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Further investigation into the contractive condition in $\mathcal{D}$-generalized metric spaces with the transitivity

Kanokwan Sawangsup[a], Wutiphol Sintunavarat[b,*]

[a]Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50300, Thailand

[b]Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand

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Abstract

In this paper, we present fixed point results in $\mathcal{D}$-generalized metric spaces endowed with a transitive relation that is not necessarily a partial order. We also give two examples with numerical results to support our main results while fixed-point results in the literature are not applicable. Moreover, we introduce some new notions for consideration of the multidimensional fixed point results in the $\mathcal{D}$-generalized metric spaces.

Keywords: Fixed point of N-order, $\mathcal{D}$-generalized metric space, transitive relation
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1 Introduction

One of the very famous fixed point result in complete metric spaces is the Banach contraction principle, which was introduced by Banach in his thesis in 1922. This principle is a very popular tool for guaranteeing the existence and uniqueness of solution of considerable problems arising in several branches of Mathematics. In 2003, Ran and Reuring [10] presented fixed point results on partially ordered metric spaces and also applied such results to nonlinear matrix equations. Moreover, there exist several results showing existence and uniqueness of fixed point on complete metric space endowed with various binary relations (see [2, 8, 9, 14, 15]). In 2015, Alam and Imdad [1] presented fixed point result on complete metric spaces endowed with an arbitrary binary relation. Under universal relation, Alam and Imdad's result can be reduced to Banach contraction principle, which is a classical and powerful tool in nonlinear analysis.

In 1987, Guo and Lakshmikantham [4] introduced the notion of a coupled fixed point. Subsequently, Bhaskar and Lakshmikantham [3] defined the concept of the mixed monotone property and used it to prove the coupled fixed point theorems in partially ordered metric spaces. In 2010, Samet and Vetro [12] introduced the notion of fixed point of N-order as natural extension of the notion of coupled fixed point and they also established results for fixed point of N-order in complete metric spaces.

*Corresponding author
Email addresses: kanokwan.sws@rmutl.ac.th, juneks.ks@gmail.com (Kanokwan Sawangsup), wutiphol@mathstat.sci.tu.ac.th, wutiphol@gmail.com (Wutiphol Sintunavarat)

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On the other hand, several generalizations of standard metric spaces have appeared. One of the generalizations of standard metric spaces was introduced by Jleli and Samet [5]. The such space is called a $\mathcal{D}$-generalized metric space, which recovers a large class of topological spaces including standard metric spaces, dislocate metric spaces, b-metric spaces, and modular metric spaces. Moreover, they defined new notions for the such space and also extend some well-known fixed point including Banach contraction principle and a fixed point result due to Ran and Reuring [10].

In this paper, we define new notions under an arbitrary binary relation in the $\mathcal{D}$-generalized metric spaces and present fixed point results in the such spaces endowed with a transitive relation, which are the weaker version of Jleli and Samet's fixed point results [5]. We also provide an example that supports our main result while fixed point result in literature are not applicable. Moreover, we establish some tools for multidimensional mappings and use it to prove the existence and uniqueness of the fixed point of N-order in the $\mathcal{D}$-generalized metric spaces.

2 Preliminaries

Throughout this paper, X, $\mathbb{N}$ and $\mathbb{N}_0$ denote a nonempty set, the set of positive integers and the set of nonnegative integers, respectively. We start our consideration by giving a brief review of the definitions and basic properties of a $\mathcal{D}$-generalized metric space.

Let X be a nonempty set and $\mathcal{D} : X \times X \to [0, +\infty]$ be a mapping. For every $x \in X$, we will use the following symbol:

$$C(\mathcal{D}, X, x) := \left\{ \{x_n\} \subset X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0 \right\}. \tag{2.1}$$

Definition 2.1. [5] We say that $\mathcal{D} : X \times X \to [0, +\infty]$ is a $\mathcal{D}$-generalized metric on a nonempty set X if it satisfies the following conditions:

($\mathcal{D}_1$) for every $(x, y) \in X \times X$, we have
$$\mathcal{D}(x, y) = 0 \Rightarrow x = y;$$

($\mathcal{D}_2$) for every $(x, y) \in X \times X$, we have
$$\mathcal{D}(x, y) = \mathcal{D}(y, x);$$

($\mathcal{D}_3$) there exists $\mathcal{C} > 0$ such that $(x, y) \in X \times X$,
$$\{x_n\} \in C(\mathcal{D}, X, x) \Rightarrow \mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

In this case, we say the pair $(X, \mathcal{D})$ is a $\mathcal{D}$-generalized metric space.

Remark 2.2. [5] Obviously, if the set $C(\mathcal{D}, X, x)$ is empty for every $x \in X$, then $(X, \mathcal{D})$ is a $\mathcal{D}$-generalized metric space if and only if ($\mathcal{D}_1$) and ($\mathcal{D}_2$) are satisfied.

Remark 2.3. Note that the class of $\mathcal{D}$-generalized metric space is always larger than the class of standard metric spaces, dislocate metric spaces, b-metric spaces, dislocate b-metric spaces, and modular metric spaces (see the detail in [5]).

The following examples are $\mathcal{D}$-generalized metric spaces.

Example 2.4 ([13]). 1. Let $X = [0, \infty]$, and let $\mathcal{D} : X \times X \to [0, \infty]$ be defined as follows:

$$\mathcal{D}(x, y) = \begin{cases} x + y & \text{if at least one of } x \text{ and } y \text{ is } 0; \\ 1 + x + y & \text{otherwise.} \end{cases}$$

2. Let $X = [0, 1]$, and let $\mathcal{D} : X \times X \to [0, \infty]$ be defined as follows:

$$\mathcal{D}(x, y) = \begin{cases} x + y & \text{if at least one of } x \text{ and } y \text{ is } 0; \\ \frac{x+y}{3} & \text{otherwise.} \end{cases}$$

Definition 2.5. [5] Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We say that $\{x_n\}$ $\mathcal{D}$-converges to x if

$$\{x_n\} \in C(\mathcal{D}, X, x).$$

Proposition 2.6. [5] Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space, $\{x_n\}$ be a sequence in X, and $x, y \in X$. If $\{x_n\}$ $\mathcal{D}$-converges to x and $\{x_n\}$ $\mathcal{D}$-converges to y, then $x = y$.

Definition 2.7. [5] Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence if

$$\lim_{m,n\to\infty} \mathcal{D}(x_n, x_{n+m}) = 0.$$

Definition 2.8. [5] Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space. It is said to be $\mathcal{D}$-complete if every Cauchy sequence in X is convergent to some element in X.

Definition 2.9. [5] Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space. A mapping $T : X \to X$ is called a weak continuous mapping if the following condition hold: if $\{x_n\} \subset X$ is $\mathcal{D}$-convergent to $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ is $\mathcal{D}$-convergent to Tx (as $k \to \infty$).

Now, we recall the definitions and basic properties of a binary relation as follow:

Definition 2.10. [7] A *binary relation on* X is a nonempty subset $\Re$ of $X \times X$. The binary relation $\Re$ is *transitive* if $(x, z) \in \Re$ for all $x, y, z \in X$ such that $(x, y) \in \Re$ and $(y, z) \in \Re$.

If $(x, y) \in \Re$, we also denote it by $x\Re y$, and we say that "x is $\Re$-related to y".

Definition 2.11. [1] Let $\Re$ be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are $\Re$-comparative if either $(x, y) \in \Re$ or $(y, x) \in \Re$. We denote it by $[x, y] \in \Re$.

Definition 2.12. [1] Let X be a nonempty set. Given a mapping $T : X \to X$, a binary relation $\Re$ defined on X is called T-*closed* if for any $x, y \in X$, $(x, y) \in \Re \Longrightarrow (Tx, Ty) \in \Re$.

The previous property is equivalent to say that T is $\Re$-*nondecreasing* (see, for instance, [11]).

Definition 2.13. [6] Let $\Re$ a binary relation on a nonempty set X. For $x, y \in X$, *a path of length* k (where k is a natural number) in $\Re$ from x to y is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subseteq X$ satisfying the following conditions:

- (i) $z_0 = x$ and $z_k = y$;
- (ii) $(z_i, z_{i+1}) \in \Re$ for all $i \in \{0, 1, 2, ..., k-1\}$.

We denote by $\Upsilon(x, y, \Re)$ the family of all paths in $\Re$ from x to y.

Notice that a path of length k involves $k + 1$ elements of X, although they are not necessarily distinct.

In this paper, we will use

$$\text{Fix}(T) := \{z \in X : z \text{ is a fixed point of } T : X \to X\}.$$

If $\Re$ is a binary relation on a nonempty set X and $T : X \to X$ is a mapping and, let us denote by $X(T; \Re)$ the set of all points $x \in X$ satisfying $(x, Tx) \in \Re$.

3 Fixed point results

In this section, we start our investigation by introduce proposition which will be useful in later.

Proposition 3.1. If $(X, \mathcal{D})$ is a $\mathcal{D}$-generalized metric space, $\Re$ is a binary relation on X, T is a self-mapping on X, then the following contractivity conditions are equivalent:

(*i*) $\mathcal{D}(Tx, Ty) \leq \alpha \mathcal{D}(x, y), \quad \forall x, y \in X$ with $(x, y) \in \Re$;

(*ii*) $\mathcal{D}(Tx, Ty) \leq \alpha \mathcal{D}(x, y), \quad \forall x, y \in X$ with $[x, y] \in \Re$,

for some $\alpha \in [0, 1)$.

Proof . The implication $(ii) \Rightarrow (i)$ is trivial. Conversely, assume that (i) holds. Take $x, y \in X$ with $[x, y] \in \Re$. If $(x, y) \in \Re$, then (ii) directly follows from (i). Now, suppose that $(y, x) \in \Re$, then using the symmetry of $\mathcal{D}$ and (i), we obtain

$$\mathcal{D}(Tx, Ty) = \mathcal{D}(Ty, Tx) \leq \alpha \mathcal{D}(y, x) = \alpha \mathcal{D}(x, y).$$

This shows that $(i) \Rightarrow (ii)$. This completes the proof. □

For every $x \in X$, we will use

$$\delta(\mathcal{D}, T, x) = \sup\{\mathcal{D}(T^i(x), T^j(x)) : i, j \in \mathbb{N}\}.$$

Now, we stare and prove our main results as follow.

Theorem 3.2. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space, $\Re$ a transitive relation on X, and T a self-mapping on X. Suppose that the following conditions hold:

(*a*) $(X, \mathcal{D})$ is complete;

(*b*) there exists $x_0 \in X(T; \Re)$ such that $\delta(\mathcal{D}, T, x_0) < \infty$;

(*c*) $\Re$ is T-closed;

(*d*) there exists $\alpha \in [0, 1)$ such that

$$\mathcal{D}(Tx, Ty) \leq \alpha \mathcal{D}(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \Re; \tag{3.1}$$

(*e*) T is weak continuous.

Then T has a fixed point. Moreover, for each $x_0 \in X(T; \Re)$ with $\delta(\mathcal{D}, T, x_0) < \infty$ and $n \in \mathbb{N}$, the sequence $\{T^n x_0\}$ is convergent to the fixed point of T.

Proof . Let x_0 be an arbitrary point in $X(T; \Re)$ with $\delta(\mathcal{D}, T, x_0) < \infty$. Put $x_n = T^n x_0$ for all $n \in \mathbb{N}_0$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N}_0$, then x_{n^*} is a fixed point of T and the proof is completed. Thus suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Since $(x_0, Tx_0) \in \Re$, using assumption (c), we have

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), \ldots, (T^n x_0, T^{n+1}x_0), \ldots \in \Re$$

so that

$$(x_n, x_{n+1}) \in \Re \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Since $\Re$ is a transitive relation, for each $p, q \in \mathbb{N}_0$ with $p < q$, we have

$$(x_{p+1}, x_{q+1}) \in \Re.$$

For every $i, j \in \mathbb{N}$, using assumption (d), we have

$$\begin{aligned}\mathcal{D}(T^{n+1+i}x_0, T^{n+1+j}x_0) &\leq \alpha\mathcal{D}(T^{n+i}x_0, T^{n+j}x_0)\\ &\leq \alpha^2\mathcal{D}(T^{n-1+i}x_0, T^{n-1+j}x_0)\\ &\vdots\\ &\leq \alpha^{n+1}\mathcal{D}(T^i x_0, T^j x_0) \quad \text{for all } \ n \in \mathbb{N}_0.\end{aligned}$$

Then

$$\delta(\mathcal{D}, T, T^{n+1}x_0) \leq \alpha^{n+1}\delta(\mathcal{D}, T, x_0) \quad \text{for all } \ n \in \mathbb{N}_0.$$

For every $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, and $m \geq 2$, we have

$$\begin{aligned}\mathcal{D}(T^{n+1}x_0, T^{n+m}x_0) &\leq \delta(\mathcal{D}, T, T^{n+1}x_0)\\ &\leq \alpha^{n+1}\delta(\mathcal{D}, T, x_0).\end{aligned}$$

Since $\delta(\mathcal{D}, T, x_0) < \infty$ and $\alpha \in [0, 1)$, we have

$$\lim_{n\to\infty} \mathcal{D}(T^{n+1}x_0, T^{n+m}x_0) = 0, \tag{3.2}$$

which implies that $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence. Since $(X, \mathcal{D})$ is $\mathcal{D}$-complete, there exists some $x^* \in X$ such that $\{x_n\}$ is $\mathcal{D}$-convergent to x^*. Since f is weak continuous, there exists a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ such that $\{T^{n_{k+1}}x_0\}$ is $\mathcal{D}$-convergent to Tx^* as $k \to \infty$. By the uniqueness of limit, we obtain $x^* = Tx^*$, that is, x^* is a fixed point of T. □

Example 3.3. Let $X = [0, \infty)$ and the $\mathcal{D}$-generalized metric $\mathcal{D} : X \times X \to [0, \infty)$ be defined by $\mathcal{D}(x, y) = |\, x - y \,|^2$ for all $x, y \in X$. Thus, the $\mathcal{D}$-generalized metric space $(X, \mathcal{D})$ is complete. Define a binary relation $\Re$ on X by

$$\Re = \left\{(x, y) \in X \times X : x, y \in \left[0, \frac{1}{1 - \ln\sqrt{2}}\right) \ \text{ with } \ x < y\right\}.$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 1 + x\ln\sqrt{2} & \text{if } 0 \leq x < \frac{1}{1-\ln\sqrt{2}};\\ x & \text{if } \frac{1}{1-\ln\sqrt{2}} \leq x \leq 9;\\ \frac{x^2+x+9}{x+2} & \text{if } x \geq 9.\end{cases}$$

Since T is a continuous mapping, it follows that T is an also weak continuous mapping. Moreover, there exists $0.98 \in X$ such that $(0.98, T(0.98)) \in \Re$ and the condition $\delta(\mathcal{D}, T, x_0) < \infty$ is also satisfied. Now, we have to show that $\Re$ is T-closed. Assume that $x, y \in X$ such that $(x, y) \in \Re$. Then $x, y \in \left[0, \frac{1}{1-\ln\sqrt{2}}\right]$ and $x < y$. It yields

$$0 \leq 1 + x\ln\sqrt{2} < 1 + y\ln\sqrt{2} < \frac{1}{1 - \ln\sqrt{2}}.$$

This means that $Tx, Ty \in \left[0, \frac{1}{1-\ln\sqrt{2}}\right]$ and $Tx < Ty$ and so $(Tx, Ty) \in \Re$. This implies that $\Re$ is T-closed.

Next, we will show that the condition (3.1) holds with $\left(\ln\sqrt{2}\right)^2$. Let x, y be arbitrary elements in X with $(x, y) \in \Re$. Then we have

$$\begin{aligned}\mathcal{D}(Tx, Ty) &= |Tx - Ty|^2\\ &= \left|\left(1 + x\ln\sqrt{2}\right) - \left(1 + y\ln\sqrt{2}\right)\right|^2\\ &= \left|(\ln\sqrt{2})\,(x - y)\right|^2\\ &\leq \left(\ln\sqrt{2}\right)^2 |x - y|^2\\ &= \alpha\,|x - y|^2.\end{aligned}$$

Therefore, all the conditions of Theorem 3.2 are satisfied and so we get the existence of fixed point of T. In this case, T has infinite fixed points.

We can see some numerical experiments for approximate the fixed point of T in Figure 1. Furthermore, the convergence behavior of these iteration is shown in Figure 2.

x_0	0.080000	0.380000	0.680000	0.980000
x_1	1.027726	1.131698	1.235670	1.339642
x_2	1.356183	1.392217	1.428251	1.464285
x_3	1.470017	1.482506	1.494994	1.507482
x_4	1.509469	1.513797	1.518125	1.522454
x_5	1.523142	1.524642	1.526142	1.527642
x_6	1.527881	1.528401	1.528921	1.529440
x_7	1.529523	1.529703	1.529883	1.530064
x_8	1.530092	1.530155	1.530217	1.530280
x_9	1.530290	1.530311	1.530333	1.530355
x_{10}	1.530358	1.530365	1.530373	1.530380
x_{11}	1.530382	1.530384	1.530387	1.530389
x_{12}	1.530390	1.530391	1.530392	1.530393
x_{13}	1.530393	1.530393	1.530393	1.530394
x_{14}	1.530394	1.530394	1.530394	1.530394
x_{15}	1.530394	1.530394	1.530394	1.530394
x_{16}	1.530394	1.530394	1.530394	1.530394
x_{17}	1.530394	1.530394	1.530394	1.530394
x_{18}	1.530394	1.530394	1.530394	1.530394
x_{19}	1.530394	1.530394	1.530394	1.530394
x_{20}	1.530394	1.530394	1.530394	1.530394
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 1: Iterates of Picard iterations in Example 3.3.

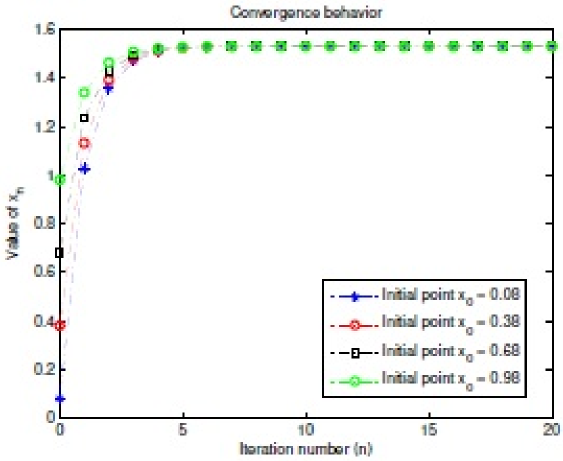

Figure 2: Convergence behavior in Example 3.3.

Remark 3.4. We note that Jleli and Samet's results in [5] are not applicable in Example 3.3 since the Jleli and Samet's k-contractive condition does not hold for each $x, y \in X$. That is, Theorem 3.3 in [5] cannot be used in this case. Furthermore, a binary relation $\Re$ in Example 3.3 is not a partial order on X and thus Theorem 5.5 in [5] cannot be used in this case.

Example 3.5. Let $X = [0,1]$ and the $\mathcal{D}$-generalized metric $\mathcal{D} : X \times X \to [0, \infty)$ be defined by

$$\mathcal{D}(x,y) = \begin{cases} x+y & \text{if at least one of } x \text{ or } y \text{ is } 0; \\ \frac{x+y}{3} & \text{otherwise.} \end{cases}$$

Thus, the $\mathcal{D}$-generalized metric space $(X, \mathcal{D})$ is complete. Set $r_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Define a binary relation $\Re$ on X by

$$\Re = \{(r_i, r_j) : 1 < i < j\} \cup \{(0, r_m) : m \in \mathbb{N}/\{1\}\}.$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0; \\ \frac{x}{2} & \text{if } 0 < x \leq \frac{1}{2}; \\ x^2 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Since T is a continuous mapping, it follows that T is an also weak continuous mapping. Moreover, there exists $\frac{1}{2} \in X$ such that $(\frac{1}{2}, T(\frac{1}{2})) = (\frac{1}{2}, \frac{1}{4}) \in \Re$ and the condition $\delta(\mathcal{D}, T, x_0) < \infty$ is also satisfied. Now, we have to show that $\Re$ is T-closed. Assume that $x, y \in X$ such that $(x, y) \in \Re$. If $(x, y) = (r_i, r_j) = (\frac{1}{i}, \frac{1}{j})$ for all $1 < i < j$, then $(Tx, Ty) = (\frac{1}{2i}, \frac{1}{2j}) \in \Re$ for all $1 < i < j$. If $(x, y) = (0, r_m) = (0, \frac{1}{m})$ for all $m > 1$, then $(Tx, Ty) = (0, \frac{1}{2m}) \in \Re$ for all $m > 1$. This implies that $\Re$ is T-closed. Next, we will show that the condition (3.1) holds with $\frac{1}{2} \leq \alpha < 1$. Let x, y be arbitrary elements in X with $(x, y) \in \Re$.

Case I: Consider $(x, y) = (r_i, r_j) = (\frac{1}{i}, \frac{1}{j})$ for all $1 < i < j$. Then

$$\mathcal{D}(Tx, Ty) = \frac{Tx + Ty}{3} = \frac{1}{2}\left(\frac{x+y}{3}\right) \leq \alpha \left(\frac{x+y}{3}\right) = \alpha \mathcal{D}(x, y).$$

Case II: Consider $(x, y) = (0, r_m) = (0, \frac{1}{m})$ for all $m > 1$. Then

$$\mathcal{D}(Tx, Ty) = Tx + Ty = 0 + \frac{1}{2m} \leq \alpha \left(0 + \frac{1}{m}\right) = \alpha \mathcal{D}(x, y).$$

Therefore, all the conditions of Theorem 3.2 are satisfied and so we get the existence of fixed point of T. In this case, point 0 and 1 are fixed points of T.

x_0	$1/2$	$1/3$	$1/4$	$1/5$
x_1	0.250000	0.166667	0.125000	0.100000
x_2	0.125000	0.083333	0.062500	0.050000
x_3	0.062500	0.041667	0.031250	0.025000
x_4	0.031250	0.020833	0.015625	0.012500
x_5	0.015625	0.010417	0.007813	0.006250
x_6	0.007813	0.005208	0.003906	0.003125
x_7	0.003906	0.002604	0.001953	0.001563
x_8	0.001953	0.001302	0.000977	0.000781
x_9	0.000977	0.000651	0.000488	0.000391
x_{10}	0.000488	0.000326	0.000244	0.000195
x_{11}	0.000244	0.000163	0.000122	0.000098
x_{12}	0.000122	0.000081	0.000061	0.000049
x_{13}	0.000061	0.000041	0.000031	0.000024
x_{14}	0.000031	0.000020	0.000015	0.000012
x_{15}	0.000015	0.000010	0.000008	0.000006
x_{16}	0.000008	0.000005	0.000004	0.000003
x_{17}	0.000004	0.000003	0.000002	0.000002
x_{18}	0.000002	0.000001	0.000001	0.000001
x_{19}	0.000001	0.000001	0.000000	0.000000
x_{20}	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3: Iterates of Picard iterations in Example 3.5.

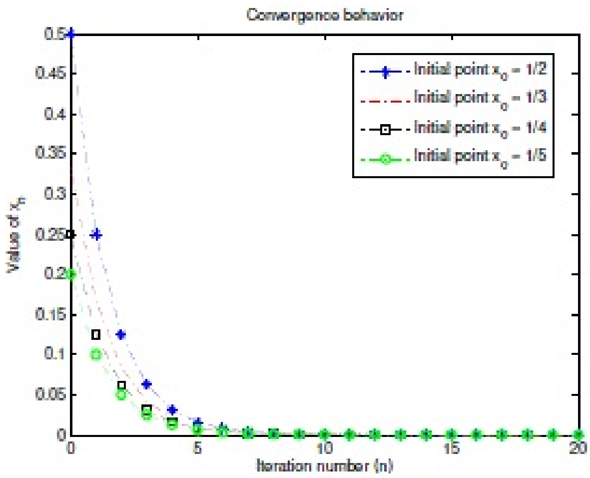

Figure 4: Convergence behavior in Example 3.5.

Remark 3.6. We note that Banach contraction principle are not applicable in Example 3.5. Moreover, the Jleli and Samet's k-contractive condition does not hold for each $x, y \in X$. So, Jleli and Samet's results in [5] are not applicable in Example 3.5. Furthermore, a binary relation $\Re$ in Example 3.5 is not a partial order on X and thus Theorem 5.5 in [5] cannot be used in this case.

Next, we introduce a definition of $\mathcal{D}$-self-closed as follows:

Definition 3.7. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space. A binary relation $\Re$ defined on X is called $\mathcal{D}$-self-closed if whenever $\{x_n\}$ is an $\Re$-preserving sequence and $\{x_n\} \in C(\mathcal{D}, X, x)$ for some $x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \Re$ for all $\alpha \in \mathbb{N}_0$.

In the following result we avoid the continuity of T.

Theorem 3.8. Theorem 3.2 also holds if we replace hypothesis (e) by the following one

(e') $\Re$ is $\mathcal{D}$-self-closed.

Proof . Following the proof of the previous theorem, we know that $\{x_n\}$ is an $\Re$-preserving sequence and is $\mathcal{D}$-convergent to some point $x^* \in X$. Since $\Re$ is $\mathcal{D}$-self-closed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x^*] \in \Re$ for all $k \in \mathbb{N}_0$. Using condition (d), Proposition 3.1, $[x_{n_k}, x^*] \in \Re$, and $\{x_{n_k}\}$ is $\mathcal{D}$-convergent to x^*, we obtain

$$\mathcal{D}(x_{n_k+1}, Tx^*) = \mathcal{D}(Tx_{n_k}, Tx^*) \leq \alpha\mathcal{D}(x_{n_k}, x^*) \tag{3.3}$$

for all k large enough. Letting $k \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \mathcal{D}(x_{n_k+1}, Tx^*) = 0, \tag{3.4}$$

which implies that x_{n_k+1} is $\mathcal{D}$-convergent to Tx^*. By the uniqueness of limit, we obtain $x^* = Tx^*$, that is, x^* is a fixed point of T. □

The following theorem guarantees the uniqueness of the point in Theorems 3.2 and 3.8.

Theorem 3.9. In addition to the hypothesis of Theorem 3.2 (respectively, Theorem 3.8), suppose that $\Upsilon(x, y, \Re)$ is nonempty and $\mathcal{D}(x, y) < \infty$ for all $x, y \in \mathrm{Fix}(T)$. Then T has a unique fixed point.

Proof . Take $x, y \in \mathrm{Fix}(T)$ and hence $\mathcal{D}(x, y) < \infty$. Since $\Upsilon(x, y, \Re)$ is nonempty, there exists a path (say $\{z_0, z_1, \ldots, z_k\}$) of some finite length $k \in \mathbb{N}$ for x to y so that

$$z_0 = x, z_k = y, (z_i, z_{i+1}) \in \Re$$

for each i $(0 \leq i \leq k-1)$. This implies that $(x, y) \in \Re$ and so

$$\mathcal{D}(x, y) = \mathcal{D}(Tx, Ty) \leq \alpha\mathcal{D}(x, y),$$

which implies that $\mathcal{D}(x, y) = 0$ (since $\alpha \in [0, 1)$). Therefore, T has a unique fixed point. □

4 Some multidimensional results

In this section we illustrate how to obtain multidimensional results from the described results in Section 3 by involving very simple tools. Let N denote a positive integer. We will denote by X^N the Cartesian product $X \times X \times \overset{N}{\cdots} \times X$.

Definition 4.1 ([12]). Let X be a nonempty set and let $T : X^N \to X$ be a given mapping. An element $(x_1, x_2, \ldots, x_N) \in X^N$ is said to be a *fixed point of N-order* of the mapping T if

$$\begin{cases} T(x_1, x_2, \ldots, x_N) = x_1, \\ T(x_2, x_3, \ldots, x_N, x_1) = x_2, \\ \quad\vdots \\ T(x_N, x_1, \ldots, x_{N-1}) = x_N. \end{cases}$$

In this section, we will use

$$\mathrm{FixN}(T) := \{z \in X^N : z \text{ is a fixed point of } N\text{-order of } T : X^N \to X\}.$$

Given a binary relation $\Re$ on X, we will denote by $\Re^N$ the binary relation on the product space X^N defined by:

$$((x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N)) \in \Re^N \quad \Leftrightarrow \quad (x_1, y_1) \in \Re, (x_2, y_2) \in \Re, ..., (x_N, y_N) \in \Re.$$

If $T : X^N \to X$ is a mapping, let us denote by $X^N(T; \Re^N)$ the set of all points $(x_1, x_2, ..., x_N) \in X^N$ such that

$$\Big((x_1, x_2, ..., x_N), (T(x_1, x_2, ..., x_N), T(x_2, x_3, \ldots, x_N, x_1), \ldots, T(x_N, x_1, \ldots, x_{N-1}))\Big) \in \Re^N,$$

that is,

$$(x_i, T(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1})) \in \Re \quad \text{for each } i \in \{1, 2, \ldots, N\}.$$

Definition 4.2. Let X be a nonempty set. If $N \geq 2$ and $T : X^N \to X$ is a mapping, a binary relation $\Re$ on X is called T_N-*closed* if for any $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in X^N$,

$$\left\{\begin{array}{c}(x_1, y_1) \in \Re, \\ (x_2, y_2) \in \Re, \\ \vdots \\ (x_N, y_N) \in \Re\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(T(x_1, x_2, \ldots, x_N), T(y_1, y_2, \ldots, y_N)) \in \Re, \\ (T(x_2, x_3, \ldots, x_N, x_1), T(y_2, y_3, \ldots, y_N, y_1)) \in \Re, \\ \vdots \\ (T(x_N, x_1, \ldots, x_{N-1}), T(y_N, y_1, \ldots, y_{N-1})) \in \Re\end{array}\right\}.$$

Next, we introduce some notions for consideration the multidimensional results in the $\mathcal{D}$-generalized metric spaces.

Let X^N be a nonempty set and $\mathcal{D}^N : X^N \times X^N \to [0, +\infty]$ be a given mapping. For every $(x_1, x_2, \ldots, x_N) \in X^N$, let us define the set

$$C(\mathcal{D}^N, X^N, (x_1, x_2, \ldots, x_N)) \quad := \quad \left\{\{(x_n^1, x_n^2, \ldots, x_n^N)\} \subset X^N : \lim_{n\to\infty} \mathcal{D}^N((x_n^1, x_n^2, \ldots, x_n^N), (x_1, x_2, \ldots, x_N)) = 0\right\}.$$

Definition 4.3. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space, $\{(x_n^1, x_n^2, \ldots, x_n^N)\}$ be a sequence in X^N, and $(x_1, x_2, \ldots, x_N) \in X^N$. We say that $\{(x_n^1, x_n^2, \ldots, x_n^N)\}$ $\mathcal{D}^N$-converges to $(x_1, x_2, \ldots, x_N)$ if

$$\{(x_n^1, x_n^2, \ldots, x_n^N)\} \in C(\mathcal{D}^N, X^N, (x_1, x_2, \ldots, x_N)).$$

Definition 4.4. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space and $i \in \{1, 2, \ldots, N\}$. A mapping $T : X^N \to X$ is called a weak continuous mapping if the following condition hold: if $\{(x_n^i, x_n^{i+1}, \ldots, x_n^N, x_n^1, x_n^2, \ldots, x_n^{i-1})\} \subset X^N$ is $\mathcal{D}^N$-convergent to $(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1}) \in X^N$, then there exists a subsequence

$$\{(x_{n_k}^i, x_{n_k}^{i+1}, \ldots, x_{n_k}^N, x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^{i-1})\}$$

of $\{(x_n^i, x_n^{i+1}, \ldots, x_n^N, x_n^1, x_n^2, \ldots, x_n^{i-1})\}$ such that $\{T(x_{n_k}^i, x_{n_k}^{i+1}, \ldots, x_{n_k}^N, x_{n_k}^1, x_{n_k}^2, \ldots, x_{n_k}^{i-1})\}$ is $\mathcal{D}$-convergent to

$$T(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1}) \text{ as } k \to \infty.$$

Let us denote by $G_T^N : X^N \to X^N$ the mapping

$$G_T^N(x_1, x_2, \ldots, x_N) = \Big(T(x_1, x_2, ..., x_N), T(x_2, x_3, \ldots, x_N, x_1), \ldots, T(x_N, x_1, \ldots, x_{N-1})\Big).$$

The following results guarantee that multidimensional notions can be interpreted in terms of G_T^N.

Lemma 4.5. Let X be a nonempty set. Given $N \geq 2$ and $T : X^N \to X$, a point $(x_1, x_2, ..., x_N) \in X^N$ is a fixed point of N-order of the mapping T if and only if it is a fixed point of G_T^N.

Lemma 4.6. Let X be a nonempty set. Given $N \geq 2$ and $T : X^N \to X$, a binary relation $\Re$ defined on X is T_N-closed if and only if the binary relation $\Re^N$ defined on X^N is G_T^N-closed.

Lemma 4.7. Let X be a nonempty set. Given $N \geq 2$ and $T : X^N \to X$, a point $(x_1, x_2, \ldots, x_N) \in X^N(T; \Re^N)$ if and only if $(x_1, x_2, ..., x_N) \in X^N(G_T^N; \Re^N)$.

Lemma 4.8. Let $(X, \mathcal{D})$ be a $\mathcal{D}$-generalized metric space. Consider the product space X^N. Suppose that $\mathcal{D}^N$: $X^N \times X^N \to \mathbb{R}$ given by:

$$\mathcal{D}^N(A, B) = \sum_{i=1}^{N} \mathcal{D}(a_i, b_i)$$

for all $A = (a_1, a_2, \ldots, a_N), B = (b_1, b_2, \ldots, b_N) \in X^N$. If

$$\limsup_{n\to\infty} \mathcal{D}(x_n, x) + \limsup_{n\to\infty} \mathcal{D}(y_n, y) \leq \limsup_{n\to\infty} [\mathcal{D}(x_n, x) + \mathcal{D}(y_n, y)] \tag{4.1}$$

for each sequence $\{x_n\}, \{y_n\} \subseteq X$ and $x, y \in X$, then the following properties hold.

1. $(X^N, \mathcal{D}^N)$ is also a $\mathcal{D}$-generalized metric space.

2. Let $\{A_n = (a_n^1, a_n^2, \dots, a_n^N)\}$ be a sequence on X^N and let $A = (a_1, a_2, \dots, a_N) \in X^N$. Then $\{A_n\}$ $\mathcal{D}^N$-converges to A if, and only if, $\{a_n^i\}$ $\mathcal{D}$-converges to a_i for all $i \in \{1, 2, \dots, N\}$.

3. If $\{A_n = (a_n^1, a_n^2, \dots, a_n^N)\}$ is a sequence in X^N, then $\{A_n\}$ is $\mathcal{D}^N$-Cauchy if, and only if, $\{a_n^i\}$ is $\mathcal{D}$-Cauchy for all $i \in \{1, 2, \dots, N\}$.

4. $(X, \mathcal{D})$ is $\mathcal{D}$-complete if, and only if, $(X^N, \mathcal{D}^N)$ is $\mathcal{D}^N$-complete.

Proof .

1. The properties $(\mathcal{D}_1)$ and $(\mathcal{D}_2)$ are trivial. Let us show that $(\mathcal{D}_3)$ also holds. Let $(a_1, a_2, \dots, a_N), (b_1, b_2, \dots, b_N) \in X^N$. Suppose that $\{(a_n^1, a_n^2, \dots, a_n^N)\} \in C\left(\mathcal{D}^N, X^N, (a_1, a_2, \dots, a_N)\right)$. Then

$$\begin{aligned} \mathcal{D}^N\left((a_1, a_2, \dots, a_N), (b_1, b_2, \dots, b_N)\right) &= \sum_{i=1}^{N} \mathcal{D}(a_i, b_i) \\ &\leq C \sum_{i=1}^{N} \limsup_{n\to\infty} \mathcal{D}(a_n^i, a_i) \\ &\leq C \limsup_{n\to\infty} \sum_{i=1}^{N} \mathcal{D}(a_n^i, a_i). \end{aligned}$$

This implies that $(X^N, \mathcal{D}^N)$ is a $\mathcal{D}$-generalized metric space.

2. Notice that for all $n \in \mathbb{N}$, and for all $i \in \{1, 2, \dots, N\}$,

$$\mathcal{D}^N(A_n, A) = \sum_{i=1}^{N} \mathcal{D}(a_n^i, a_i).$$

So, if $\{A_n\}$ $\mathcal{D}^N$-converges to A. This means that $\{A_n\} \in C(\mathcal{D}^N, X^N, A)$. It follows that

$$0 = \lim_{n\to\infty} \mathcal{D}^N(A_n, A) = \lim_{n\to\infty} \sum_{i=1}^{N} \mathcal{D}(a_n^i, a_i). \tag{4.2}$$

Therefore,

$$\lim_{n\to\infty} \mathcal{D}(a_n^i, a_i) = 0 \quad \text{for all} \ \ i \in \{1, 2, \dots, N\} \tag{4.3}$$

and so $\{a_n^i\}$ $\mathcal{D}$-converges to a_i for all $i \in \{1, 2, \dots, N\}$. Conversely, if $\{a_n^i\}$ $\mathcal{D}$-converges to a_i for all $i \in \{1, 2, \dots, N\}$. This mean that $\{a_n\} \in C(\mathcal{D}, X, a_i)$ for all $i \in \{1, 2, \dots, N\}$. So, it easy to see that $\{A_n\}$ $\mathcal{D}^N$-converges to A.

3. Similarly, it can be proved that, for all $n, k \in \mathbb{N}$, all $i \in \{1, 2, \dots, N\}$,

$$\mathcal{D}^N(A_n, A_k) = \sum_{i=1}^{N} \mathcal{D}(a_n^i, a_k^i).$$

Therefore, if $\{A_n\}$ is a $\mathcal{D}^N$-Cauchy sequence, then $\{a_n^i\}$ is a $\mathcal{D}$-Cauchy sequence for all $i \in \{1, 2, \dots, N\}$. The converse is similarly.

4. It follows from the last two items.

□

Next, we show how to use Theorem 3.2 in order to deduce multidimensional fixed point results that guarantee existence of fixed points of N-order.

Theorem 4.9. Let $(X,\mathcal{D})$ be a complete metric space satisfying the condition (4.1), $\Re$ be a transitive relation on X, and $T : X^N \to X$ be a mapping. Suppose that the following conditions hold:

(*a*) $(X^N,\mathcal{D}^N)$ is $\mathcal{D}^N$-complete;

(*b*) there exists $(x_0^1,x_0^2,\dots,x_0^N) \in X^N(T;\Re^N)$ such that $\delta(\mathcal{D}^N,G_T^N,(x_0^1,x_0^2,...,x_0^N)) < \infty$;

(*c*) $\Re$ is T_N-closed;

(*d*) there exists $\alpha \in [0,1)$ such that

$$\begin{aligned}&\sum_{i=1}^{N}\mathcal{D}(T\left(x_i,x_{i+1},\dots,x_N,x_1,x_2,\dots,x_{i-1}\right),T\left(y_i,y_{i+1},\dots,y_N,y_1,y_2,\dots,y_{i-1}\right))\\ \leq\ &\alpha\sum_{i=1}^{N}\mathcal{D}(\left(x_i,x_{i+1},\dots,x_N,x_1,x_2,\dots,x_{i-1}\right),\left(y_i,y_{i+1},\dots,y_N,y_1,y_2,\dots,y_{i-1}\right))\end{aligned}$$

for all $(x_1,x_2,\dots,x_N),(y_1,y_2,\dots,y_N) \in X$ with $((x_1,x_2,\dots,x_N),(y_1,y_2,\dots,y_N)) \in \Re^N$;

(*e*) T is weak continuous.

Then T has a fixed point of N-order.

Proof . By items 1 and 4 of Lemma 4.8, a $\mathcal{D}$-generalized metric space (X^N,D^N) is $\mathcal{D}^N$-complete. By Lemma 4.6, the binary relation $\Re^N$ defined on X^N is G_T^N-closed. Assume that $(x_0^1,x_0^2,...,x_0^N) \in X^N(T;\Re^N)$ such that $\delta(\mathcal{D}^N,G_T^N,(x_0^1,x_0^2,...,x_0^N)) < \infty$. By Lemma 4.7, we obtain $(x_0^1,x_0^2,...,x_0^N) \in X^N(G_T^N;\Re^N)$ such that

$$\delta(\mathcal{D}^N,G_T^N,(x_0^1,x_0^2,...,x_0^N)) < \infty.$$

Since T is weak continuous, we get G_T^N is also weak continuous. Now, let $A=(a_1,a_2,\dots,a_N),B=(b_1,b_2,\dots,b_N) \in X^N$ with $(A,B) \in \Re^N$. Then

$$\begin{aligned}\mathcal{D}^N(G_T^NA,G_T^NB),) &= \mathcal{D}^N(G_T^N(a_1,a_2,\dots,a_N),G_T^N(b_1,b_2,\dots,b_N))\\ &= \mathcal{D}^N\begin{pmatrix}\left(T(a_1,a_2,\dots,a_N),T(a_2,a_3,\dots,a_N,a_1),\dots,T(a_N,a_1,\dots,a_{N-1})\right),\\ \left(T(b_1,b_2,\dots,b_N),T(b_2,b_3,\dots,b_N,b_1),\dots,T(b_N,b_1,\dots,b_{N-1})\right)\end{pmatrix}\\ &= \sum_{i=1}^{N}\mathcal{D}(T(a_i,a_{i+1},\dots,a_N,a_1,a_2,\dots,a_{i-1}),T(b_i,b_{i+1},\dots,b_N,b_1,b_2,\dots,b_{i-1}))\\ &\leq \alpha\sum_{i=1}^{N}\mathcal{D}(a_i,b_i)\\ &= \alpha\mathcal{D}^N(A,B).\end{aligned}$$

Applying Theorem 3.2, there exists $\hat{X}=(\hat{x_1},\hat{x_2},\dots,\hat{x_N}) \in X^N$ such that $G_T^N(\hat{X})=\hat{X}$, that is, $(\hat{x_1},\hat{x_2},\dots,\hat{x_N})$ is a fixed point of G_T^N. Using Lemma 4.5, $(\hat{x_1},\hat{x_2},\dots,\hat{x_N})$ is a fixed point of N-order of T. This completes the proof. □

By using Theorem 3.8 with a similar technique as in the proof of Theorem 4.9, we get the following existence result of fixed point of N-order.

Theorem 4.10. Let $(X,\mathcal{D})$ be a complete metric space satisfying the condition (4.1), $\Re$ be a transitive relation on X, and $T : X^N \to X$ be a mapping. Suppose that the following conditions hold:

(*a*) $(X^N,\mathcal{D}^N)$ is $\mathcal{D}^N$-complete;

(*b*) there exists $(x_0^1,x_0^2,...,x_0^N) \in X^N(T;\Re^N)$ such that $\delta(\mathcal{D}^N,G_T^N,(x_0^1,x_0^2,...,x_0^N)) < \infty$;

(*c*) $\Re$ is T_N-closed;

(d) there exists $\alpha \in [0,1)$ such that

$$\begin{aligned} & \sum_{i=1}^{N} \mathcal{D}(T\left(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1}\right), T\left(y_i, y_{i+1}, \ldots, y_N, y_1, y_2, \ldots, y_{i-1}\right)) \\ \leq \quad & \alpha \sum_{i=1}^{N} \mathcal{D}(\left(x_i, x_{i+1}, \ldots, x_N, x_1, x_2, \ldots, x_{i-1}\right), \left(y_i, y_{i+1}, \ldots, y_N, y_1, y_2, \ldots, y_{i-1}\right)) \end{aligned}$$

for all $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in X$ with $((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N)) \in \Re^N$;

(e) $\Re^N$ is $\mathcal{D}^N$-self-closed.

Then T has a fixed point of N-order.

By using Theorem 3.9, we get the following uniqueness result of fixed point of N-order.

Theorem 4.11. In addition to the hypothesis of Theorem 4.9 (respectively, Theorem 4.10), suppose that $\Upsilon((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N), \Re^N)$ is nonempty and $\mathcal{D}((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N)) < \infty$ for each $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in \mathrm{FixN}(T)$. Then T has a unique fixed point of N-order.

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