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Distributional solution for semilinear system involving fractional gradient and a numerical example

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Abstract

In this research, a semilinear fractional system involving a new operator is tackled. The existence of a distributional solution is demonstrated and the Leray-Schauder degree method is used to deal with the existence of this system. For the uniqueness of the solution, we use the contraction principle with some assumptions made on the semilinear term Φ_1 and Φ_2. Then, using an example and the finite difference method a numerical investigation of this system is conducted.

Keywords: distributional solution, semilinear elliptic system, Leray-Schauder degree, Riesz fractional gradient, homotopy invariance, partial differential equations
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1 Introduction

Classical partial differential equations can be generalized to fractional partial differential equations (FPDEs). In recent years, fractional differential equations have received a lot of attention from researchers and this was due to its applications in various field, such as: image processing [11], mechanics [3], biophysics [4], finance [14].

In this research, we study existence and uniqueness results for a new class of semi-linear fractional system by using the Leray-Schauder degree theory. This problem has been studied in [1], where the authors studied the existence of weak solution for a semilinear elliptic system of non-local equation involving the fractional Laplacian. Also, S. Dob et al in [7] study the existence and the uniqueness of weak solution for the non-linear fractional elliptic system using fixed point theorem, we notice that the problem mentioned abouve can be seen as the fractional setting of the problem in [13], where H. lakhal et al. studies existence of weak solutions in the case of the classical Laplacian. What distinguishes our work from the rest of the works mentioned previously is that, a new class of fractional problem is used. This new class is so interesting that it owns new and spry features, in addition to that; it gives more accurate and clearer results in the numerical study and the best weak formula where it appears as the main element in it. Unlike fractional Laplacian. It is also, the fractional gradient similar to the ordinary gradient in appearance.

The fractional Laplacian has many of definitions, regardless of the well-known fractional Laplacian one. In our work, the concept of distributional Riesz fractional gradient is used. In [18, 19] Shieh and Spector have considerd a

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new class of fractional partial differential equation based on the distributional Riez fractional derivatives, and they examined the the existence and the uniqueness of results of the linear fractional problem relating the distributional Riesz fractional gradient, proving it with Lax-Milgram theorem. Studies on these issues have continued since then (see for example [15, 16, 2, 17]).
Now, we define the fractional gradient γ-gradient D^γ of order $\gamma \in (0,1)$, as follows:
$\triangleright$ for $\vartheta \in L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, we have

$$D_j^\gamma \vartheta = \frac{\partial}{\partial x_j}(I_{1-\gamma} * \vartheta), \quad 0<\gamma<1, \quad j=1,\dots,d,$$

where $\dfrac{\partial}{\partial x_j}$ is taken in the distributional sense, for all $v \in C_c^\infty(\mathbb{R}^d)$,

$$\left\langle \frac{\partial^\gamma \vartheta}{\partial x_j^\gamma}, v \right\rangle = -\left\langle (I_{1-\gamma} * \vartheta), \frac{\partial v}{\partial x_j} \right\rangle = -\int_{\mathbb{R}^d} (I_{1-\gamma} * \vartheta)\frac{\partial v}{\partial x_j} dx,$$

$\triangleright$ In addition, for $\vartheta \in C_c^\infty(\Omega)$, we wrote γ-gradient (D^γ) (see Section 1, p.3 in [18]) by

$$D^\gamma \vartheta = I_{1-\gamma} * D\vartheta.$$

with I_γ denoting the Riesz potential of order γ, $0<\gamma<1$:

$$(I_\gamma * \vartheta)(x) = c_{d,1-\gamma}\int_{\mathbb{R}^d} \frac{\vartheta(y)}{|x-y|^{d-\gamma}} dy, \text{ which } c_{d,\gamma} = 2^\gamma \pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d+\gamma+1}{2})}{\Gamma(\frac{1-\gamma}{2})}.$$

For more detaile about fractional gradient s-gradient see ([15, 5, 18, 19, 20]) and the referenses therien. As it is shown in [18]. D^γ has nice properties for $\vartheta \in C_c^\infty(\mathbb{R}^d)$, namely it coincides with the fractional Laplacian as follows:

$$(-\Delta)^\gamma \vartheta = -D^\gamma . D^\gamma \vartheta.$$

Where, for $0<\gamma<1$,

$$(-\Delta)^\gamma \vartheta(x) = c_{d,\gamma}^2 \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} \frac{\vartheta(x)-\vartheta(y)}{|x-y|^{d+2\gamma}}\mathcal{X}_\epsilon(x,y)dy = \frac{1}{2}c_{d,\gamma}^2 \int_{\mathbb{R}^d} \frac{\vartheta(x+y)+\vartheta(x-y)-2\vartheta(x)}{|y|^{d+2\gamma}}dy.$$

Furthermore, Schikorra et al. in [21] found that $D^\gamma.(D^\gamma u(x))$ is a fractional particular type of 2-Laplacian, which is defined by

$$div^\gamma(D^\gamma u(x)) = D^\gamma.(D^\gamma u(x)) = \sum_{j=1}^{d} \frac{\partial^\gamma}{\partial x_j^\gamma}\frac{\partial^\gamma}{\partial x_j^\gamma}u(x).$$

In the present research, we apply Leray-Schauder degree theory to prove some existence results in a Bessel Potential space. This work was inspired by [16], where we developed an equation into the following semilinear fractional elliptic system

$$\begin{cases} -D^\gamma.(D^\gamma u) + \Phi_1(x,u,v) = \Psi_1(x) & \text{in } \Omega, \\ -D^\gamma.(D^\gamma v) + \Phi_2(x,u,v) = \Psi_2(x) & \text{in } \Omega, \\ u=v=0 & \text{on } \mathbb{R}^d/\Omega. \end{cases} \tag{1.1}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded open domain, with Lipshitz boundary, $\gamma \in (0,1)$ with $2\gamma < d$, $(\Psi_1,\Psi_2) \in (L^2(\Omega)\times L^2(\Omega))$ and $\Phi_1(x,k,p), \Phi_2(x,k,p) : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ measurable on $x\in\Omega$ and continuous on $k,p \in \mathbb{R}$. As well $\{-D^\gamma.(D^\gamma(u))\}$ is a nonlocal operator defined in [15] in the duality sense. The main benefit of this nonlocal operator is its rotational invariance. It is well-known in harmonic analysis that the only higher dimensional operator is translational and rotational invariant is the Riesz operator.

The problem studied in [16] is a special case of the problem (1.1) where the authors studied the existence and the uniqueness results for semilinear fractional problem involving the non-local operator $-D^\gamma.(D^\gamma(u))$. Then, Abada et al. in [2] were pioneers in studying the existence and the uniqueness of distributional solution for a nonlinear problem including the distributional Riesz derivative and they suggest Leray-Schauder degree theorem. At last, Slimani et al. in [17] used fixed point theorem to prove the existence result for convection-reaction fractional problem. It is worth noting that our problem is more broad than the previous work mentioned above, it is complementary.

The outline of the article is as follows. First, in the coming section we give some preliminary results. Section 3 contains a fixed point formulation of our system and the statement of results. In section 4 we prove existence of the distributional solution for the system (1.1) by using the Leray-Schauder degree theory, and in section 5 we give some assumptions on functions Φ_1 and Φ_2 to prove the uniqueness of distributional solution. In section 6 we present a numerical example to illustrate the usefulness of the finite difference method. Finally, we end this research by a conclusion.

2 Preliminaries

In this section, we will introduce some fractional Sobolev spaces $X^{\gamma,2}(\mathbb{R}^d)$, $L^{\gamma,2}(\mathbb{R}^d)$ and $L_0^{\gamma,2}(\Omega)$ in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. For more details (see [18, 15]).

Definition 2.1. Let $\gamma \in (0,1)$. if $\vartheta \in C_c^\infty(\mathbb{R}^d)$, we define the space

$$X^{\gamma,2}(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{X^{\gamma,2}(\mathbb{R}^d)}},$$

where the norm

$$\|\vartheta\|^2_{X^{\gamma,2}(\mathbb{R}^d)} = \|\vartheta\|^2_{L^2(\mathbb{R}^d)} + \|D^\gamma \vartheta\|^2_{L^2(\mathbb{R}^d)}.$$

Definition 2.2. Let $\gamma \in \mathbb{R}_+$. We will defined The Bessel potential space as follows:

$$L^{\gamma,2}(\mathbb{R}^d) = g_\gamma(L^2(\mathbb{R}^d)).$$

Where the Bessel potential g_γ are defined (see Section 2, p.7 in [18]) follow:

$$g_\gamma(x) = \frac{1}{(4\pi)^{\frac{\gamma}{2}}\Gamma(\frac{\gamma}{2})} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{\delta}} e^{\frac{-\delta}{4\pi}} \delta^{\frac{\gamma-d}{2}} \frac{d\delta}{\delta},$$

in the sense that every $\vartheta \in L^{\gamma,2}(\mathbb{R}^d)$ can written as

$$\vartheta = g_\gamma * f,$$

for some $f \in L^2(\mathbb{R}^d)$. With norm $\|\vartheta\|_{L^{\gamma,2}(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$.

Definition 2.3. Let $\Omega \subset \mathbb{R}^d$ be open, $\gamma \in (0,1)$. We define the space

$$L_0^{\gamma,2}(\Omega) = \{\vartheta \in L^{\gamma,2}(\mathbb{R}^d); \vartheta = 0 \text{ in } \mathbb{R}^d/\Omega\}.$$

Theorem 2.4. (see [18]).If γ is a non-negative integer and $1 < p < \infty$, then $L^{\gamma,p}(\mathbb{R}^d)$ coinsides with the space $W^{\gamma,p}(\mathbb{R}^d)$, the norms in the two spaces being equivalent. This conclusion holds for any real $\gamma \in (0,1)$ if $p = 2$.

Remark 2.5. We notice that, from Theorem 2.4 records the result of Calderón mentioned in the introduction in the integer setting, and more generally, in combination with (Theorem 1.7 in [18]) shows that,

$$X^{\gamma,2}(\mathbb{R}^d) = L^{\gamma,2}(\mathbb{R}^d) = W^{\gamma,2}(\mathbb{R}^d).$$

Taking the space $L^2(\Omega) \times L^2(\Omega)$, with the norm

$$\|(\vartheta, v)\|_{L^2(\Omega)\times l^2(\Omega)} = \|\vartheta\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}.$$

Next, we consider the space $L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$, with the norme denote by $\|(.,.)\|_{L_0^{\gamma,2}(\Omega)\times L_0^{\gamma,2}(\Omega)}$

$$\|(\vartheta, v)\|_{L_0^{\gamma,2}(\Omega)\times L_0^{\gamma,2}(\Omega)} = \|\vartheta\|_{L_0^{\gamma,2}(\Omega)} + \|v\|_{L_0^{\gamma,2}(\Omega)}.$$

Proposition 2.6. (Fractional Poincaré inequality, Lemma 3.2 of [15]). Let $\gamma \in (0,1)$. Then there exists a constant $C_P = C(d,\Omega)/\gamma > 0$ such that

$$\|\vartheta\|_{L^2(\Omega)} \leq C_P \|D^\gamma \vartheta\|_{L^2(\mathbb{R}^d)},$$

for all $\vartheta \in L_0^{\gamma,2}(\Omega)$.

Remark 2.7. Our remark is that:

- From the Proposition 2.6, we conclude that the norms $\big(\|\vartheta\|^2_{L^2(\mathbb{R}^d)} + \|D^\gamma \vartheta\|_{L^2(\mathbb{R}^d)}\big)^{\frac{1}{2}}$ and $\|D^\gamma \vartheta\|_{L^2(\mathbb{R}^d)}$ are equivalent norms in $L_0^{\gamma,2}(\Omega)$.
- The space $L_0^{\gamma,2}(\Omega)$ with the inner product

$$\big\langle \vartheta, v \big\rangle_{L_0^{\gamma,2}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} D^\gamma \vartheta . D^\gamma v dx,$$

  is a Hilbert space.

Theorem 2.8. [18] (Fractional Sobolev inequality). Let $1 < p < \infty$ and $\gamma \in (0,1)$ be such that $\gamma p < d$. Then there exists a constant $C = C(d,p,\gamma) > 0$ such that

$$\|\vartheta\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|D^\gamma \vartheta\|_{L^p(\mathbb{R}^d)},$$

for all $\vartheta \in L^{\gamma,p}(\mathbb{R}^d)$, where $p^* = \frac{dp}{d-\gamma p}$.

Proposition 2.9. [15] For $0 < \gamma \leq 1$ and $1 \leq q \leq 2^*$, where $2^* = \dfrac{2d}{d-2\gamma}$. Then, by the Sobolev-Poincaré inequalities, we have the embeddings

$$L_0^{\gamma,2}(\Omega) \hookrightarrow L^q(\Omega).$$

We recall that those embeddings are compact for $1 \leq q < 2^*$.

Remark 2.10. For given $\vartheta, v \in L_0^{\gamma,2}(\Omega)$ and $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a (not necessarily symmetric) matrix, $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary, we put the bilinear form

$$b_A(\vartheta, v) = \int_{\mathbb{R}^d} A D^\gamma \vartheta(x) . D^\gamma v(x) dx.$$

Depending on the expansion of the results in [6], [9], [10], [12] and [20] we obtain

$$\big\langle -D^\gamma . (A D^\gamma \vartheta), v \big\rangle = \int_{\mathbb{R}^N} A D^\gamma \vartheta D^\gamma v,$$

when $A D^\gamma \vartheta \in [L^2(\mathbb{R}^d)]^d$.

Proposition 2.11. [15]$\Big(b_A, L_0^{\gamma,2}(\Omega)\Big)$ is a closed, coercive, regular Dirichlet form.

Definition 2.12. We say that $u, v \in L_0^{\gamma,2}(\Omega)$ is a distributional solution for the problem (1.1) if for any $\phi_1, \phi_2 \in L_0^{\gamma,2}(\Omega)$, we have

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u(x) . D^\gamma \phi_1(x) dx = \lambda \int_\Omega \Psi_1(x)\phi_1(x) dx - \lambda \int_\Omega \Phi_1(x,u,v)\phi_1(x) dx, & \forall \phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} D^\gamma v(x) . D^\gamma \phi_2(x) dx = \lambda \int_\Omega \Psi_2(x)\phi_2(x) dx - \lambda \int_\Omega \Phi_2(x,u,v)\phi_2(x) dx, & \forall \phi_2 \in L_0^{\gamma,2}(\Omega). \end{cases} \tag{2.1}$$

3 Fixed point formulation of the proplem (1.1) and Statement results

In this section, in order to prove the result, let us assume the semilinear function $\Phi_1(x,k,p)$ and $\Phi_2(x,k,p)$ satisfies the following hypothesis

(H1) Growth hypothesis

$$|\Phi_1(x,k,p)| \leq a_1(x) + h_1|k| + r_1|p| \quad \forall k,p \in \mathbb{R}, \quad \text{a.e. } x \in \Omega,$$
$$|\Phi_2(x,k,p)| \leq a_2(x) + h_2|k| + r_2|p| \quad \forall k,p \in \mathbb{R}, \quad \text{a.e. } x \in \Omega,$$

where $a_1, a_2 \in L^2(\Omega)$, and $h_1, h_2, r_1, r_2 \in \mathbb{R}^+$.

(H2) Sign hypothesis:

$$\Phi_1(x,k,p)k \geq 0 \quad \forall k,p \in \mathbb{R} \quad \text{a.e} \quad x \in \Omega.$$
$$\Phi_2(x,k,p)p \geq 0 \quad \forall k,p \in \mathbb{R} \quad \text{a.e} \quad x \in \Omega.$$

$\triangleright$ Now we will write the following linear problem.

For $\overline{u}, \overline{v} \in L^2(\Omega)$, we define the following linear problem

$$\begin{cases} -D^\gamma.(D^\gamma u) = \lambda\Psi_1(x) - \lambda\Phi_1(x,\overline{u},\overline{v}) & \text{in } \Omega, \\ -D^\gamma.(D^\gamma v) = \lambda\Psi_2(x) - \lambda\Phi_2(x,\overline{u},\overline{v}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^d/\Omega, \end{cases} \tag{3.1}$$

where $\Psi_1, \Psi_2 \in L^2(\Omega)$.

Proposition 3.1. Thanks to hypothesis (H1), the problem (3.1) has a unique distributional solution $(u,v) \in L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$.

Proof . For all $(\overline{u},\overline{v}) \in L^2(\Omega) \times L^2(\Omega)$, we have $\Phi_1(.,\overline{u},\overline{v}), \Phi_2(.,\overline{u},\overline{v}) \in L^2(\Omega)$. We say $(u,v) \in L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$ is a distributional solution of (3.1) if

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u(x).D^\gamma\phi_1(x)dx = \lambda\int_\Omega \Psi_1(x)\phi_1(x)dx - \lambda\int_\Omega \Phi_1(x,\overline{u},\overline{v})\phi_1(x)dx, & \forall\phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} D^\gamma v(x).D^\gamma\phi_2(x)dx = \lambda\int_\Omega \Psi_2(x)\phi_2(x)dx - \lambda\int_\Omega \Phi_2(x,\overline{u},\overline{v})\phi_2(x)dx, & \forall\phi_2 \in L_0^{\gamma,2}(\Omega). \end{cases} \tag{3.2}$$

With

$$a_1(.,.):\ L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega) \to \mathbb{R}$$
$$(u,\phi_1) \mapsto a_1(u,\phi_1) = \int_{\mathbb{R}} D^\gamma u(x).D^\gamma\phi_1(x)dx,$$
$$l_1(.):\ L_0^{\gamma,2}(\Omega) \to \mathbb{R}$$
$$w \mapsto l(\phi_1) = \lambda\int_\Omega \Psi_1(x)\phi_1(x)dx - \lambda\int_\Omega \Phi_1(x,\overline{u},\overline{v})\phi_1(x)dx \quad \forall\lambda \in [0,1].$$

And

$$a_2(.,.):\ L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega) \to \mathbb{R}$$
$$(v,\phi_2) \mapsto a_2(v,\phi_2) = \int_{\mathbb{R}} D^\gamma v(x).D^\gamma\phi_2(x)dx,$$
$$l_2(.):\ L_0^{\gamma,2}(\Omega) \to \mathbb{R}$$
$$w \mapsto l(\phi_2) = \lambda\int_\Omega \Psi_2(x)\phi_2(x)dx - \lambda\int_\Omega \Phi_2(x,\overline{u},\overline{v})\phi_2(x)dx \quad \forall\lambda \in [0,1].$$

Next, we prove the bilinear form $a_1(.,.)$ and $a_2(.,.)$ are coercive, $\forall v \in L_0^{\gamma,2}(\Omega)$

$$\begin{aligned} a_1(v,v) &= \int_{\mathbb{R}} |D^\gamma v(x)|^2 dx \\ &= \|v\|^2_{L_0^{\gamma,2}(\Omega)}, \end{aligned}$$

thus, $a_1(.,.)$ is coercive.
In the same way, we prove that $a_2(.,.)$ is coercive.
Then, for prove the bilinear form $a_1(.,.), a_2(.,.)$ and the linear form $l_1(.), l_2(.)$ are continuous. Using the Cauchy-Schwarz inequality, for all $u, \phi_1 \in L_0^{\gamma,2}(\Omega)$

$$\begin{aligned} |a_1(u,\phi_1)| &\leq \|D^\gamma u\|_{L^2(\mathbb{R}^d)} \|D^\gamma \phi_1\|_{L^2(\mathbb{R}^d)} \\ &= \|u\|_{L_0^{\gamma,2}(\Omega)} \|\phi_1\|_{L_0^{\gamma,2}(\Omega)}. \end{aligned}$$

Therefore $a_1(.,.)$ is continuous.
For all $\phi_1 \in H_0^\gamma(\Omega)$. According to Proposition 2.6 and the Hölder inequality and hypothesis (H1), we have

$$\begin{aligned} |l_1(\phi_1)| &\leq \|\Psi_1\|_{L^2(\Omega)} \|\phi_1\|_{L^2(\Omega)} + \int_\Omega |\Phi_1(x,\overline{u},\overline{v})||\phi_1(x)| dx \\ &\leq C_P \|\Psi_1\|_{L^2(\Omega)} \|\phi_1\|_{L_0^{\gamma,2}(\Omega)} + C_P \left(\|a_1\|_{L^2(\Omega)} + h_1 \|\overline{u}\|_{L^2(\Omega)} + r_1 \|\overline{v}\|_{L^2(\Omega)}\right) \|\phi_1\|_{L_0^{\gamma,2}(\Omega)}. \end{aligned}$$

Finally

$$|l_1(\phi_1)| \leq C_P \left(\|a_1\|_{L^2(\Omega)} + h_1 \|\overline{u}\|_{L^2(\Omega)} + r_1 \|\overline{v}\|_{L^2(\Omega)} + \|\Psi\|_{L^2(\Omega)}\right) \|\phi_1\|_{L_0^{\gamma,2}(\Omega)}, \tag{3.3}$$

hence, $l_1(.)$ is continuous.
In the same way, we prove that $a_2(.,.)$ and $l_2(.)$ are continous. As result, we may apply the Lax-Milgram theorem and we conclude the problem (3.1) has a unique distributional solution $(u,v) \in L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$. $\square$

The following theorem is main result

Theorem 3.2. Under the hypothesis (H1) and (H2), the problem (1.1) has a distributional solution $(u,v) \in L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$.

The following operator is well defined $H : [0,1] \times L^2(\Omega) \times L^2(\Omega) \to L_0^{\gamma,2}(\Omega) \times L_0^{\gamma,2}(\Omega)$ such that $H(\lambda, \overline{u}, \overline{v}) = (u,v)$ where (u,v) is the solution of (3.1).
We observe that problem (1.1) is equivalent to the problem

$$\begin{cases} (u,v) \in L^2(\Omega) \times L^2(\Omega), \\ H(1,\overline{u},\overline{v}) = (\overline{u},\overline{v}). \end{cases} \tag{3.4}$$

4 Several Lemmas and Main Result

In the present section, we will use the Leray-Shauder degree theory to obtain existence result of the system (1.1).

Lemma 4.1. (Priori estimate) We will show that
$\exists R > 0$, $\forall (\overline{u},\overline{v}) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} H(\lambda,\overline{u},\overline{v}) = (\overline{u},\overline{v}) \\ \lambda \in [0,1], (\overline{u},\overline{v}) \in L^2(\Omega) \times L^2(\Omega) \end{cases} \Rightarrow \|(\overline{u},\overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R+1.$$

Proof . Let $H(\lambda,\overline{u},\overline{v}) = (u,v) = (\overline{u},\overline{v})$, $\forall \lambda \in [0,1]$, which mean that

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma \overline{u}(x).D^\gamma \phi_1(x) dx = \lambda \int_\Omega \Psi_1(x)\phi_1(x) dx - \lambda \int_\Omega \Phi_1(x,\overline{u},\overline{v})\phi_1(x) dx, \quad \forall \phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} D^\gamma \overline{v}(x).D^\gamma \phi_2(x) dx = \lambda \int_\Omega \Psi_2(x)\phi_2(x) dx - \lambda \int_\Omega \Phi_2(x,\overline{u},\overline{v})\phi_2(x) dx, \quad \forall \phi_2 \in L_0^{\gamma,2}(\Omega). \end{cases} \tag{4.1}$$

We take $\phi_1(x) = \overline{u}(x)$ and $\phi_2(x) = \overline{v}(x)$, according the hypothesis (H2), we obtain

$$\begin{cases} \int_{\mathbb{R}} |D^\gamma \overline{u}(x)|^2 dx & \leq \|\Psi_1\|_{L^2(\Omega)} \|\overline{u}\|_{L^2(\Omega)} \\ \int_{\mathbb{R}} |D^\gamma \overline{v}(x)|^2 dx & \leq \|\Psi_2\|_{L^2(\Omega)} \|\overline{v}\|_{L^2(\Omega)}. \end{cases}$$

Moreover, from the Proposition 2.6, implies that

$$\begin{cases} \|\overline{u}\|_{L^2(\Omega)} & \leq C_P^2 \|\Psi_1\|_{L^2(\Omega)} \\ \|\overline{v}\|_{L^2(\Omega)} & \leq C_P^2 \|\Psi_2\|_{L^2(\Omega)}, \end{cases} \tag{4.2}$$

taking the sum of the two inequalities of (4.2), we obtain

$$\|(\overline{u}, \overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_P^2 \|\Psi_1\|_{L^2(\Omega)} + C_P^2 \|\Psi_2\|_{L^2(\Omega)}, \tag{4.3}$$

thus

$$\|(\overline{u}, \overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R + 1.$$

We can deduce from this that for all $t \in [0,1]$ there are no solutions to the equation $H(t, \overline{u}, \overline{v}) = (\overline{u}, \overline{v})$ in the boundary of the sphere $B(0, R+1) = \{v \in L^2(\Omega) : \|(\overline{u}, \overline{v})\|_{L^2(\Omega)\times L^2(\Omega)} < R + 1.\}$. ☐

Lemma 4.2. Under the hypothesis (H1), H: $[0,1] \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is continuous.

Proof . Let $\{\lambda_n, \overline{u}_n, \overline{u}_n\}_{n\in\mathbb{N}} \subset [0,1] \times L^2(\Omega) \times L^2(\Omega)$ which converges to $(\lambda, \overline{u}, \overline{v})$ in $[0,1] \times L^2(\Omega) \times L^2(\Omega)$ when $n \to +\infty$. We will show that $H(\lambda_n, \overline{u}_n, \overline{v}_n)$ converges to $H(\lambda, \overline{u}, \overline{v})$, we pose $H(\lambda_n, \overline{u}_n, \overline{v}_n) = (u_n, v_n)$ and $H(\lambda, \overline{u}, \overline{v}) = (u, v)$, we obtain

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u_n(x).D^\gamma \phi_1(x) dx = \int_\Omega \lambda_n \Psi_1(x)\phi_1(x) dx - \int_\Omega \lambda_n \Phi_1(x, \overline{u}_n, \overline{v}_n)\phi_1(x) dx, & \forall \phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} D^\gamma v_n(x).D^\gamma \phi_2(x) dx = \int_\Omega \lambda_n \Psi_2(x)\phi_2(x) dx - \int_\Omega \lambda_n \Phi_2(x, \overline{u}_n, \overline{v}_n)\phi_2(x) dx, & \forall \phi_2 \in L_0^{\gamma,2}(\Omega), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u(x).D^\gamma \phi_1(x) dx = \int_\Omega \lambda \Psi_1(x)\phi_1(x) dx - \int_\Omega \lambda \Phi_1(x, \overline{u}, \overline{v})\phi_1(x) dx, & \forall \phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} D^\gamma v(x).D^\gamma \phi_2(x) dx = \int_\Omega \lambda \Psi_2(x)\phi_2(x) dx - \int_\Omega \lambda \Phi_2(x, \overline{u}, \overline{v})\phi_2(x) dx, & \forall \phi_2 \in L_0^{\gamma,2}(\Omega), \end{cases} \tag{4.5}$$

We make the diffirence between (4.4) and (4.5), we obtain

$$\begin{cases} \int_{\mathbb{R}^d} \left(D^\gamma u_n - D^\gamma u\right).D^\gamma \phi_1 dx = \int_\Omega (\lambda_n - \lambda)\Psi_1 \phi_1 dx - \int_\Omega \left(\lambda \Phi_1(x, \overline{u}, \overline{v}) - \lambda_n \Phi_1(x, \overline{u}_n, \overline{v}_n)\right) \phi_1 dx, \\ \hfill \forall \phi_1 \in L_0^{\gamma,2}(\Omega). \\ \int_{\mathbb{R}^d} \left(D^\gamma v_n - D^\gamma v\right).D^\gamma \phi_2 dx = \int_\Omega (\lambda_n - \lambda)\Psi_2 \phi_2 dx - \int_\Omega \left(\lambda \Phi_2(x, \overline{u}, \overline{v}) - \lambda_n \Phi_2(x, \overline{u}_n, \overline{v}_n)\right) \phi_2 dx, \\ \hfill \forall \phi_2 \in L_0^{\gamma,2}(\Omega). \end{cases} \tag{4.6}$$

We take $\phi_1(x) = u_n(x) - u(x)$ and $\phi_2(x) = v_n(x) - v(x)$ and apply the Cauchy-Schwarz inequality and Proposition 2.6, we obtain

$$\begin{cases} \|u_n - u\|_{L^2(\Omega)} \leq C_P^2 \left(|\lambda_n - \lambda| \|\Psi_1\|_{L^2(\Omega)} + \|\lambda \Phi_1(., \overline{u}, \overline{v}) - \lambda_n \Phi_1(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right) \\ \|v_n - v\|_{L^2(\Omega)} \leq C_P^2 \left(|\lambda_n - \lambda| \|\Psi_2\|_{L^2(\Omega)} + \|\lambda \Phi_2(., \overline{u}, \overline{v}) - \lambda_n \Phi_2(., \overline{u}_n, \overline{v}_n)\|_{L^2(\Omega)}\right), \end{cases} \tag{4.7}$$

we have $(\overline{u}_n, \overline{v}_n) \to (\overline{u}, \overline{v})$ in $L^2(\Omega) \times L^2(\Omega)$ implies that

$$\begin{cases} \overline{u}_n \to \overline{u} & \text{a.e on } \Omega \\ |\overline{u}_n| < G & \text{a.e on } \Omega, \\ \text{or } G \in L^2(\Omega), \end{cases} \quad \text{and} \quad \begin{cases} \overline{v}_n \to \overline{v} & \text{a.e on } \Omega \\ |\overline{v}_n| < K & \text{a.e on } \Omega, \\ \text{or } K \in L^2(\Omega), \end{cases}$$

from the hypothesis (H1), we obtain

$$\begin{cases} \Phi_1(x, \overline{u}_n, \overline{v}_n) & \to \Phi_1(x, \overline{u}, \overline{v}), \qquad a.e\text{on } \Omega \\ |\Phi_1(x, \overline{u}_n, \overline{v}_n)| & < a(x) + h_1 G + r_1 K \quad \in L^2(\Omega) \qquad a.e \text{ on } \Omega, \end{cases}$$

and

$$\begin{cases} \Phi_2(x,\overline{u}_n,\overline{v}_n) & \to \Phi_2(x,\overline{u},\overline{v}), \qquad a.e \text{on } \Omega \\ |\Phi_2(x,\overline{u}_n,\overline{v}_n)| & < a(x)+h_2G+r_2K \quad \in L^2(\Omega) \qquad a.e \text{ on } \Omega, \end{cases}$$

From Lebesgue convergence theorem, $\Phi_1(x,\overline{u}_n,\overline{v}_n) \to \Phi_1(x,\overline{u},\overline{v})$ in $L^2(\Omega)$ and we have $(\lambda_n)_{n\in\mathbb{N}}$ converges to λ when $n\to+\infty$. Therfore (u_n,v_n) converges to (u,v) in $L^2(\Omega)\times L^2(\Omega)$. So H is continuous from $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)\times L^2(\Omega)$. □

Lemma 4.3. Under the hypothesis (H1), $\{H(\lambda,\overline{u},\overline{v}), t\in[0,1], (\overline{u},\overline{v})\in \overline{B}_{R+1}\}$ is relatively compact in $L^2(\Omega)\times L^2(\Omega)$.

Proof . Let $(\lambda_n)_{n\in\mathbb{N}}\subset[0,1]$ and $(\overline{u}_n,\overline{v}_n)_{n\in\mathbb{N}}\subset \overline{B}_{R+1}$

$$\begin{cases} \left|\int_{\mathbb{R}^d} D^\gamma u_n(x).D^\gamma u_n(x)dx\right| & = \left|\int_\Omega \lambda_n\Psi_1(x)u_n(x)dx - \int_\Omega \lambda_n\Phi_1(x,\overline{u}_n,\overline{v}_n)u_n(x)dx\right| \\ \left|\int_{\mathbb{R}^d} D^\gamma v_n(x).D^\gamma v_n(x)dx\right| & = \left|\int_\Omega \lambda_n\Psi_2(x)v_n(x)dx - \int_\Omega \lambda_n\Phi_2(x,\overline{u}_n,\overline{v}_n)v_n(x)dx\right|, \end{cases}$$

applying the Cauchy-Schwarz inequality, implies that

$$\begin{cases} \|u_n\|^2_{L^2(\Omega)} \le \|\Psi_1\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} + \|\Phi_1(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} \\ \|v_n\|^2_{L^2(\Omega)} \le \|\Psi_2\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)} + \|\Phi_2(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)}\|v_n\|_{L^2(\Omega)}. \end{cases}$$

From the hypothesis (H1) the sequence $\{\Phi_1(x,\overline{u}_n,\overline{v}_n)\}_{n\in\mathbb{N}}$ and $\{\Phi_2(x,\overline{u}_n,\overline{v}_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$ and the Proposition 2.6, we obtain

$$\begin{cases} \|u_n\|_{L_0^{\gamma,2}(\Omega)} \le C_P\left(\|\Psi_1\|_{L^2(\Omega)} + \|\Phi_1(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)}\right). \\ \|v_n\|_{L_0^{\gamma,2}(\Omega)} \le C_P\left(\|\Psi_2\|_{L^2(\Omega)} + \|\Phi_2(.,\overline{u}_n,\overline{v}_n)\|_{L^2(\Omega)}\right). \end{cases} \tag{4.8}$$

By the combination of two inequalities of the equation (4.8), we obtain

$$\|(u_n,v_n)\|_{L_0^{\gamma,2}(\Omega)\times L_0^{\gamma,2}(\Omega)} \le M,$$

where $M = C_P\left(\|\Psi_1\|_{L^2(\Omega)}+R_1\right)+C_P\left(\|\Psi_2\|_{L^2(\Omega)}+R_2\right)$. Consequently $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in $L_0^{\gamma,2}(\Omega)\times L_0^{\gamma,2}(\Omega)$ so $(u_n,v_n)\rightharpoonup(u,v)$ in $L_0^{\gamma,2}(\Omega)\times L_0^{\gamma,2}(\Omega)$, according to Proposition 2.9 we conclude that there is a subsequence of $\{(u_{n_k},v_{n_k})\}_{k\in\mathbb{N}}$ which converges to (u,v) in $L^2(\Omega)\times L^2(\Omega)$. □

Now we show the proof of Theorem 3.2, where it given existence of distributional solution for problem (1.1).

Proof .[Proof of Theorem 3.2] Thanks to the previous lemmas 4.1,4.2 and 4.3 we concluded that $d(I_d-H(t,.,.),\bar{B}_{R+1},0)$ well defined, by the homotopy invariance property, we find

$$\begin{aligned} d(I_d-H(1,.,.),\overline{B}_{R+1},0) &= d(I_d-H(0,.,.),\overline{B}_{R+1},0) \\ &= d(I_d,\overline{B}_{R+1},0) = 1\neq 0, \end{aligned}$$

therefore

$$(\overline{u},\overline{v}) - H(1,\overline{u},\overline{v}) = 0 \Leftrightarrow (\overline{u},\overline{v}) = H(1,\overline{u},\overline{v}).$$

Hence we have showed that $(\overline{u},\overline{v})$ is a solution of (1.1). □

5 Uniqueness of distributional solution

In this section, we will make some assumption about the functions Φ_1 and Φ_2 to prove the uniqueness of distributional solution the problem (1.1).

There are Φ_1 and Φ_2 Lipschitz continuous functions with respect to the second variable, that is means there exists constants $c_1,c_2\in\mathbb{R}^+$ for almost every $x\in\Omega$ and for any $k=(k_1,k_2), \tilde{k}=(\tilde{k}_1,\tilde{k}_2), l=(l_1,l_2), \tilde{l}=(\tilde{l}_1,\tilde{l}_2)\in L^2(\Omega)\times L^2(\Omega)$,

$$\begin{cases} \|\Phi_1(x,k)-\Phi_1(x,\tilde{k})\|_{L^2(\Omega)} \le c_1\|k-\tilde{k}\|_{L^2(\Omega)\times L^2(\Omega)} \\ \|\Phi_2(x,l)-\Phi_2(x,\tilde{l})\|_{L^2(\Omega)} \le c_1\|l-\tilde{l}\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{5.1}$$

As a result, proving that H is contraction is sufficient to show that H admet a unique fixed point

Lemma 5.1. The operator H is contraction from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ for all $\lambda \in [0,1]$.

Proof . Let $(\overline{u}_1, \overline{u}_2), (\overline{v}_1, \overline{v}_2) \in L^2(\Omega) \times L^2(\Omega)$ and for all $\lambda \in [0,1]$, we have

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u_1(x).D^\gamma \phi_1(x)dx = \lambda \int_\Omega \Psi_1(x)\phi_1(x)dx - \lambda \int_\Omega \Phi_1(x, \overline{u}_1, \overline{v}_1)\phi_1(x)dx \\ \int_{\mathbb{R}^d} D^\gamma v_1(x).D^\gamma \phi_2(x)dx = \lambda \int_\Omega \Psi_2(x)\phi_2(x)dx - \lambda \int_\Omega \Phi_2(x, \overline{u}_1, \overline{v}_1)\phi_2(x)dx, \end{cases}$$

and

$$\begin{cases} \int_{\mathbb{R}^d} D^\gamma u_2(x).D^\gamma \phi_1(x)dx = \lambda \int_\Omega \Psi_1(x)\phi_1(x)dx - \lambda \int_\Omega \Phi_1(x, \overline{u}_2, \overline{v}_2)\phi_1(x)dx \\ \int_{\mathbb{R}^d} D^\gamma v_2(x).D^\gamma \phi_2(x)dx = \lambda \int_\Omega \Psi_2(x)\phi_2(x)dx - \lambda \int_\Omega \Phi_2(x, \overline{u}_2, \overline{v}_2)\phi_2(x)dx. \end{cases}$$

We make the difference between the two previous systems and we take $\phi_1 = u_1 - u_2$, $\phi_2 = v_1 - v_2$ and apply Cauchy-Schwarz inequality, we ontain

$$\begin{cases} \|u_1 - u_2\|^2_{L_0^{\gamma,2}(\Omega)} \leq \|\Phi_1(., \overline{u}_1, \overline{v}_1) - \Phi_1(., \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)} \\ \|v_1 - v_2\|^2_{L_0^{\gamma,2}(\Omega)} \leq \|\Phi_2(., \overline{u}_1, \overline{v}_1) - \Phi_2(., \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)} \|v_1 - v_2\|_{L^2(\Omega)} \end{cases}$$

Thanks to Proposition 2.6, and hypothesis (5.1), we get

$$\begin{cases} \|u_1 - u_2\|_{L^2(\Omega)} \leq C_P c_1 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)} \\ \|v_1 - v_2\|_{L^2(\Omega)} \leq C_P c_2 \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)}, \end{cases} \tag{5.2}$$

by adding the two inequalities of (5.2), we arrive at

$$\|(u_1, v_1) - (u_2, v_2)\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_P(c_1 + c_2)\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)}.$$

That is means

$$\|H(\lambda, \overline{u}_1, \overline{v}_1) - H(\lambda, \overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)} \leq C_P(c_1 + c_2)\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|_{L^2(\Omega)\times L^2(\Omega)}$$

we conclude that if $C_P(c_1 + c_2) < 1$ then H is a contraction.
Finally, we arrived to the following result: H is a contraction if $C_P(c_1 + c_2) < 1$ and applying Banach contraction principle Theorem it results that H admits a unique fixed point $(\overline{u}, \overline{v}) \in L^2(\Omega) \times L^2(\Omega)$, hence the problem (1.1) admet a unique distributional solution.

□

6 Numerical Example

The numerical example are based on the finite difference method (FDM) for the numerical approximation of the solution to the nonlinear fractional elliptic system (1.1), the next results were shown after a lot of mathematical calculations. For more informations (see [8])

In this Section, we present the numerical simulations, we use the following system

$$\begin{cases} -D^{\frac{1}{2}}.(D^{\frac{1}{2}}u(x,y)) + \Phi_1(x, y, u(x,y), v(x,y)) = \Psi_1(x,y) & \text{in} \quad \Omega, \\ -D^{\frac{1}{2}}.(D^{\frac{1}{2}}v(x,y)) + \Phi_2(x, y, u(x,y), v(x,y)) = \Psi_2(x,y) & \text{in} \quad \Omega, \\ u = v = 0 \qquad \text{on} \quad \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{6.1}$$

where $\Omega =]0,1[\times]0,1[$, the functions Φ_1 and Φ_2 are written in terms of u and v as follows

$$\Phi_1(x, y, u(x,y), v(x,y)) = -4x(y-1)^2 u^{\frac{1}{2}} - 4x^2(y-1)v^{\frac{1}{2}},$$

$$\Phi_2(x, y, u(x,y), v(x,y)) = -4y^2(x-1)v^{\frac{1}{2}} - 4y(x-1)^2 u^{\frac{1}{2}},$$

and we can write the second terms Ψ_1 and Ψ_2 as next

$$\Psi_1(x,y) = -4x^2y^2(y-1) - 8x^2(y-1)y^2(x-1) - 4x^4(y-1) + 8x^4y(y-1),$$

$$\Psi_1(x,y) = -4yx^2(x-1)^2 - 8yx^2(x-1)^2(y-1) - 4y^3(x-1)^2 + 8y^3x(x-1)^2.$$

In this particular case, the solution can be computed exactly and it reads as follows,

$$u(x,y) = x^4(y-1)^4,$$
$$v(x,y) = y^4(x-1)^4.$$

The computational results are shown for our model in Table 1. We can notice that in the two-dimensional case, the predicted convergence rate is reached shortly. Since the computations are lengthy, we have tested our system only with some numbers of steps N. In Fig. 3, we show the computational errors evaluated for different values of N.

/	/	u	v	/
N	h	max-error	max-error	r
35	0.0278	1.3650×10^{-11}	1.1657×10^{-14}	0.0417
40	0.0244	3.0754×10^{-12}	3.5565×10^{-15}	0.0366
45	0.0217	1.0939×10^{-12}	1.2084×10^{-15}	0.0326
60	0.0164	5.3331×10^{-13}	6.9390×10^{-17}	0.0246
65	0.0152	3.6230×10^{-13}	2.7303×10^{-17}	0.0227

Table 1: Computational error and estimated convergence rate r for the matrix transformation method applied to the finite difference approximation of (6.1).

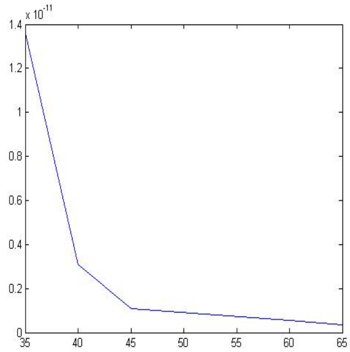

Figure 1: Plot of the absolute error.

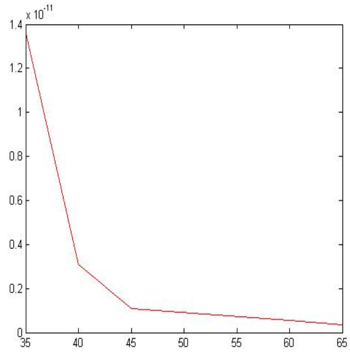

Figure 2: Plot of the absolute error of u.

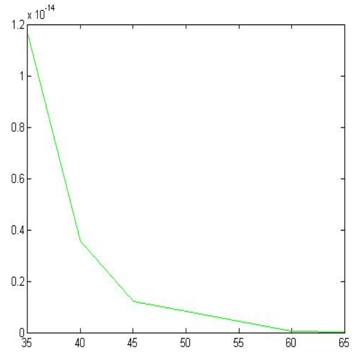

Figure 3: Plot of the absolute error of v.

7 Conclusion

In this research, we study the existence and the uniqueness of solution for semilinear fractional elliptic system involving new operator. From the positive reports we received from evaluation of [16]. Some future works we will continue the research in this line, extending the study from Lebegue space to Morrey spaces $L^{p,\lambda}$ for suitable p and λ and including also the numerical study of systems.

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