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Long-time solvability of functional differential equations with “supremum” involving increasing nonlinearity at infinite time

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Abstract

In this paper, we investigate a class of initial value problems of nonlinear fractional differential equations with state deviating arguments, delayed impulses and supremum on the half line. A global existence-uniqueness result is obtained using the theory of fixed points in uniform spaces, which is different from what is commonly used in such studies. Our result is obtained in a setting where classical variants of the fixed point theorems frequently used in the literature are inapplicable, and moreover without any bounding restriction with respect to time, neither on the solution nor on the reaction part of the problem. Two examples illustrating our main findings are also given.

Keywords: Impulsive functional differential equations, fractional calculus, uniform space, non self mapping, fixed point theorem
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1 Introduction

Differential equations with ”maxima” model many real world processes (in automotic control theory for example) whose the present state depends on its maximal value on certain time interval [8, 25]. In addition, as is observed in numerous fields of science and thechnology, when the simulated process is subject to abrupt changes in certain moments of time, impulsive differential equations are used [9, 21].

Recently, some existence results and stability properties of solutions for first and second order impulsive differential equations with supremum (IDESs) are investigated in [15, 17, 16, 26]. However, to the best of our knowledge, in the fractional case, there are not so many contributions on this type of equations, except in a few publications such as [27, 30], where some integral inequalities, existence and finite-time stability results for Hadamard type impulsive differential equations with maxima are studied. Thus, we found a lack of interst in the existence of solutions for this type of problem in the current literature. Especially since most of the results concerning the stability of solutions for (IDESs) are subordinated to the existence of solutions [17, 27].

Motivated by the above considerations, and in a sequel of our previous work [24], with some generalizations, we treat in this paper, the following initial value problem, of nonlinear fractional differential equation, with state deviating

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arguments, delayed impulses and a supremum:

$$ {}^{C}D_{0^+}^{\alpha} u\left(t\right) = f\left(t, \sup_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\tau_1(t)), \ldots, u(t-\tau_N(t))\right), t\in J_k,\, k\in\mathbb{N} \tag{1.1} $$

$$ \Delta u(t_k) = I_k\left(u\left(t_k - \rho_k(t_k)\right)\right),\ k\in\mathbb{N}^* \tag{1.2} $$

$$ u\left(t\right) = \phi\left(t\right), \quad t\le 0, \tag{1.3} $$

where ${}^{C}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha\in\,]0,1[$, N is a positive integer, a, b and τ_i (with $0\le i\le N$) are real continuous functions defined on $\mathbb{R}_+ = [0, +\infty[$ subject to conditions which will be specified later, $f:\mathbb{R}_+\times\mathbb{R}^{1+N}\longrightarrow\mathbb{R}$ is a nonlinear continuous function, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of u at $t = t_k$ and $\{t_k\}$ is a sequence of points in $\mathbb{R}_+$ such that $t_k < t_{k+1}$ for $k\in\mathbb{N}^*$ and $\lim\limits_{k\to\infty} t_k = \infty$, $J_0 = [0, t_1]$ and $J_k =]t_k, t_{k+1}]$ $(k\in\mathbb{N}^*)$, the initiale condition $\phi :]-\infty, 0]\longrightarrow\mathbb{R}$ is a continuous function such that $\phi\left(0\right) = \phi_0 > 0$.

It should be noted that $(1.1)-(1.3)$ contains as special cases a wide class of fractional differential equations, including those with and without impulses, with and without delayes, with and without supremum, for a bounded or unbounded time interval, considered in different works such as [11, 22, 24, 29]. Note also that most existence and uniqueness results for ordinary or fractional differential equations with and without impulses, require that the nonlinear part of the problem under consideration satisfies the Lipschitz condition with bounded arguments (see, e.g.,[3, 4, 23, 24] and the references therein). Keeping in mind to overcome this restriction, we aim throught this work, to obtain an existence-uniqueness result of $(1.1)-(1.3)$ and the similar problem without impulses (3.6)-(3.7) given below, for a wider class of nonlinearities that includes those satisfying the classical Lipschitz conditionas a special case (see Remark 3.1).

Although some types of differential and integral equations with supremum are investigated in [10, 12, 18] and the references therein, it is worth to mention that those existence results are inapplicable to (3.6)-(3.7). This is mainly due to the fact that the unbounded interval of time in (3.6)-(3.7) (which is not the case in the mentioned papers), fundamentally affects the functional setting used in the previously cited papers, and the possibility of generalizing these results to the unbounded case is not straightforward and requires further investigations. In this sens, the result of Corollary 3.7 is new and may complements some previously known results in the literature.

To our knowledge, most existence results for differential and integral equations of ordinary or fractional order are based on various tools in metric or normed spaces such as the fixed point theorems, the method of lower and upper solutions, etc. However, in the framework of uniform spaces, theses tools are used only in some few papers such as [6, 24, 28]. In this context, we also aim to contribute in this field of applications of the fixed point theory in uniform spaces, in particual when the classical variants of this theory frequently used in the literature, fail to be applicable directly (see Remark 3.6 (i)). This explains the approach chosen in this work, different from what is commonly used in such studies.

The rest of the paper is organized as follows. In Section 2, we recall the definition of the fractional derivatives, we introduce the tools used in this work and we give an auxiliary lemma. In Section 3, we prove the existence-uniqueness result for (1.1)-(1.3) and the similar problem without impulses. Finally, in Section 4 we provide two examples to illustrate the applicability of our theoretical results.

2 Preliminaries

Let us recall the notion of the fractional derivatives. For further details on some essential related properties, we refer to [13, 19].
Let n be a positive integer, α the positive real such that $n-1<\alpha\le n$ and d^n/dt^n the classical derivative operator of order n.

Definition 2.1. [[13, 19]] The Riemann-Liouville fractional integral, and the Riemann-Liouville fractional derivative, of a real function u defined on $\mathbb{R}_+$ of order α, are defined respectively by

$$ I_{0^+}^{\alpha} u\left(t\right) := \frac{1}{\Gamma\left(\alpha\right)}\int_0^t (t-s)^{\alpha-1} u\left(s\right) ds, \qquad t>0, $$

$$ D_{0^+}^{\alpha} u\left(t\right) := \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} u\left(t\right) := \frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1} u\left(s\right) ds, \qquad t>0, $$

where $\Gamma(.)$ is the Gamma function, provided that the right hand sides exist.

Definition 2.2. [[13, 19]] The Caputo fractional derivative of a real function u defined on $\mathbb{R}_+$ of order α, noted by ${}^{C}D^{\alpha}_{0+}$, is defined by

$$ {}^{C}D^{\alpha}_{0+}u(t) := \left(D^{\alpha}_{0+}\left[u - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!}(.)^k \right] \right)(t), \qquad t > 0, $$

provided that the right hand side exists point wise.

In what follows we give a fixed point theorem for non-self mappings in uniform spaces [28, Theorem 1], on which our main result is based. We also define some useful and related concepts and notations which can be found in [5, 6, 28].

Let X be a Hausdorff sequentially complete locally convex topological vector space and $\mathcal{P} = \{P_K : K \in \mathcal{K}\}$ be a saturated family of semi-norms, generating the topology of X, where $\mathcal{K}$ is an index set. For further details on locally convex spaces, we refer to [20].
For $\mathbf{E} \subset X$ we denote by $\partial\mathbf{E}$ the boundary of $\mathbf{E}$. Let $j : \mathcal{K} \longrightarrow \mathcal{K}$ be a mapping from the index set into itself, and $j^n(K) = j(j^{n-1}(K))$ stands for the n^{th} iterate of the mapping j, where $j^0(K) = K$.

Definition 2.3 ([5, 28]). A nonempty closed subset $\mathbf{E}$ of X is said to be j-bounded, if for every $u, v \in \mathbf{E}$ and $K \in \mathcal{K}$

$$ C(K, u, v) := sup\left\{ P_{j^n(K)}(u - v) : n = 0, 1, 2, \dots \right\} < \infty. \tag{2.1} $$

Definition 2.4 ([6, 28]). A nonlinear mapping $F : \mathbf{E} \longrightarrow X$ is said to be j-contractive if for any $K \in \mathcal{K}$

$$ \exists L_K \in [0, 1[\text{ such that } \quad P_K(Fu - Fv) \le L_K \cdot P_{j(K)}(u - v), \quad \forall u, v \in \mathbf{E} $$

Definition 2.5 ([7]). X is said to be metrically convex with respect to some semi-norm $P_K \in \mathcal{K}$, if for any u, v in X (with $u \neq v$), there exists a point w in X $(u \neq w \neq v)$ such that

$$ P_K(u - w) + P_K(w - v) = P_K(u - v). \tag{2.2} $$

Theorem 2.6. [28, Theorem 1] Assume that X is metrically convex with respect to every semi-norm. Let $\mathbf{E}$ be a j-bounded subset of X and $F : \mathbf{E} \longrightarrow X$ be a j-contractive mapping such that

(i) For every $K \in \mathcal{K}$ the following series is convergent

$$ \sum_{n=0}^{\infty} 2^{2n} L_K L_{j(K)} \dots L_{j^n(K)}. \tag{2.3} $$

(ii) $F(\partial\mathbf{E}) \subset \mathbf{E}$.

Then F has a unique fixed point in $\mathbf{E}$.

2.1 Concept of solutions for (1.1)-(1.3)

Because of the non-local character of fractional order differential operators, there is a fundamental dependence between the fractional derivative and the lower integration bound considered in its definition. Consequently, this leads to different concepts of solutions for impulsive fractional differential equations. Currently, there are mainly two approaches (see [1, 2]). We will precise in Definition 2.7 below, the approach adopted in this work.

Let X be the following locally convex sequentially complete Hausdorff space of real valued piecewise continuous functions defined on $\mathbb{R}$:

$$ X = \Big\{ \quad u : \mathbb{R} \to \mathbb{R} : u\,|_{\mathbb{R}_-} \in \mathcal{C}(\mathbb{R}_-) \text{ and for every } k \in \mathbb{N} : \\ u\,|_{J_k} \in \mathcal{C}(J_k) \text{ with } u(t_k^+) \text{ finite} \Big\}. \tag{2.4} $$

and $\{P_K : K \in \mathcal{K}\}$ be the saturated family of semi-norms, generating the topology of X, defined by

$$ P_K(u) = \sup_{t \in K}\left\{ e^{-\lambda t}\,|u(t)| \right\}, \tag{2.5} $$

where K runs over the set of all compact subsets of $\mathbb{R}$ denoted by $\mathcal{K}$, and λ is a positive real number to be specified later.

In the following definition of solutions, the approach A_1 in [2] is adapted to (1.1)-(1.3).

Definition 2.7. [[2]] A function $u \in X$ is said to be a solution of $(1.1)-(1.3)$, if it satifies (1.3), moreover $u = u_0 \mid_{J_0}$, and for every $k \in \mathbb{N}^*$: $u = u_k \mid_{J_k}$, where $u_0 \in \mathcal{C}\left(]-\infty, t_1]\right)$ is the solution of

$$\begin{cases} {}^C D^{\alpha}_{0^+} u\left(t\right) = f\left(t, \sup\limits_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\tau_1(t)),\ldots,u(t-\tau_N(t))\right), & t\in]0,t_1] \\ u\left(t\right) = \phi\left(t\right), & t\leq 0 \end{cases}$$

and for $k \in \mathbb{N}^*$: $u_k \in \mathcal{C}\left(]-\infty, t_{k+1}]\right)$ is the solution of

$$\begin{cases} {}^C D^{\alpha}_{0^+} u\left(t\right) = f\left(t, \sup\limits_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\tau_1(t)),\ldots,u(t-\tau_N(t))\right), & t\in[0,t_{k+1}] \\ u(t_k^+) = u(t_k^-) + I_k\left(u\left(t_k^- - \rho_k(t_k^-)\right)\right), \\ u\left(t\right) = \phi\left(t\right), & t< 0 \end{cases}$$

2.2 Equivalent integral equation of (1.1)-(1.3)

To formulate an equivalent integral equation of $(1.1)-(1.3)$, we need the following assumptions.

(H_1) a, b are continuous functions such that

(i) $\forall t \geq 0$: $0 \leq \overline{a} \leq a(t) \leq b(t)$, with $\overline{a} = \inf a(t)$, $\exists \overline{b}$ such that $\overline{a} < \overline{b}$, $a(t) = \overline{a}$, $b(t) = \overline{b} : \forall t \in \left[0, \overline{b}\right]$ and $b(t) \leq t$ for $t > \overline{b}$.

(ii) $\overline{b} < t_1$

(H_2) τ_i $(1 \leq i \leq N)$ is a continuous function such that

(i) $\exists \tau > 0 :\ \tau_i(t) > t - \tau,\ \ \forall t > 0.$

(ii) $\tau < t_1$, and if $\tau_i(t) \in \{t_k\}_{k=1}^{\infty}$, then $t - \tau_i(t) \in \{t_k\}_{k=1}^{\infty}$.

(iii) $\exists h_i > 0$ such that: $\tau_i(t) \geq t,\ \forall t \in [0, h_i]$ and $\tau_i(t) < t$, $\forall t \in]h_i, +\infty[$.

(H_3) For every $k \in \mathbb{N}^* : \rho_k(t) > 0, \quad \forall t > 0$

Using the reproduction of the proof of [24, Lemma 1] for an interval of time equals $]-\infty, t_1]$ instead of whole $\mathbb{R}$, in addition of [14, Lemma 2.6] with a slight adaptation, we get the equivalent integral equation to (1.1)-(1.3) given by the following lemma.

Lemma 2.8. Let f be a continuous function and $(H_1), (H_2.(i), (ii)), (H_3)$ hold true. Then, $u \in X$ is a solution of (1.1)-(1.3) if and only if u is a solution of the following integral equation

$$u(t) = \begin{cases} \phi_0 + \displaystyle\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \sup\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\tau_1(s)),\ldots,u(s-\tau_N(s))\right) ds, & t\in J_0, \\ \phi_0 + \displaystyle\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \sup\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\tau_1(s)),\ldots,u(s-\tau_N(s))\right) ds & \\ \quad + \displaystyle\sum_{i=1}^{k} I_i(u(t_i^- - \rho_i(t_i^-))), & t\in J_k : k\in\mathbb{N}^* \\ \phi(t), & t\leq 0 \end{cases} \tag{2.6}$$

3 Main Results

To prove our main results, we need the following further assumptions.

(H_4) There exist positive real valued functions L_1 and L_2 defined on $\mathbb{R}_+$, satisfying

(i) $\left|f\left(t,\xi,x_1,\ldots,x_N\right) - f\left(t,\eta,y_1,\ldots,y_N\right)\right| \leq L_1\left(t\right)\left|\xi - \eta\right| + L_2\left(t\right)\sum\limits_{i=1}^{N}\left|x_i - y_i\right|$ whenever the left hand side is defined.

(ii) For $\lambda > 0$ and $\nu := 1 + 1/\alpha$, we have

$$R_\lambda := \int_0^{+\infty} L_1^\nu(s)\, e^{-\nu\lambda(s-b(s))} ds < \infty, \text{ and } S_\lambda = \int_0^{+\infty} L_2^\nu(s)\, e^{-\nu\lambda s} ds < \infty.$$

(H_5) $\forall k \in \mathbb{N}^*$, $\exists l_k > 0$ such that: $|I_k(x) - I_k(y)| \leq l_k |x - y|$, $\forall x, y \in \mathbb{R}$

(H_6) There exists a positive constant $M > \phi_0$ such that

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, M, x_1, ..., x_N) \leq \frac{M - \phi_0}{\overline{b}^\alpha}, \ \forall (t,x_1,...,x_N) \in [0, \overline{b}] \times \mathbb{R}^N.$$

(H_7) f is a non negative function and moreover, $\exists \iota \in]\phi_0, M[$ such that

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, \iota, x_1, ..., x_N) > \frac{M - \phi_0}{|\overline{b} - \max\limits_{1\leq i\leq N} h_i|^\alpha}, \forall (t,x_1,...,x_N) \in [0, \overline{b}] \times \left(]\phi_0, \phi_0 + \tfrac{\iota - \phi_0}{\overline{b}}\tau[\right)^N.$$

Remark 3.1. Note that if L_1 and L_2 are bounded functions then the requirement stated in $(H_4.(ii))$ is satisfied (recall that $b(t) \leq t$ for $t > \overline{b}$). However, we point out that with (H_4), not only L_1 and L_2 but even f is allowed to increase indefinitely with respect to t (an example is given in the last section). So, hypothesis (H_4) includes a wide range of functions compared to the classical Lipschitz condition.

We denote by $\mathbf{E}_{\phi,M}$ the subset of X defined by

$$\mathbf{E}_{\phi,M} = \left\{u \in X : u(t) = \phi(t) \text{ for } t \leq 0, \text{ and } u(t) \leq M \text{ for } t \in [\overline{a}, \overline{b}]\right\},$$

where $\overline{a}, \overline{b}$ and M are the constants given by (H_1) and (H_6). $\mathbf{E}_{\phi,M}$ is closed and its boundary is:

$$\partial\mathbf{E}_{\phi,M} = \left\{u \in X : u(t) = \phi(t) \text{ for } t \leq 0 \text{ and } \max_{t\in[\overline{a}, \overline{b}]} u(t) = M\right\}.$$

Note that the extremities of the interval on which the supremum is taken, a and b depend on the time t. This is motivated by applications in the theory of neutral functional differential equations [5, 6, 28], but it is also used to force the corresponding operator to be contractive in the sens of Definition 2.4. Following this idea, let us define a map $j : \mathcal{K} \longrightarrow \mathcal{K}$ by

$$j(K) := \begin{cases} K & \text{if } K_+ = \emptyset, \\ [0, \max\{K_m, \tau, \overline{b}\}] & \text{if } K_+ \neq \emptyset, \end{cases} \tag{3.1}$$

where $K_+ := K \cap]0, +\infty[$, $K_m = \sup K$.

Lemma 3.2. [24, Remark 2] For j defined by (3.1), the set $\mathbf{E}_{\phi,M}$ is j-bounded.

Throughout the remaining of this paper, F denotes the operator defined on $\mathbf{E}_{\phi,M}$ by the right hand side of (2.6). Under some appropriate hypotheses, the following lemma asserts the non-self nature of this operator.

Lemma 3.3. Under the hypotheses $(H_1) - (H_3)$, F maps $\mathbf{E}_{\phi,M}$ into X and moreover, (H_7) ensure that it does not map $\mathbf{E}_{\phi,M}$ into itself provided that

$$\max_{1\leq i\leq N} h_i < \overline{b} \tag{3.2}$$

Proof . Recall first that the functions f, τ_i $(1 \leq i \leq N)$, a, b and ρ_k $(k \in \mathbb{N}^*)$ are all continuous. Let now $u \in \mathbf{E}_{\phi,M} \subset X$, then $u|_{J_k} \in \mathcal{C}(J_k)$, so in view of $(H_1) - (H_3)$, $Fu \in \mathcal{C}(J_k)$ as a composition of continuous functions on J_k. Since for every $k \in \mathbb{N}^*$, $u(t_k^+)$ exists, then $Fu(t_k^+)$ exists also as a limite of a composotion of functions, which admit each of them a limit in t_k^+. Hence $Fu \in X$.
Let now $v_0 \in \mathbf{E}_{\phi,M}$ be a function defined by means of the constant ι given in (H_7) such that for $t \in [0, \overline{b}]$:

$$v_0(t) = \phi_0 + \frac{(\iota - \phi_0)t}{\overline{b}}$$

According to (H_1), $(H_2.(i),(iii))$, (H_7) and (3.2), we get

$$\begin{aligned} Fv_0\left(\bar{b}\right) &= \phi_0 + \int_0^{\bar{b}} \frac{\left(\bar{b}-s\right)^{\alpha-1}}{\Gamma(\alpha)} f\left(s,\iota,v_0(s-\tau_1(s)),\ldots,v_0(s-\tau_N(s))\right) ds \\ &> \phi_0 + \alpha\, \frac{M-\phi_0}{(\bar{b}-\max\limits_{1\le i\le N} h_i)^{\alpha}} \int_{\max\limits_{1\le i\le N} h_i}^{\bar{b}} (\bar{b}-s)^{\alpha-1} ds = M \end{aligned}$$

Which means that $Fv_0 \notin \mathbf{E}_{\phi,M}$ and completes the proof. □

Proposition 3.4. Let $(H_1)-(H_5)$ be satisfied. Then, for each $u,v\in\mathbf{E}_{\phi,M}$ and every $K\in\mathcal{K}$ the following inequality holds true:

$$P_K\left(Fu-Fv\right) \le C_\lambda P_{j(K)}\left(u-v\right), \tag{3.3}$$

where

$$C_\lambda = \frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)\,\lambda^{\frac{\alpha^2}{\alpha+1}}}\left(R_\lambda^{\frac{\alpha}{\alpha+1}} + N\,e^{\lambda\tau}\,S_\lambda^{\frac{\alpha}{\alpha+1}}\right) + \sum_{k\ge 1} l_k$$

Proof . For $K_+=\emptyset$, we have

$$P_K\left(Fu-Fv\right) = P_K\left(\phi_0-\phi_0\right) = 0,$$

and thus (3.3) is satisfied.

Now for $K_+\neq\emptyset$, let $t\in K_+$. Using (H_1), $(H_2.(iii))$, (H_3), $(H_4.(i))$ and (H_5), we obtain

$$\begin{aligned} |Fu(t)-Fv(t)| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L_1(s) \left| \sup_{\sigma\in[a(s),b(s)]} u(\sigma) - \sup_{\sigma\in[a(s),b(s)]} v(\sigma)\right| ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L_2(s) \sum_{i=1}^N |u(s-\tau_i(s)) - v(s-\tau_i(s))| ds \\ &+ \sum_{t_k<t} l_k \left|u(t_k^- - \rho(t_k^-)) - v(t_k^- - \rho(t_k^-))\right| \\ &\le \int_0^t L_1(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sup_{\sigma\in[a(s),b(s)]} |u(\sigma)-v(\sigma)| ds \\ &+ \sum_{i\in\{1,\ldots,N: h_i\le t\}} \int_{h_i}^t L_2(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(r_i(s)) - v(r_i(s))| ds \\ &+ \sum_{t_k<t} l_k \left|u(t_k^- - \rho(t_k^-)) - v(t_k^- - \rho(t_k^-))\right| \\ &\le \int_0^t L_1(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} \max_{\sigma\in[\bar{a},\max\{K_m,\bar{b}\}]} e^{-\lambda\sigma} |u(\sigma)-v(\sigma)| ds \\ &+ \sum_{i\in\{1,\ldots,N: h_i\le t\}} \int_{h_i}^t L_2(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} e^{-\lambda r_i(s)} |u(r_i(s)) - v(r_i(s))| ds \\ &+ \sum_{t_k<t} l_k.e^{\lambda t} \max_{\delta\in[0,\,t]} e^{-\lambda\delta} |u(\delta)-v(\delta)|, \end{aligned}$$

where $r_i(s) = s-\tau_i(s)$. Taking into account (3.1) together with $(H_2.(i),(iii))$, we get:

$$|Fu(t)-Fv(t)| \le$$
$$\left[\int_0^t L_1(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} ds + \sum_{i\in\{1,\ldots,N: h_i\le t\}} \int_{h_i}^t L_2(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda r_i(s)} ds + \sum_{t_k<t} l_k.e^{\lambda t}\right] P_{j(K)}(u-v).$$

Now, multiplying both sides of the above inequality by $e^{-\lambda t}$, we obtain:

$$e^{-\lambda t}\left|Fu\left(t\right)-Fv\left(t\right)\right|\leq\left[\int_{0}^{t}L_{1}\left(s\right)\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}e^{-\lambda(t-b(s))}ds\right.$$

$$+\sum_{i\in\{1,\dots,N:h_i\leq t\}}\int_{h_i}^{t}L_{2}\left(s\right)\frac{\left(t-s\right)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-r_i(s))}ds$$

$$\left.+\sum_{t_k<t}l_k\right]P_{j(K)}\left(u-v\right)$$

The change of variable $x=\lambda\left(t-s\right)$, leads to:

$$e^{-\lambda t}\left|Fu\left(t\right)-Fv\left(t\right)\right|\leq$$

$$\left[\tfrac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_{0}^{\lambda t}L_1(t-\frac{x}{\lambda})\,x^{\alpha-1}e^{-x}e^{-\lambda((t-\frac{x}{\lambda})-b(t-\frac{x}{\lambda}))}dx\,+\right.$$

$$\tfrac{1}{\lambda^{\alpha}\Gamma(\alpha)}\sum_{i\in\{1,\dots,N:h_i\leq t\}}\int_{0}^{\lambda(t-t_i)}L_2(t-\frac{x}{\lambda})\,x^{\alpha-1}e^{-x}e^{-\lambda\tau_i\left(t-\frac{x}{\lambda}\right)}dx$$

$$\left.+\sum_{t_k<t}l_k\right]P_{j(K)}\left(u-v\right)$$

According to $(H_4.(ii))$, Hölder's inequality toghether with $(H_2.(i))$, give:

$$\begin{aligned}
e^{-\lambda t}\left|Fu\left(t\right)-Fv\left(t\right)\right| \;&\leq\left[\frac{1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\left(\int_{0}^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(\int_{0}^{\lambda t}L_1^{\nu}(t-\frac{x}{\lambda})e^{-\nu\lambda((t-\frac{x}{\lambda})-b(t-\frac{x}{\lambda}))}dx\right)^{\frac{1}{\nu}}\right.\\
&\quad+\frac{1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\sum_{i\in\{1,\dots,N:h_i\leq t\}}\left(\int_{0}^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\\
&\quad\left.\left(\int_{0}^{\lambda t}L_2^{\nu}(t-\frac{x}{\lambda})e^{-\nu\lambda\tau_i\left(t-\frac{x}{\lambda}\right)}dx\right)^{\frac{1}{\nu}}+\sum_{k\geq 1}l_k\right]P_{j(K)}\left(u-v\right)\\
&\leq\left[\frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}}{\Gamma(\alpha)\,\lambda^{\frac{\alpha^2}{\alpha+1}}}\left(R_{\lambda}^{\frac{1}{\nu}}+N\,e^{\lambda\tau}\,S_{\lambda}^{\frac{1}{\nu}}\right)+\sum_{k\geq 1}l_k\right]P_{j(K)}\left(u-v\right)
\end{aligned}$$

Finally, by taking the supremum on K, this yields immediately to (3.3) and completes the proof. □

The following theorem provides sufficient conditions for the existence-uniqueness result of (1.1)-(1.3).

Theorem 3.5. Let hypotheses $(H_1)-(H_7)$ and condition (3.2) hold true. If the following relation is satisfied:

$$\max\{\tau,\,\bar{b}\}^{\frac{\alpha^2}{\alpha+1}}\,\frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)}\,\left(R_{1/\max\{\tau,\bar{b}\}}^{\frac{\alpha}{\alpha+1}}+N\,e\,S_{1/\max\{\tau,\bar{b}\}}^{\frac{\alpha}{\alpha+1}}\right)+\sum_{k\in\mathbb{N}^*}l_k<\frac{1}{4},\tag{3.4}$$

then the problem (1.1)-(1.3) admits a unique global solution in $\mathbf{E}_{\phi,M}$.

Proof . Recall that, in view of Lemma 2.8, the solutions of (1.1)-(1.3) are the fixed points of the operator F. Hence it is sufficient to show that all conditions of Theorem 2.6 are fulfilled.

For any u, v in X (with $u \neq v$), let $\beta \in]0,1[$, then $w_\beta = (1-\beta)u + \beta v$ is such that $u \neq w \neq v$. A straightforward computation yields to (2.2) and hence X is metrically convex with respect to every semi-norm.

Note first that the series (2.3) in our case turns into

$$\sum_{n=0}^{\infty} 2^{2n} C_\lambda^{n+1} \tag{3.5}$$

Let us now choose $\lambda = 1/\max\{\tau, \overline{b}\}$ in (2.5). In view of Proposition 3.4, condition (3.4), means that (3.3) holds true with $C_\lambda < 1/4$. Consequently F is j-contractive and moreover the series (3.5) converges.

Let now $u \in \partial \mathbf{E}_{\phi,M}$, then for $\overline{a} \leq t \leq \overline{b}$, (H_1) and (H_6) give:

$$\begin{aligned} Fu\,(t) \quad &= \phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, M, u(s-\tau_1(s)), \ldots, u(s-\tau_N(s))\right) ds \\ &\leq \phi_0 + \frac{\alpha(M-\phi_0)}{\overline{b}^\alpha} \int_0^t (t-s)^{\alpha-1}\, ds \leq M, \end{aligned}$$

which means that $F(\partial \mathbf{E}_{\phi,M}) \subset \mathbf{E}_{\phi,M}$ and completes the proof. □

Remark 3.6.

(i) We emphasize here, that under the hypotheses of Theorem 3.5 and in view of Lemma 3.3, the operator F is a non self mapping. Thus, the classical variants of fixed point theorems frequently used in the literature, fail to be applicable directly.

(ii) In addition to the existence and uniqueness result for solutions of (1.1)-(1.3), Theorem 3.5 also provides (with appropriate adaptation) a generalization and improvements of [24, Theorem 1] given by Corollary 3.7 below. The generalization is due to the assuption (H_1) that allows the supremum to be taken on very long time intervals for some required necessity. A first aspect of improvement is due to (H_4) (see Remark 3.1) and a second is due to the condition (3.8) which is less restrictive than (21) in [24] for values of α close to 1 in the particular case where L_1 and L_2 in (H_4) are constants.

Corollary 3.7. Under hypotheses $(H_1.(i))$, $(H_2.(i),(iii))$, (H_4), (H_6), (H_7) and condition (3.2), the problem

$${}^{C}D_{0^+}^{\alpha} u\,(t) = f\left(t, \max_{\sigma \in [a(t),\, b(t)]} u(\sigma), u(t-\tau_1(t)), \ldots, u(t-\tau_N(t))\right), \qquad t > 0 \tag{3.6}$$

$$u\,(t) = \phi\,(t)\,, \quad t \leq 0, \tag{3.7}$$

admits a unique continuous solution in $\mathbf{A}_{\phi,M}$ provided that

$$\max\{\tau, \overline{b}\}^{\frac{\alpha^2}{\alpha+1}} \frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)} \left(R_{1/\max\{\tau,\overline{b}\}}^{\frac{\alpha}{\alpha+1}} + N\,e\,S_{1/\max\{\tau,\overline{b}\}}^{\frac{\alpha}{\alpha+1}}\right) < \frac{1}{4}. \tag{3.8}$$

Where

$$\mathbf{A}_{\phi,M} = \left\{u \in \mathcal{C}\,(\mathbb{R}) : u\,(t) = \phi\,(t) \text{ for } t \leq 0, \text{ and } u\,(t) \leq M \text{ for } t \in [\overline{a},\, \overline{b}]\right\}$$

Proof . By omitting the impulsive condition (1.2), that is when for every $k \in \mathbb{N}^* : I_k = 0$, the problem (1.1)-(1.3) is reduced to (3.6)-(3.7). Furthermore, the hypotheses of Theorem 3.5 reduces to those of Corollary 3.7.
Note also that in this particular case, condition (1.2) with $I_k = 0$, implies that:

$$\Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k\left(u\left(t_k - \rho_k(t_k)\right)\right) = 0.$$

Which means that the space X defined in (2.4) becomes $\mathcal{C}\,(\mathbb{R})$ and consequently, the subset $\mathbf{E}_{\phi,M}$ becomes $\mathbf{A}_{\phi,M}$.
Now, since the condition (3.4) includes as a special case the condition (3.8) in which $l_k = 0$ ($k \in \mathbb{N}^*$), the result follows immediately from Theorem 3.5. □

4 Examples

We conclude with two examples illustrating the validity of our findings including the improvements explained in Remark 3.6.

Example 1. Let us consider the following problem

$$\begin{aligned} {}^{C}D^{0.5}u(t) = & \frac{4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}}{3\times10^{-1}+2\times10^{-3}\left|\sup\limits_{\sigma\in[a(t),\,b(t)]} u(\sigma)\right|} \\ & + \frac{5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}}{1+3\times10^{-4}\left|u\left(3\times10^{-4}-\frac{3\times10^{-4}+12\times10^{-8}}{1+t}\right)\right|},\ t>0,\ t\neq10^{k},\ k\in\mathbb{N}^{*} \end{aligned} \tag{4.1}$$

$$\Delta u(10^{k}) = \frac{1}{4ek(k+1)\left(1+\left|u\left(-\frac{1}{5k}-\frac{3\times10^{k}}{4+4\times10^{k}}\right)\right|\right)}, \qquad k\in\mathbb{N}^{*} \tag{4.2}$$

$$u(t) = t+1, \quad t\leq 0 \tag{4.3}$$

where

$$a(t)=\begin{cases} 10^{-1}: & 0\leq t\leq\frac{1}{2} \\ 2\times10^{-1}t: & t\geq\frac{1}{2} \end{cases} \quad\text{and}\quad b(t)=\begin{cases} 1/2: & 0\leq t\leq\frac{1}{2} \\ (2t+1)/4: & t\geq\frac{1}{2} \end{cases}$$

The problem (4.1)-(4.3) is identified to (1.1)-(1.3) with:

$$\begin{aligned} &\alpha=0.5,\ \ N=1,\ \ \tau_1(t)=t-3\times10^{-4}+\frac{3\times10^{-4}\left(1+4\times10^{-4}\right)t}{(1+t)}, \\ &f(t,x,y)=\frac{4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}}{3\times10^{-1}+2\times10^{-3}|x|}+\frac{5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}}{1+3\times10^{-4}|y|},\ \ \phi(t)=t+1 \end{aligned} \tag{4.4}$$

and

$$\{t_k\}_{k\in\mathbb{N}^*}=\{10^k\}_{k\in\mathbb{N}^*},\quad I_k(x)=\frac{1}{4ek(k+1)\ (1+|x|)},\quad \rho_k(t)=t+\frac{1}{5k}+\frac{3t}{4+4t}$$

It is not hard to see that hypotheses $(H_1)-(H_3)$ are satisfied with

$$\overline{a}=10^{-1},\ \overline{b}=\frac{1}{2},\ \tau=3\times10^{-4}\ \text{and}\ \ h_1=4\times10^{-4}$$

Let us verify (H_4), we have

$$\begin{aligned} |f(t,\xi,x)-f(t,\eta,y)| \leq\ & \frac{2\times4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}\times10^{-3}\left(|\eta|-|\xi|\right)}{\left(3\times10^{-1}+2\times10^{-3}|\xi|\right)\left(3\times10^{-1}+2\times10^{-3}|\eta|\right)} \\ & + \frac{3\times5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}\times10^{-4}\left(|y|-|x|\right)}{\left(1+3\times10^{-4}|x|\right)\left(1+3\times10^{-4}|y|\right)} \\ & \leq 9.44\times10^{-2}\left(\tfrac{t}{3}+1\right)^{\frac{1}{3}}|\xi-\eta| \\ & +16.29\times10^{-4}\left(\tfrac{2t}{15}+\tfrac{1}{3}\right)^{\frac{1}{3}}|x-y| \end{aligned}$$

So, $(H_4.(i))$ is fulfilled with

$$L_1(t)=9.44\times10^{-2}\left(\frac{t}{3}+1\right)^{\frac{1}{3}} \quad\text{and}\quad L_2(t)=16.29\times10^{-4}\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}$$

A straightforward computation leads to

$$R_\lambda = 8.42421124\times10^{-4}\left(\frac{63\lambda+18}{162\lambda^2}+\frac{9\lambda+1}{27\lambda^2}e^{\frac{3\lambda}{2}}\right) \quad\text{and}\quad S_\lambda = 4.32278119\times10^{-9}\left(\frac{15\lambda+2}{135\lambda^2}\right)$$

Which means that $(H_4.(ii))$ is satisfied too. Now, for all $k \in \mathbb{N}^*$ we have:

$$\begin{aligned}
|I_k(x) - I_k(y)| &\leq \frac{1}{4ek(k+1)}\left|\frac{1}{1+|x|} - \frac{1}{1+|y|}\right| \\
&\leq \frac{1}{4ek(k+1)}\frac{|x-y|}{(1+|x|)(1+|y|)} \\
&\leq \frac{1}{4ek(k+1)}|x-y|
\end{aligned}$$

So (H_5) is verified with $l_k = \dfrac{1}{4ek(k+1)}$.

In addition, there exists $M = 15.2 > \phi_0 = 1$, such that for every $(t,x) \in [0, \frac{1}{2}] \times \mathbb{R}$, we have

$$f(t, M, x) \leq \frac{4.25\left(\frac{7}{6}\right)^{\frac{1}{3}}}{3\times 10^{-1} + 2\times 10^{-3}\times M} + 5.43\left(\tfrac{2}{5}\right)^{\frac{1}{3}} \simeq 17.5422869253$$

and

$$\alpha.\Gamma(\alpha).\frac{M-\phi_0}{\bar{b}^{\alpha}} = 0,5.\sqrt{\pi}\frac{15.2-1}{0.5^{0.5}} \simeq 17.7970607499$$

Consequently (H_6) holds true.

To check (H_7), let us choose $\iota = \phi_0 + 10^{-9}$, then we have for every $(t,x) \in [0, \frac{1}{2}] \times \left[\phi_0, \phi_0 + \frac{\iota-\phi_0}{\bar{b}}\tau\right]$:

$$\begin{aligned}
f(t,\iota,x) &> \frac{4.25}{3\times 10^{-1} + 2\times 10^{-3}\times \iota} \\
&+ \frac{5.43\left(\frac{1}{3}\right)^{\frac{1}{3}}}{1 + 3\times 10^{-4}\left(\phi_0 + \frac{\iota-\phi_0}{\bar{b}}\tau\right)} \simeq 17.8377994017
\end{aligned}$$

and

$$\alpha.\Gamma(\alpha)\frac{M-\phi_0}{|\bar{b}-h_1|^{\alpha}} = 0,5.\sqrt{\pi}.\frac{15.2-1}{(0,4996)^{\frac{1}{2}}} \simeq 17.8041838483$$

Since moreover f is clearly non negative, hypothesis (H_7) holds true.

Furthermore, (3.2) is clearly satisfied and

$$\max\{\tau,\bar{b}\}^{\frac{\alpha^2}{\alpha+1}}\frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)}\left(R^{\frac{\alpha}{\alpha+1}}_{1/\max\{\tau,\bar{b}\}} + NeS^{\frac{\alpha}{\alpha+1}}_{1/\max\{\tau,\bar{b}\}}\right) + \sum_{k\in\mathbb{N}^*} l_k \simeq 0.0928213$$

That is, all conditions of Theorem 3.5 are fulfilled, and consequently (4.1)-(4.3) has a unique solution in $\mathbf{E}_{\phi,M}$, where

$$\begin{aligned}
\mathbf{E}_{\phi,M} = \Big\{ u \in X : u(t) = t+1 \quad \text{for} \quad t \leq 0 \quad \text{and} \\
u(t) \leq 15.2 \quad \text{for } t \in [10^{-1}, 1/2]\Big\}
\end{aligned}$$

Note that f increases indefinitely with respect to t as it is pointed out in Remark 3.1.

Example 2. Let $(\mathcal{P})$ be the special form of (3.6)-(3.7) corresponding to α, N, τ_1, f, and ϕ given by (4.4) with

$$a(t) = 10^{-1} \quad \text{and} \quad b(t) = \begin{cases} \frac{1}{2} & 0 \leq t \leq \frac{1}{2} \\ 0.2 - \frac{1}{0.999\,t} & t \geq \frac{1}{2} \end{cases} \tag{4.5}$$

From the discussion done in Example 1, we immediately deduce that all conditions of Corollary 3.7 are satisfied and consequently the problem on question admits a unique continuous solution in $\mathbf{A}_{\phi,M}$, where

$$\mathbf{A}_{\phi,M} = \Big\{ u \in \mathcal{C}(\mathbb{R}) : u(t) = t+1 \quad \text{for} \quad t \leq 0 \quad \text{and} \\ u(t) \leq 15.2 \quad \text{for} \;\; t \in \left[10^{-1},\, 1/2\right] \Big\}$$

Note that $b(t)$ in (4.5) is bounded by $\bar{b} = \frac{1}{2}$, that is [24, Theorem 1] covers $(\mathcal{P})$, but fails to be applicable as f do not satisfies the Lipschitz condition with constant arguments.

5 Conclusion

In this paper, we investigate a class of initial value problems of nonlinear fractional differential equations with state deviating arguments, delayed impulses and supremum on the half line.

These equations appear in the modeling of many real-world processes (in automatic control theory, for example). In this type of considered equations, what determines the solution at a time t, is the current state but also its maximum value over a certain past time interval. After converting the problem into an equivalent integral equation, by means of some techniques from the context of generalized contractions, appropriate conditions on the data, lead to a global existence-uniqueness result. This, without any bounding restriction with respect to time, neither on the solution nor on the reaction part of the problem.

It should be noted that our approach is based mainly on the theory of fixed points in uniform spaces, which is rarely used in this field. Our result is obtained in a setting where classical variants of the fixed point theorems frequently used in the literature are inapplicable, and moreover allowing the nonlinear part of the problem to increase indefinitely with time.

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