb{R}^+$ with $r < 1$ and diam $f(x) \leq M \|x\|_X^r$. Then

$$\operatorname{diam}\left(\frac{1}{2^n} f(2^n x)\right) \leq \frac{1}{2^n} M \|2^n x\|_X^r = 2^{-n(1-r)} M \|x\|_X^r .$$

Since $2^{-(1-r)} < 1$,

$$\lim_{n\to\infty} \operatorname{diam}\left(\frac{1}{2^n} f(2^n x)\right) \leq \lim_{n\to\infty} 2^{-n(1-r)} M \|x\|_X^r = 0$$

showing that $F(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ is a singleton set, and Theorem 4.1 is proved. $\square$

Corollary 4.2. Let $0 < p < 1$ and $\theta \geq 0$ be real numbers. If $f : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h\left(f(x+y), f(x) \oplus f(y)\right) \leq k_1 h(f(x+y) \oplus f(x-y), 2f(x)) + \theta\left(\|x\|_X^p + \|y\|_X^p\right) \tag{4.14}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $F : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(f(x), F(x)\right) \leq \left(\frac{2\theta}{(1-k_1)(2-2^p)}\right)\|x\|_X^p \tag{4.15}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\operatorname{diam} f(x) \leq M\|x\|_X^p$$

for all $x \in X$, then $F(x)$ is a singleton set.

Proof . Define $\varphi(x, y) := \theta\left(\|x\|_X^p + \|y\|_X^p\right)$. Then

$$\varphi\left(\frac{x}{2}, \frac{x}{2}\right) = \theta\left(\left\|\frac{x}{2}\right\|_X^p + \left\|\frac{x}{2}\right\|_X^p\right) = \frac{1}{2^p}\varphi(x, x),$$

and so

$$\varphi(x, x) = 2^p\varphi\left(\frac{x}{2}, \frac{x}{2}\right) = 2 \cdot 2^{p-1} \cdot \varphi\left(\frac{x}{2}, \frac{x}{2}\right).$$

Choosing $L = 2^{p-1}$, then

$$h(f(x), F(x)) \leq \left(\frac{1}{2(1-k_1)(1-L)}\right)\varphi(x, x) = \left(\frac{2\theta}{(1-k_1)(2-2^p)}\right)\|x\|_X^p$$

for all $x \in X$, as desired. □

Theorem 4.3. Let $\varphi : X^2 \to [0, \infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y) \tag{4.16}$$

for all $x, y \in X$. If $f : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h\left(f(x+y), f(x) \oplus f(y)\right) \leq k_1 h(f(x+y) \oplus f(x-y), 2f(x)) + \varphi(x, y) \tag{4.17}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $F : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(f(x), F(x)\right) \leq \left(\frac{L}{2(1-k_1)(1-L)}\right)\varphi(x, x) \tag{4.18}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 1$ and M such that

$$\operatorname{diam} f(x) \leq M\|x\|_X^r$$

for all $x \in X$, then $F(x)$ is a singleton set.

Proof . Setting $y = x$ in (4.17), we obtain

$$h(f(2x), 2f(x)) \leq k_1 h(f(2x), 2f(x)) + \varphi(x, x),$$

since range of f is convex, and so

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \leq \left(\frac{1}{1-k_1}\right)\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{4.19}$$

for all $x \in X$. Using the same arguments as in the proof of Theorem 4.1, constructing the set

$$S := \{g : X \to C_{cb}(Y) \mid g(0) = \{0\}\}$$

with its generalized metric

$$d(g,h) := \inf\{\mu \in (0,\infty) \mid h(g(x), f(x)) \le \mu \cdot \varphi(x,x) \quad \forall x \in X\},$$

where, as usual, $\inf \emptyset = +\infty$. Then the pair (S,d) is a complete generalized metric space; see also [4]. Also, define a linear mapping $\hat{J} : S \to S$ by

$$\hat{J}g(x) = 2g\left(\frac{x}{2}\right) \tag{4.20}$$

for all $x \in X$ and for each $g \in S$. Then it is easily verify that $\hat{J}$ is a contractive mapping; indeed, if $d(g,f) = \varepsilon$, then one can show that $d(\hat{J}g, \hat{J}f) \le Ld(g,f)$ by using (4.16) and (4.20). Moreover, the inequalities (4.16) and (4.19) give

$$d(f, \hat{J}f) \le \frac{L}{2(1-k_1)}. \tag{4.21}$$

The rest of the proof of the theorem is similar to that of Theorem 4.1. □

Corollary 4.4. Let $p > 1$ and $\theta \ge 0$ be real numbers. If $f : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h\left(f(x+y), f(x) \oplus f(y)\right) \le k_1 h(f(x+y) \oplus f(x-y), 2f(x)) + \theta\left(\|x\|_X^p + \|y\|_X^p\right) \tag{4.22}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $F : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(f(x), F(x)\right) \le \left(\frac{2\theta}{(1-k_1)(2^p-2)}\right) \|x\|_X^p \tag{4.23}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\operatorname{diam} f(x) \le M \|x\|_X^p$$

for all $x \in X$, then $F(x)$ is a singleton set.

Proof . Taking $\varphi(x,y) := \theta\left(\|x\|_X^p + \|y\|_X^p\right)$ and choosing $L = 2^{1-p}$, we have

$$h(f(x), F(x)) \le \left(\frac{L}{2(1-k_1)(1-L)}\right) \varphi(x,x) = \left(\frac{2\theta}{(1-k_1)(2^p-2)}\right) \|x\|_X^p$$

for all $x \in X$, as desired. □

Using the same arguments as in the proofs of Theorems 4.1 and 4.3, and using Theorem 3.2, it is easily verify that the following results hold whose analogous proofs are omitted.

Theorem 4.5. Let $\varphi : X^2 \to [0,\infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{4.24}$$

for all $x, y \in X$. If $g : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $g(0) = \{0\}$ and

$$h\left(g(x+y) \oplus g(x-y), 2g(x)\right) \le k_2 h(g(x+y), g(x) \oplus g(y)) + \varphi(x,y) \tag{4.25}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $G : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(g(x), G(x)\right) \le \left(\frac{1}{2(1-k_2)(1-L)}\right) \varphi(x,x) \tag{4.26}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 1$ and M such that

$$\operatorname{diam} g(x) \le M \|x\|_X^r$$

for all $x \in X$, then $G(x)$ is a singleton set.

Corollary 4.6. Let $0 < p < 1$ and $\theta \geq 0$ be real numbers. If $g : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $g(0) = \{0\}$ and

$$h\left(g(x+y) \oplus g(x-y), 2g(x)\right) \leq k_2 h(g(x+y), g(x) \oplus g(y)) + \theta\left(\|x\|_X^p + \|y\|_X^p\right) \tag{4.27}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $G : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(g(x), G(x)\right) \leq \left(\frac{2\theta}{(1-k_2)(2-2^p)}\right) \|x\|_X^p \tag{4.28}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\text{diam } g(x) \leq M \|x\|_X^p$$

for all $x \in X$, then $G(x)$ is a singleton set.

Theorem 4.7. Let $\varphi : X^2 \to [0, \infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x, y) \leq \frac{L}{2} \varphi\left(2x, 2y\right) \tag{4.29}$$

for all $x, y \in X$. If $g : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $g(0) = \{0\}$ and

$$h\left(g(x+y) \oplus g(x-y), 2g(x)\right) \leq k_2 h(g(x+y), g(x) \oplus g(y)) + \varphi(x, y) \tag{4.30}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $G : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(g(x), G(x)\right) \leq \left(\frac{L}{2(1-k_2)(1-L)}\right) \varphi(x, x) \tag{4.31}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 1$ and M such that

$$\text{diam } g(x) \leq M \|x\|_X^r$$

for all $x \in X$, then $G(x)$ is a singleton set.

Corollary 4.8. Let $p > 1$ and $\theta \geq 0$ be real numbers. If $g : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $g(0) = \{0\}$ and

$$h\left(g(x+y) \oplus g(x-y), 2g(x)\right) \leq k_2 h(g(x+y), g(x) \oplus g(y)) + \theta\left(\|x\|_X^p + \|y\|_X^p\right) \tag{4.32}$$

for all $x, y \in X$, then there exists a unique additive set–valued mapping $G : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(g(x), G(x)\right) \leq \left(\frac{2\theta}{(1-k_2)(2^p-2)}\right) \|x\|_X^p \tag{4.33}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\text{diam } g(x) \leq M \|x\|_X^p$$

for all $x \in X$, then $G(x)$ is a singleton set.

4.2 Stability of Quadratic Set–Valued k–Functional Inequalities

We now arrive at the investigations of the stability of inequalities (1.14), (1.15) involving the quadratic mappings.

Theorem 4.9. Let $\varphi : X^3 \to [0, \infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x, y, z) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{4.34}$$

for all $x, y, z \in X$. If $n : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $n(0) = \{0\}$ and

$$\begin{aligned} &h\left(n(x+y) \oplus n(x-y), 2n(x) \oplus 2n(y)\right) \\ &\leq k_3 h(n(x+y+z) \oplus n(x+y-z) \oplus n(y+z-x) \oplus n(x+z-y), 4[n(x) \oplus n(y) \oplus n(z)]) + \varphi(x, y, z) \end{aligned} \tag{4.35}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $H : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(n(x), H(x)\right) \leq \left(\frac{1}{4(1-2k_3)(1-L)}\right) \varphi(x, x, 0) \tag{4.36}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 2$ and M such that

$$\operatorname{diam} n(x) \leq M \left\|x\right\|_X^r$$

for all $x \in X$, then $H(x)$ is a singleton set.

Proof . Letting $y = x$ and $z = 0$ in (4.35), by the convexity of range of n and $n(0) = \{0\}$, we have

$$h\left(n(2x), 4n(x)\right) \leq k_3 h(2n(2x), 8n(x)) + \varphi(x, x, 0), \tag{4.37}$$

and so

$$(1-2k_3)h(n(2x), 4n(x)) \leq \varphi(x, x, 0) \tag{4.38}$$

for all $x \in X$. That is,

$$h\left(n(2x), 4n(x)\right) \leq \left(\frac{1}{1-2k_3}\right) \varphi(x, x, 0) \tag{4.39}$$

for all $x \in X$. Constructing the set

$$S := \{g : X \to C_{cb}(Y) \mid g(0) = \{0\}\}$$

with its generalized metric

$$d(g, h) := \inf\{\mu \in (0, \infty) \mid h(g(x), f(x)) \leq \mu \cdot \varphi(x, x, 0) \;\; \forall x \in X\},$$

where, as usual, $\inf \emptyset = +\infty$. Then it is easy to show that the pair (S, d) is a complete generalized metric space; see also [4]. Next, define a linear mapping $J : S \to S$ by

$$Jg(x) = \frac{1}{4} g\left(2x\right) \tag{4.40}$$

for all $x \in X$ and for each $g \in S$. Then for given $g, f \in S$ with $d(g, f) = \varepsilon$, we have

$$h(g(x), f(x)) \leq \varepsilon \varphi(x, x, 0)$$

and so

$$h(Jg(x), Jf(x)) = h\left(\frac{1}{4} g\left(2x\right), \frac{1}{4} f(2x)\right) = \frac{1}{4} h(g(2x), f(2x)) \leq \frac{\varepsilon}{4} \varphi(2x, 2x, 0) \leq \varepsilon L \varphi(x, x, 0)$$

yielding that $d(Jg, Jf) \leq \varepsilon L$. We now arrived at

$$d(Jg, Jf) \leq L d(g, f) \tag{4.41}$$

for all $g, f \in S$, showing that J is a contractive mapping with a contractive constant $L < 1$. Moreover, the inequalities (4.39) and (4.40) give

$$h(n(x), Jn(x)) \leq \left(\frac{1}{4(1-2k_3)}\right) \varphi(x, x, 0), \tag{4.42}$$

and so

$$d(n, Jn) \leq \frac{1}{4(1-2k_3)}. \tag{4.43}$$

Theorem 2.3 thus now ensures that there exists a mapping $H : X \to (C_{cb}(Y), \oplus, h)$ satisfying the following assertions:

(i) H is a fixed point of J, i.e., $H(x) = JH(x) = \frac{1}{4}H(2x)$ and so

$$H(2x) = 4H(x) \tag{4.44}$$

for all $x \in X$. Moreover, the mapping H is unique in the set

$$K := \{g \in S \ : \ d(f, g) < \infty\}. \tag{4.45}$$

This means that H is a unique mapping, which satisfies (4.44), in which there exists $C \in (0, \infty)$ such that

$$h(n(x), H(x)) \leq C\varphi(x, x, 0)$$

for all $x \in X$;

(ii) $d(J^k n, H) \to 0$ as $k \to \infty$. This gives

$$H(x) = \lim_{k\to\infty} \left(\frac{1}{4}\right)^k n\left(2^k x\right) \tag{4.46}$$

for all $x \in X$;

(iii) $d(n, H) \leq \left(\frac{1}{1-L}\right) d(n, Jn)$. This implies that

$$d(n(x), H(x)) \leq \frac{1}{4(1-2k_3)(1-L)} \tag{4.47}$$

for all $x \in X$, and the desired (4.36) follows.

The rest of the proof of the theorem is similar to that of Theorem 4.1. □

Corollary 4.10. Let $0 < p < 2$ and $\theta \geq 0$ be real numbers. If $n : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $n(0) = \{0\}$ and

$$\begin{aligned} &h\left(n(x+y) \oplus n(x-y), 2n(x) \oplus 2n(y)\right) \\ &\leq k_3 h(n(x+y+z) \oplus n(x+y-z) \oplus n(y+z-x) \oplus n(x+z-y), 4[n(x) \oplus n(y) \oplus n(z)]) \\ &\quad + \theta\left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p\right) \end{aligned} \tag{4.48}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $H : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(n(x), H(x)\right) \leq \left(\frac{2\theta}{(1-2k_3)(4-2^p)}\right) \|x\|_X^p \tag{4.49}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\operatorname{diam} n(x) \leq M \|x\|_X^p$$

for all $x \in X$, then $H(x)$ is a singleton set.

Proof . Taking $\varphi(x, y, z) := \theta\left(\|x\|_X^p + \|y\|_X^p + \|x\|_X^p\right)$ and choosing $L = 2^{p-2}$, the assertion follows immediately. □

Theorem 4.11. Let $\varphi : X^3 \to [0, \infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x, y, z) \leq \frac{L}{4}\varphi\left(2x, 2y, 2z\right) \tag{4.50}$$

for all $x, y, z \in X$. If $n : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $n(0) = \{0\}$ and

$$\begin{aligned} &h\left(n(x+y) \oplus n(x-y), 2n(x) \oplus 2n(y)\right) \\ &\leq k_3 h(n(x+y+z) \oplus n(x+y-z) \oplus n(y+z-x) \oplus n(x+z-y), 4[n(x) \oplus n(y) \oplus n(z)]) \\ &\quad + \varphi(x, y, z) \end{aligned} \tag{4.51}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $H : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(n(x), H(x)\right) \leq \left(\frac{L}{4(1-2k_3)(1-L)}\right)\varphi(x,x,0) \tag{4.52}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 2$ and M such that

$$\operatorname{diam} n(x) \leq M\left\|x\right\|_X^r$$

for all $x \in X$, then $H(x)$ is a singleton set.

Proof . Putting $y = x$ and $z = 0$ into (4.51), we obtain

$$h\left(n(x), 4n\left(\frac{x}{2}\right)\right) \leq \left(\frac{1}{1-2k_3}\right)\varphi\left(\frac{x}{2},\frac{x}{2},0\right) \tag{4.53}$$

for all $x \in X$. Using the same arguments as in the proof of Theorem 4.9, constructing the set

$$S := \{g : X \to C_{cb}(Y) \mid g(0) = \{0\}\}$$

with its generalized metric

$$d(g,h) := \inf\{\mu \in (0,\infty) \mid h(g(x), f(x)) \leq \mu \cdot \varphi(x,x,0) \ \ \forall x \in X\},$$

where, as usual, $\inf \emptyset = +\infty$. Then the pair (S,d) is a complete generalized metric space; see also [4]. Also, define a linear mapping $\hat{J} : S \to S$ by

$$\hat{J}g(x) = 4g\left(\frac{x}{2}\right) \tag{4.54}$$

for all $x \in X$ and for each $g \in S$. Then it is easily show that $\hat{J}$ is a contractive mapping with the contractive constant $L < 1$; indeed, if $d(g,f) = \varepsilon$, then one can show that $d(\hat{J}g, \hat{J}f) \leq Ld(g,f)$. Moreover, the inequalities (4.50) and (4.53) give

$$d(n, \hat{J}n) \leq \frac{L}{4(1-2k_3)}. \tag{4.55}$$

The rest of the proof of the theorem is similar to that of Theorem 4.1. □

Corollary 4.12. Let $p > 2$ and $\theta \geq 0$ be real numbers. If $n : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying

$$\begin{aligned} &h\left(n(x+y) \oplus n(x-y), 2n(x) \oplus 2n(y)\right) \\ &\leq k_3 h(n(x+y+z) \oplus n(x+y-z) \oplus n(y+z-x) \oplus n(x+z-y), 4[n(x) \oplus n(y) \oplus n(z)]) \\ &\quad + \theta\left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p\right) \end{aligned} \tag{4.56}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $H : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(n(x), H(x)\right) \leq \left(\frac{2\theta}{(1-2k_3)(2^p-4)}\right)\|x\|_X^p \tag{4.57}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\operatorname{diam} n(x) \leq M\left\|x\right\|_X^p$$

for all $x \in X$, then $H(x)$ is a singleton set.

Proof . Taking $\varphi(x,y,z) := \theta\left(\|x\|_X^p + \|y\|_X^p + \|x\|_X^p\right)$ and choosing $L = 2^{2-p}$, the assertion follows immediately. □

Using the same arguments as in the proofs of Theorems 4.9 and 4.11, and using Theorem 3.3, it is easily verify that the following results clearly hold whose analogous proofs are omitted.

Theorem 4.13. Let $\varphi : X^3 \to [0,\infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x,y,z) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{4.58}$$

for all $x, y, z \in X$. If $\ell : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $\ell(0) = \{0\}$ and

$$\begin{aligned} &h(\ell(x+y+z) \oplus \ell(x+y-z) \oplus \ell(y+z-x) \oplus \ell(z+x-y), 4[\ell(x) \oplus_4 \ell(y) \oplus \ell(z)]) \\ &\leq\ k_4 h(\ell(x+y) \oplus \ell(x-y), 2\ell(x) \oplus 2\ell(y)) + \varphi(x,y,z) \end{aligned} \tag{4.59}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $L : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(\ell(x), L(x)\right) \leq \left(\frac{1}{4(2-k_4)(1-L)}\right) \varphi(x,x,0) \tag{4.60}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 2$ and M such that

$$\operatorname{diam} \ell(x) \leq M \left\|x\right\|_X^r$$

for all $x \in X$, then $L(x)$ is a singleton set.

Corollary 4.14. Let $p < 2$ and $\theta \geq 0$ be real numbers. If $f : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $\ell(0) = \{0\}$ and

$$\begin{aligned} &h(\ell(x+y+z) \oplus \ell(x+y-z) \oplus \ell(y+z-x) \oplus \ell(z+x-y), 4[\ell(x) \oplus_4 \ell(y) \oplus \ell(z)]) \\ &\leq\ k_4 h(\ell(x+y) \oplus \ell(x-y), 2\ell(x) \oplus 2\ell(y)) + \theta\left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p\right) \end{aligned} \tag{4.61}$$

for all $x, y, z \in X$, then there exists a unique additive set–valued mapping $L : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(\ell(x), L(x)\right) \leq \left(\frac{2\theta}{(2-k_4)(4-2^p)}\right) \|x\|_X^p \tag{4.62}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\operatorname{diam} \ell(x) \leq M \left\|x\right\|_X^p$$

for all $x \in X$, then $L(x)$ is a singleton set.

Theorem 4.15. Let $\varphi : X^3 \to [0,\infty)$ be a fixed function for which there exists $L < 1$ such that

$$\varphi(x,y,z) \leq \frac{L}{4}\varphi\left(2x, 2y, 2z\right) \tag{4.63}$$

for all $x, y, z \in X$. If $\ell : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $\ell(0) = \{0\}$ and

$$\begin{aligned} &h(\ell(x+y+z) \oplus \ell(x+y-z) \oplus \ell(y+z-x) \oplus \ell(z+x-y), 4[\ell(x) \oplus_4 \ell(y) \oplus \ell(z)]) \\ &\leq\ k_4 h(\ell(x+y) \oplus \ell(x-y), 2\ell(x) \oplus 2\ell(y)) + \varphi(x,y,z) \end{aligned} \tag{4.64}$$

for all $x, y, z \in X$, then there exists a unique quadratic set–valued mapping $L : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(\ell(x), L(x)\right) \leq \left(\frac{L}{4(2-k_4)(1-L)}\right) \varphi(x,x,0) \tag{4.65}$$

for all $x \in X$. Moreover, if there exists positive real numbers $r < 2$ and M such that

$$\operatorname{diam} \ell(x) \leq M \left\|x\right\|_X^r$$

for all $x \in X$, then $L(x)$ is a singleton set.

Corollary 4.16. Let $p > 2$ and $\theta \geq 0$ be real numbers. If $\ell : X \to (C_{cb}(Y), \oplus, h)$ is a mapping satisfying $\ell(0) = \{0\}$ and

$$\begin{aligned} & h(\ell(x+y+z) \oplus \ell(x+y-z) \oplus \ell(y+z-x) \oplus \ell(z+x-y), 4[\ell(x) \oplus_4 \ell(y) \oplus \ell(z)]) \\ \leq \; & k_4 h(\ell(x+y) \oplus \ell(x-y), 2\ell(x) \oplus 2\ell(y)) + \theta \left(\|x\|_X^p + \|y\|_X^p + \|z\|_X^p\right) \end{aligned} \tag{4.66}$$

for all $x, y, z \in X$, then there exists a unique additive set–valued mapping $F : X \to (C_{cb}(Y), \oplus, h)$ such that

$$h\left(\ell(x), L(x)\right) \leq \left(\frac{2\theta}{(2-k_4)(2^p-4)}\right) \|x\|_X^p \tag{4.67}$$

for all $x \in X$. Moreover, if there exists positive real numbers p and M such that

$$\text{diam } \ell(x) \leq M \|x\|_X^p$$

for all $x \in X$, then $L(x)$ is a singleton set.

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