Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1891–1901
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26019.3192

Some discussion on generalizations of metric spaces in fixed point perspective

Jayesh Savaliya[a,*], Dhananjay Gopal[b], Shailesh Kumar Srivastava[a]

[a] *Department of Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat-395007, India*

[b] *Department of Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur, Chhattisgarh-495009, India*

(Communicated by Oluwatosin Temitope Mewomo)

Abstract

The motive of this paper is to provide an understanding of the role of generalizations of metric spaces in a fixed point perspective. For this purpose, the concept of the pseudo non-triangular metric is introduced. Further, we study and analyze the structure of open sets, closed sets, and other topological properties of the new metric. Then, we compare it with JS-metric, strong JS-metric as well as non-triangular metric and observe that pseudo non-triangular metric becomes the bare minimum metric structure required to prove a new fixed point theorem for contractive type mappings. Finally, we establish the Caristi type fixed point theorem, which generalizes some well-known results, including recent developments by Karapinar et al. [15].

Keywords: Pseudo non-triangular metric, Non-triangular metric, Fixed Point, Caristi-type contractive map
2020 MSC: Primary 54H25; Secondary 47H10

1 Introduction

The metric fixed point theory is a well-established topic of research. The Banach contraction principle [1] has a huge impact on the establishment of this theory. A closer look at the proofs of fixed point theorems that involve contractive type mappings in various metric spaces (generalized metric spaces) suggests that each new theorem is either aimed at enlarging the class of contraction or the metric structure or both. In the case of the further generalization of the metric, the main aim is to weaken the triangle inequality.

On the other hand, due to the immense applicability of the Banach contraction principle in various branches of science and engineering, a number of generalizations were made by various researchers(see, for instance- [18, 13, 10, 16], and references cited therein). Branciari [2] presented the generalization of metric space in the year 2000. This led to a novel technique to study the metric fixed point theory. In the same year, Hitzler and Seda [12] presented the notion of dislocated metric space, which has application in logical programming. In 2015, Jleli and Samet [14] established the JS-metric space, which is an extension of many abstract metric spaces, like standard metric space, rectangular metric space [8], b-metric space [5], dislocated metric space [14], and some others. In 2018, Gajic and Ralevic [7] introduced strong JS-metric space, a subclass of JS-metric space. The results of Sehgal and Thomas [3] in generalized

*Corresponding author
Email addresses: jdsavaliya555@gmail.com (Jayesh Savaliya), gopaldhananjay@yahoo.in (Dhananjay Gopal), shaileshiitr2010@gmail.com (Shailesh Kumar Srivastava)

Received: January 2022 *Accepted:* November 2022

metric space related to strong JS-metric space were also improved. In 2020, Khojasteh and Khandani [17] established non-triangular metric space, an extension of JS-metric in some respects. Non-triangular metrics inherently use the competence of sequence instead of triangular inequality. Motivated by Khojasteh and Khandani [17], Deshmukh and Gopal [6] discussed some topological properties of non-triangular metric space and also proved the Suzuki type Z-contraction for non-triangular spaces

Inspired by the above work, we dive deep into the role of weakening triangle inequality to form a larger class of metric space. In this regard, we introduced the pseudo non-triangular metric space and studied its basic topological properties. We then compared it with JS-metric and non-triangular metrics and observed that in the convergence of a sequence (which is the key to proving the existence of a fixed point for contractive type mappings), these metrics coincide. As a result, the pseudo non-triangular metric becomes the bare minimum metric structure required to prove a new fixed point theorem for contractive type mappings.

Throughout the paper, we will consider the set Ω is non-empty and $\mathbb{R}^+ = [0, \infty)$.

2 Generalization of Metric Spaces.

Jleli and Samet characterise JS-metric spaces as follows [14]:

Definition 2.1 ((JS-metric space) [14]). A map $\varrho : \Omega \times \Omega \to [0, \infty]$ is called a JS-metric space if it possesses the requirement stated bellows for every $a, c \in \Omega$,

(JS_1) $\varrho(a\,,\,c) = 0 \implies a = c$,

(JS_2) $\varrho(a\,,\,c) = \varrho(c\,,\,a)$,

(JS_3) There exists $K > 0$ such that $\varrho(a, c) \leq K \limsup_{n\to\infty} \varrho(a_n, c)$, for $\{a_n\} \in C(\varrho,\,\Omega,\,a)$,

where $C(\varrho,\,\Omega,\,a) = \{(a_n)|\lim_{n\to\infty} \varrho(a_n\,,\,a) = 0\}$.

If $C(\varrho,\,\Omega,\,a)$ is empty for each $a \in \Omega$, then the third property of Definition 2.1 holds trivially.

Definition 2.2 (Non-Triangular Metric Space [6]). A map $\varrho : \Omega \times \Omega \to \mathbb{R}^+$ is a non-triangular metric on Ω if it fulfils the requirements below for each $a, c \in \Omega$,

(NT_1) $\varrho(a,\,a) = 0$;

(NT_2) $\varrho(a, c) = \varrho(c,\,a)$;

(NT_3) For each $\{a_n\} \subset \Omega$ with $\{a_n\} \in C(\varrho,\,\Omega,\,a)$ and $\{a_n\} \in C(\varrho,\,\Omega,\,c)$, then we have $a = c$.

Remark 2.3. [6] For a non-triangular metric, if we let $\varrho(a,\,c) = 0$ then a constant sequence $\{a_n\} = a$ converges to a and hence, using the third property of Definition 2.2, we get $a = c$.

Theorem 2.4. [6] Let $(\Omega,\,\varrho)$ be a JS-metric space (where we consider the map $\varrho : \Omega \times \Omega \to [0,\,\infty)$ only) such that for every $a \in \Omega$. If $C(\varrho,\,\Omega,\,a)$ is non-empty, then $(\Omega,\,\varrho)$ is a non-triangular metric space.

Remark 2.5. [6] The third property of Definition 2.2 concludes that a convergent sequence has a unique limit.

Definition 2.6 (Strong JS-Metric Space [7]). The map $\varrho^* : \Omega \times \Omega \to \mathbb{R}^+$ is called a Strong JS- metric if it fulfils the requirements stated below for all $a, c \in \Omega$:

(ϱ^*_1) $\varrho^*(a, c) = 0 \Leftrightarrow a = c$;

(ϱ^*_2) $\varrho^*(a, c) = \varrho^*(c, a)$;

(ϱ_3^*) there exists positive real number K, such that for all $a, c \in \Omega$ and $\{a_n\} \in C(\varrho^*, \Omega, a), \{c_n\} \in C(\varrho^*, \Omega, c)$,

$$\varrho^*(a,c) \leq K \limsup_{n\to\infty} \varrho^*(a_n, c_n).$$

Remark 2.7. [7] If we let sequence $\{c_n\}$ be constant in a property (ϱ_3^*) of Definition 2.6, it concludes that strong JS-metric space is a subclass of JS-metric space.

Remark 2.8. [7] Usual metric space and b-metric space satisfy all the properties of strong JS-metric space, and hence, every b-metric and metric is strong JS-metric.

Now, we establish an extension of strong JS-metric space.

Definition 2.9 (Pseudo Non-Triangular Metric Space). A map $\varrho : \Omega \times \Omega \to \mathbb{R}^+$ is called a pseudo non-triangular metric on a set Ω if it possesses the requirements stated below for all $a, c \in \Omega$:

(ϱ_1) $\varrho(a, a) = 0$;

(ϱ_2) $\varrho(a,c) = \varrho(c,a)$;

(ϱ_3) For some $K > 0$, a sequence $\{a_n\} \in C(\varrho, \Omega, a)$ and a sequence $\{c_n\} \in C(\varrho, \Omega, c)$,

$$\varrho(a,c) \leq K \limsup_{n\to\infty} \varrho(a_n, c_n).$$

For a pseudo non-triangular metric space, Cauchy sequence, convergence, and completeness are define as below:

Definition 2.10. Let (Ω, ϱ) be a pseudo non-triangular metric space and sequence $\{a_n\} \subset \Omega$. Then, a sequence $\{a_n\}$ converges to $a \in \Omega$, if $\lim_{n\to\infty} \varrho(a_n, a) = 0$.

Definition 2.11. Let (Ω, ϱ) be a pseudo non-triangular metric space and sequence $\{a_n\} \subset \Omega$. Then, a sequence $\{a_n\}$ is called Cauchy sequence, if $\lim_{m\to\infty} \varrho(a_n, a_{n+m}) = 0$ for each $m \in \mathbb{N}$.

Definition 2.12. Let (Ω, ϱ) be a pseudo non-triangular metric space and sequence $\{a_n\} \subset \Omega$. Then, the space (Ω, ϱ) is called complete if every Cauchy sequence in Ω is converges to some point $a \in \Omega$.

Proposition 2.13. Each usual metric space (Ω, ϱ) is a pseudo non-triangular metric space.

Usual metric space assures property (ϱ_1) and (ϱ_2) of Definition 2.9 trivially. Now we only need to check the condition (ϱ_3) of Definition 2.9; for that, let $\{a_n\} \xrightarrow{\varrho} a$ and $\{c_n\} \xrightarrow{\varrho} c$. Using triangular inequality of standard metric, for all $n \in \mathbb{N}$,

$$\varrho(a,c) \leq \varrho(a, a_n) + \varrho(a_n, c_n) + \varrho(c_n, c),$$

$$\implies \varrho(a,c) \leq \limsup_{n\to\infty} \varrho(a_n, c_n),$$

and hence (ϱ_3) is satisfied with $K = 1$.

Definition 2.14 (b-metric space [7]). A map $\varrho : \Omega \times \Omega \mapsto \mathbb{R}^+$ is said to be a b-metric space, if it possesses the following characteristics for every $a, c, z \in \Omega$,

(b_1) $\varrho(a,c) = 0 \Leftrightarrow a = c$,

(b_2) $\varrho(a,c) = \varrho(c,a)$,

(b_3) there exists $b \geq 1$ such that $\varrho(a,c) \leq b[\varrho(a, z) + \varrho(z, c)]$.

Proposition 2.15. Pseudo non-triangular metric space is an extension of b-metric space.

Proof . Let a b-metric ϱ on a set Ω, then the properties (ϱ_1) and (ϱ_2) of Definition 2.9 are satisfied trivially. Now we need to focus on the property (ϱ_3) of a pseudo non-triangular metric space. Let $a, c \in \Omega$ with the sequence $\{a_n\} \in C(\varrho, \Omega, a)$ and sequence $\{c_n\} \in C(\varrho, \Omega, c)$, from the property (b_3), for each $n \in \mathbb{N}$

$$\varrho(a, c) \leq b\varrho(a\ , a_n) + b^2\varrho(a_n\ , c_n) + b^2\varrho(c_n\ , c).$$

Thus we have,

$$\varrho(a, c) \leq b^2\varrho(a_n\ , c_n).$$

Hence, the property (ϱ_3) is satisfied with $K = b^2$. So every b-metric is a pseudo non-triangular metric. □

Example 2.16. Let $\Omega = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{n : n \in \{0, 2, 3, 4, 5, 6\}\}$. Define $\varrho : \Omega \times \Omega \to \mathbb{R}^+$ such that

$$\varrho(a, c) = \begin{cases} 0, & \text{if } a = c. \\ 8, & \text{if } a, c \in A. \\ \frac{a}{2}, & \text{if } a \in A, c \in \{3, 4\}. \\ \frac{c}{2}, & \text{if } a \in \{3, 4\}, c \in A. \\ 4, & \text{otherwise.} \end{cases}$$

From the construction of the map, $\varrho(a, a) = 0$, which satisfies the property (ϱ_1) of Definition 2.9, which implies that constant sequences are convergent. Now from construction $\varrho(a, c) = \varrho(c, a)$, which implies property (ϱ_2) of Definition 2.9 is satisfied.

As we want to prove the map is a pseudo non-triangular metric, now we need to verify the third property of Definition 2.9 only. To prove it let $a, c \in \Omega$ and the sequence $\{a_n\} \in C(\varrho, \Omega, a)$ and sequence $\{c_n\} \in C(\varrho, \Omega, c)$. From the construction, we can notice that the convergent sequence are converges to 0 (usually) or constant sequence only.

It implies, $\varrho(a, c) \leq K \limsup_{n\to\infty} \varrho(a_n, c_n)$ for all $a, c \in \Omega$ and for some $K > 0$ whenever $\{a_n\} \to a$ and $\{c_n\} \to c$. Hence (Ω, ϱ) is a pseudo non-triangular metric space.
Now, as $\varrho(3, 4) = 4, \varrho(3, \frac{1}{n}) = \frac{1}{2n}, \varrho(\frac{1}{n}, 4) = \frac{1}{2n}$ but there does not exist a real number $b \geq 1$, for which $\varrho(3, 4) \leq b[\varrho(3, \frac{1}{n}) + \varrho(\frac{1}{n}, 4)]$. Hence given metric space is not a b-metric space.

Remark 2.17. From Definition 2.6, we can conclude every strong JS-metric space is a pseudo non-triangular metric space.

Example 2.18. Let the map $\varrho : \Omega \times \Omega \to [0, \infty)$ on a set Ω construct as $\varrho(a, c) = 5$ for every $a, c \in \Omega$. Then (JS_1) and (JS_2) of the Definition 2.1 satisfy easily. Also, $a \in \Omega$ and $\{a_n\}$ be any sequence in Ω. Then there are no convergent sequences in (Ω, ϱ) because $\lim_{n\to\infty} \varrho(a_n, a) = 5 \neq 0$. Therefore (JS_3) of Definition 2.1 holds easily. So (Ω, ϱ) is a JS-metric space, but $\varrho(a, a) = 5 \neq 0$ says that (Ω, ϱ) is not pseudo non-triangular metric space.

Theorem 2.19. Let (Ω, ϱ) be a pseudo non-triangular metric space with Hausdorffness property, which implies (Ω, ϱ) is a non-triangular metric space.

Proof . Let a pseudo non-triangular metric ϱ on a set Ω with Hausdorffness property. Then property (ϱ_1) and (ϱ_2) of Definition 2.2 is satisfied trivially. By Hausdorffness property, every convergent sequence $\{a_n\}$ has a unique limit. In other words, we can write, for each $a, c \in \Omega$ and $\{a_n\} \subset \Omega$ such that $\{a_n\} \in C(\varrho, \Omega, a)$ and $\{a_n\} \in C(\varrho, \Omega, c)$ then we have $a = c$, which is property (ϱ_3) of the definition of non-triangular metric space. Hence every pseudo non-triangular metric space is a non-triangular metric space.□

Remark 2.20. If (Ω, ϱ) be a pseudo non-triangular metric space with Hausdorffness property, then (Ω, ϱ) is a JS-metric space.

Example 2.21. Let $\Omega = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$ and $\varrho : \Omega \times \Omega \to \mathbb{R}^+$, defined as

$$\varrho(a,c) = \begin{cases} |a-c|, \text{ if } a, c \in \Omega \setminus \{1\}. \\ 0, \text{ if } a = c = 1. \\ 2a, \text{ if } a \neq 1, c = 1. \\ 2c, \text{ if } a = 1, c \neq 1. \end{cases}$$

Construction of a map implies $\varrho(a,a) = 0$ and $\varrho(a,c) = \varrho(c,a)$, which is the property (ϱ_1) and property (ϱ_2) of Definition 2.9.

Now we need to prove only the third property; for that, we consider a few cases here, one of them is $a, c \in \Omega \setminus \{1\}$, then possibilities of convergent sequences are either constant sequence or sequence converges to 0 usually. Also, as here $\varrho(1,0) = 0$, a sequence containing 1 as infinite terms also converge to 0.

For these sub-cases, let sequences $\{a_n\}$ and $\{c_n\}$ which are constant a and c, respectively, and for each $K \geq 1$,

$$|a-c| \leq K|a-c| = K \limsup_{n\to\infty} |(a_n - c_n)| = K \limsup_{n\to\infty} \varrho(a_n, c_n),$$

If the sequence $\{a_n\}$ converges to 0 contains, the term 1 for infinite times then,

$$c = \varrho(0,c) \leq 2c = \limsup_{n\to\infty} 2c_n = \limsup_{n\to\infty} \varrho(1, c_n) = \limsup_{n\to\infty} \varrho(a_n, c_n).$$

Hence, in this case, the map ϱ satisfies the property (ϱ_3) of Definition 2.9. For the second case, we consider $a = c = 1$; then the map ϱ satisfies the property (ϱ_3) of Definition 2.9 trivially.

For the last case, consider one of the a or c equals 1. We consider $a = 1$ and $c \neq 1$ then, $\varrho(1,c) = 2c$. Now, as every sequence converges to c is either a constant sequence or sequence converges to c conventionally, and the sequence containing the term 1 as infinite time converges to $c = 0$.

Now, if $c \neq 0$, then by the possibilities of sequence $\{c_n\} \to c$ satisfies the property for some $K > 0$,

$$\varrho(a,c) \leq K \limsup_{n\to\infty} \varrho(a_n, c_n).$$

If $c = 0$, then $\varrho(0,1) = 0$, and hence the map satisfies third property of Definition 2.9. Hence each case of a and c, there exists $K > 0$, such that

$$\varrho(a,c) \leq K \limsup_{n\to\infty} \varrho(a_n, c_n).$$

It concludes that (Ω, ϱ) is a pseudo non-triangular metric space.
As sequence $\{a_n\} = \{\frac{1}{n}\}$ has two different limit hence ϱ does not satisfy property (NT_3) Definition 2.2, hence (Ω, ϱ) is not a non-triangular metric space.
The map ϱ is not JS-metric as, $\varrho(1,0) = 0$ and $1 \neq 0$. Hence (Ω, ϱ) is not a JS-metric space.

Example 2.22. Let $\Omega = \mathbb{R}$ & $\varrho : \Omega \times \Omega \to \mathbb{R}^+$ constructed as following:

$$\varrho(a,c) = \begin{cases} |a-c|, \text{ if } a = 0 \text{ or } c = 0 \text{ or } a = c. \\ 1, \text{ otherwise}. \end{cases}$$

From the map ϱ, we can see that $\varrho(a,a) = 0$ & $\varrho(a,c) = \varrho(c,a)$.
In the given space, if $a = 0$ then sequence $\{a_n\}$ converges to a is $\lim_{n\to\infty} \varrho(a_n, 0) = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$, i.e. sequence $\{a_n\}$ converges to 0 conventionally.
Now, if $a \neq 0$, then the sequence $\{a_n\}$ is eventually constant only. If sequence $\{a_n\}$ is non constant then $\{a_n\}$ diverges as $\lim_{n\to\infty} \varrho(a_n, a) = 1 \neq 0$.

Now, let $a, c \in \Omega$ and a sequence $\{a_n\}$ with $\lim_{n\to\infty} \varrho(a_n, a) = 0$ and $\lim_{n\to\infty} \varrho(a_n, c) = 0$. If sequence $\{a_n\}$ is eventually constant, then it can easily verify that $a = c$. So now, we take sequence $\{a_n\}$ is non-constant. Then only possibility for a is to be 0, and $\{a_n\}$ converges to 0 conventionally. Now, if we take $c \neq 0$ then $\lim_{n\to\infty} \varrho(a_n, c) = 1 \neq 0$, which is a contradiction, then $a = c = 0$ must hold. Therefore, in any case, we get $a = c$. Hence (Ω, ϱ) is a non-triangular metric space.
For any $a \in \Omega$, $C(\varrho, \Omega, a) \neq \emptyset$. Suppose that (Ω, ϱ) is a pseudo non-triangular metric space. Now choosing $a = 0$ and $\{a_n\} = \{\frac{1}{2n}\}$ for each $n \in \mathbb{N}$, $\{a_n\}$ converges to 0 conventionally. Let $c = K + 1$ then $\{c_n\} = K + 1$, where $K > 0$ then, $\varrho(a,c) = \varrho(0, K+1) = K + 1$ and $K \limsup_{n\to\infty} \varrho(a_n, c_n) = K \limsup_{n\to\infty} \varrho(\frac{1}{2n}, K+1) = K < K + 1 = \varrho(a,c)$, which is a contradiction. Hence (Ω, ϱ) is not a pseudo non-triangular metric space.

One can notice from the above examples that the non-triangular metric and pseudo-non-triangular metric are independent concepts. Also, the concept of JS-metric is different from pseudo non-triangular metric.

3 Topology of pseudo non-triangular Metric Spaces

For a detailed study of space, it is essential to know its topological structure. We must understand the nature of the open ball, the open set, continuity, completeness, etc. of the space. In this section, we focus on some of these properties. It distinguishes the differences and similarities between various spaces and pseudo non-triangular metric spaces.
We start this section by defining the ball in pseudo non-triangular metric space.

Definition 3.1 (Ball). The ball centred at a of the radius $\epsilon > 0$ in the pseudo non-triangular metric space (Ω, ϱ) is defined as:

$$B_\epsilon(a) = \{c \in \Omega | \varrho(a, c) < \epsilon\}.$$

Definition 3.2 (Open Set). Let (Ω, ϱ) be pseudo non-triangular metric space. A set $\mathcal{U} \subset \Omega$ is open if for every $a \in \mathcal{U}$, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset \mathcal{U}$.

It is noted that the ball $B_\epsilon(a)$ is open in metric spaces. However, we cannot guaranteed that it is open in pseudo non-triangular metric space.

Example 3.3. Let $P = \{\frac{1}{n} | n \in \mathbb{N}\}$ and $Q = \{0, 3\}$ and $\Omega = P \cup Q$ and let the map $\varrho : \Omega \times \Omega \to \mathbb{R}^+$, where ϱ is defined as

$$\varrho(a, c) = \begin{cases} 0, & \text{if } a = c. \\ 1, & \text{if } a \neq c \, and \, \{a, c\} \subset P \, or \, \{a, c\} \subset Q. \\ a, & \text{if } a \in P, c \in Q. \\ c, & \text{if } c \in P, a \in Q. \end{cases}$$

It is clear from the construction of the map that $\varrho(a, a) = 0$ and $\varrho(a, c) = \varrho(c, a)$ which assures that property (ϱ_1) and property (ϱ_2) are satisfied. Now, we will check that ϱ satisfies the property (ϱ_3) of pseudo non-triangular metric space. For that we consider a and c with $\{a_n\} \to p$ and $\{c_n\} \to q$.

If $a = c$, then the map ϱ satisfies the property (ϱ_3) of Definition 2.9 trivially as $\varrho(a, c) = 0$.

Now, if $a \neq c$, we check the possibilities of sequence $\{a_n\} \to p$ and $\{c_n\} \to q$. If both the sequence $\{a_n\}$ and $\{c_n\}$ are either in P or in Q, then $\limsup_{n \to \infty} \varrho(a_n, c_n) = 1$, and hence there exist some $K > 0$, which satisfies

$$\varrho(a, c) \leq K \limsup_{n \to \infty} \varrho(a_n, c_n).$$

If we take a sequence as convergent but not eventually constant, then we need to know about the possibilities of a. For that, let $a \in Q$ then $\varrho(a_n, a) \to 0$ if $a_n \in P$ for each $n \geq m$ for some $m \in \mathbb{N}$ and $\{a_n\} \to 0$ in the usual sense. If we take $a \in P$, there is no convergent sequence that is not an eventually constant and converges to a. As $\{a_n\}$ cannot be in P after $n > m$ for some m, and if $a_n \in Q$ after $n > m$ for some m, then $\varrho(a_n, a) = a \not\to 0$; which is a contradiction. So here, convergent sequence is either constant or sequence converges to 0 in the usual sense. If the sequence is eventually constant, then it trivially satisfies the property (ϱ_3) of pseudo non-triangular metric space for $K = 1$. Now, if $a \in Q$ and $c \in P$, then $\varrho(a, c) = c$ and as $\{a_n\} \to a$ $\limsup_{n \to \infty} \varrho(a_n, a) = 1$, so in every case, ϱ satisfies the property (ϱ_3) of pseudo non-triangular metric space. Then (Ω, ϱ) is a pseudo non-triangular metric space.
We will now see that ball $B_\epsilon(a)$ need not be an open set in this space. For that let $B_1(\frac{1}{2}) = \{0, 3, \frac{1}{2}\}$, there is no $\epsilon > 0$ such that $B_\epsilon(0) \subset B_1(\frac{1}{2})$. So $B_1(\frac{1}{2})$ is not an open set.

Theorem 3.4. Let (Ω, ϱ) be pseudo non-triangular metric space. Then,

1. Ω and $\emptyset$ are open.
2. Let $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n$ be a open sets, then $\bigcap_{i=1}^{n} \mathcal{U}_i$ is open set.
3. Let $\mathcal{U}_\lambda$, where $\lambda \in \Lambda$ is open sets, then $\bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda$ is open.

Proof . 1. Let $a \in \Omega$ then $B_\epsilon(a) \subset \Omega$. Then Ω is an open set vacuously. Also, there is no element in the empty set, so it is a trivially open set.
2. Let $\{\mathcal{U}_\lambda\}$ be an arbitrary open sets and $V = \bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$. Let $a \in V$, then $a \in \mathcal{U}_\lambda$ for some λ and as $\mathcal{U}_\lambda$ is open, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset \mathcal{U}_\lambda \subset V$. Hence V is an open set.

3. Let $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n$ be open sets. For $a \in \bigcap\limits_{i=1}^{n} \mathcal{U}_i$, then a in each $\mathcal{U}_i$, $i = 1, 2, ..., n$ and as each $\mathcal{U}_i$ is open, there exists ϵ_i such that $B_{\epsilon_i}(a) \subset \mathcal{U}_i$ for every $i = 1, 2, ..., n$. Now let $\epsilon_r = \min\limits_{i=1,2,...,n} \epsilon_i$ then $B_{\epsilon_r}(a) \subset B_{\epsilon_i}(a) \subset \mathcal{U}_i$ for each i and so $B_{\epsilon_r}(a) \subset \bigcap\limits_{i=1}^{n} \mathcal{U}_i$. □

In the following example, we will see that the second condition of Theorem 3.4 is not valid for an arbitrary value of n.

Example 3.5. Let $\Omega = \mathbb{R}$ and a map $\varrho : \Omega \times \Omega \to \mathbb{R}^+$ defined as $\varrho(a, c) = |a - c|$. As ϱ is a pseudo non-triangular metric space. Hence (Ω, ϱ) is a pseudo non-triangular metric space. Consider the collection $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$. It can be seen that each $(-\frac{1}{n}, \frac{1}{n})$ is open in Ω. But as $\bigcap\limits_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$, and $\{0\}$ is not open in Ω. Therefore, the second condition of Theorem 3.4 is not true for the arbitrary value of n.

If the map ϱ is a pseudo non-triangular metric on the set Ω, then the topology $\tau(\varrho)$ on set Ω is defined as $\mathcal{U} \in \tau(\varrho)$ iff for each $a \in \Omega$, $B_\epsilon(a) \subset \mathcal{U}$ for some $\epsilon > 0$.

Definition 3.6 (Closed Set). Let the map ϱ be a pseudo non-triangular metric on the set Ω then a set $F \subset \Omega$ called a closed set if F^c is an open set.

Theorem 3.7. Let a map ϱ be a pseudo non-triangular metric on the set Ω. Then,

1. $\emptyset$ and Ω are closed.
2. Let $F_1, F_2, ..., F_n$ be closed sets then, $\bigcup\limits_{i=1}^{n} F_i$ is a closed set.
3. Let $F_{\lambda\in\Lambda}$ be closed sets. Then $\bigcap\limits_{\lambda\in\Lambda} F_\Lambda$ is a closed set.

Proof . 1. As $\emptyset^c = \Omega$ is open and $\Omega^c = \emptyset$ is also open then $\emptyset$ and Ω are closed.
2. Let $F_1, F_2, ..., F_n$ be closed sets then $F_1^c, F_2^c, ..., F_n^c$ are open then by Theorem 3.4 $\bigcap\limits_{i=1,2,...,n} F_i^c = (\bigcup\limits_{i=1,2,...,n} F_i)^c$ is open. Hence $\bigcup\limits_{i=1,2,...,n} F_i$ is closed.
3. Let $\{F_\lambda\}$ be a collection of closed sets, then F_λ^c is open set for each λ. Again from Theorem 3.4 $\bigcup\limits_{\lambda} F_\lambda^c = (\bigcap\limits_{\lambda} F_\lambda)^C$ is open. Therefore $\bigcap\limits_{\lambda} F_\lambda$ is a closed. □

Example 3.8. Let a space defined in Example 3.5, and consider the collection of closed set $\{[0, 1 - \frac{1}{n}] : n \in \mathbb{N}\}$, as $\bigcup\limits_{n\in\mathbb{N}} [0, 1 - \frac{1}{n}] = [0, 1)$, one can conclude that second property of Theorem 3.7 is not true for arbitrary values of n.

Theorem 3.9. Let (Ω, ϱ) be a pseudo non-triangular metric space, $\mathcal{U} \subset \Omega$ is said to be an open set iff for any sequence $\{a_n\} \subset \mathcal{U}^c$ with $\varrho(a_n, a) \to 0$ then $a \in \mathcal{U}^c$.

Proof . Let $\mathcal{U}$ be an open set and sequence $\{a_n\}$ in $\mathcal{U}^c$ with $\varrho(a_n, a) \to 0$. Suppose that $a \in \mathcal{U}$, and as $\mathcal{U}$ is the open set, there exist some $\epsilon > 0$ with $B_\epsilon(a) \subset \mathcal{U}$. From $\varrho(a_n, a) \to 0$ for every $\epsilon > 0$, there is a natural number $N \in \mathbb{N}$ such that $\varrho(a_n, a) < \epsilon$, for every $n \geq N$. Therefore $a_n \in B_\epsilon(a)$, implies $a_n \in \mathcal{U}$, which is a contradiction, therefore $a \in \mathcal{U}^c$.

Let $\mathcal{U}^c$ has all of its limit points, and if possible, then suppose that $\mathcal{U}$ is not an open set, then there exists $a \in \mathcal{U}$ such that every ball of any radius containing a has at least one point of $\mathcal{U}^c$, choose a point $a_n \in B_{\frac{1}{n}}(a) \cap \mathcal{U}^c$, then $\{a_n\}$ be a sequence in $\mathcal{U}^c$ with $\varrho(a_n, a) \to 0$, but a is not in $\mathcal{U}^c$ which contradicts our hypothesis. Hence set $\mathcal{U}$ is an open set. □

Lemma 3.10. Let (Ω, ϱ) be a pseudo non-triangular metric space then it is T_1 space but not Hausdorff.

Proof . Let $a, c \in \Omega$ with $a \neq c$. First, we show that set $\Omega \setminus \{a\}$ is an open set. For that define $r = \frac{\varrho(a,c)}{2}$, then $a \notin B_r(c)$. But $c \in B_r(c) \subset \Omega \setminus \{a\}$, so $\Omega \setminus \{a\}$ be open set for every $a \in \Omega$. Then $a \in \Omega \setminus \{c\}$ as $a \notin \Omega \setminus \{a\}$ and $c \notin \Omega \setminus \{c\}$. So (Ω, ϱ) is T_1 space.
From Example 2.21, we can say that in space (Ω, ϱ), convergence sequence may not have unique limits, and so (Ω, ϱ) is not Hausdorff space. □

4 Caristi type Fixed Point Theorem in pseudo non-triangular metric Spaces

In this section, we established the Carisiti type contraction and developed the example in the context of a theorem. For this, we state the following theorems needed in the sequel.

In 1974, Ciric [4] determined the refinement of the Banach contraction principle as given below:

Theorem 4.1. Let f be a self map on set Ω, where (Ω, ϱ) be a complete metric space. For a fixed constant $\alpha < 1$

$$\varrho(f(a), f(c)) \leq \alpha N(a, c),$$

for all $a, c \in \Omega$. Where $N(a, c) = max\{\varrho(a, c), \varrho(f(a), a), \varrho(f(c), c), \varrho(f(a), c), \varrho(f(c), a)\}$. Then in Ω, f contains a unique fixed point.

In 1976, Caristi [3] developed the extension of the Banach contraction principle.

Theorem 4.2. [3] Let the map ϱ be metric on a set Ω, such that it is complete and if f fulfils

$$\varrho(a, f(a)) \leq \phi(a) - \phi(f(a)),$$

for each $a \in \Omega$, then it contains a fixed point in Ω, where $\phi : \Omega \to \mathbb{R}^+$ is a lower semi-continuous.

Karapinar et al. [15] introduced the theorem using the result given by Caristi [3] and Ciric [4].

Theorem 4.3. Let f be a self-map on the set Ω, and ϱ be a complete b metric. If there exists a function $\phi : \Omega \to \mathbb{R}^+$ with

$$\varrho(a, f(a)) > 0 \Rightarrow \varrho(f(a), f(c)) \leq (\phi(a) - \phi(f(a)))N(a, c),$$

for all $a, c \in \Omega$. Then f has at least one fixed point in Ω.

By using the above theorems, we stated the new theorem as follows:

Theorem 4.4. Let (Ω, ϱ) be a complete pseudo non-triangular metric space with Hausdorffness property and $f : \Omega \to \Omega$ be a map. If there exists $a \in \Omega$ such that $\delta(\varrho, f, a) < \infty$ and let a map $\phi : X \to \mathbb{R}^+$ such that

$$\varrho(a, f(a)) > 0 \Rightarrow \varrho(f(a), f(c)) \leq (\phi(a) - \phi(f(a)))N(a, c),$$

for every $a, c \in \Omega$. Then there exists some $a \in \Omega$ such that $f(a) = a$. Where $\delta(\varrho, f, a) = sup\{\varrho(f^n(a), f^m(a)) : n, m \in \mathbb{N}\}$ For $a \in \Omega$.

Proof . Let $\{a_n\}$ be a Picard sequence defined as $a_{n+1} = f(a_n) = f^n(a_0)$. If for some $j \in \mathbb{N}$, $a_j = a_{j+1}$ then $f(a_j) = a_j$ and it will be fixed point of f.
If possible, we assume that $a_n \neq a_m$ for any $n, m \in \mathbb{N}$, and hence $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$. Now from assumption, we obtain

$$\varrho(a_{n+1}, a_n) \leq (\phi(a_n) - \phi(a_{n+1}))N(a_n, a_{n+1}),$$

for all $n \in \mathbb{N}$. From the above equation, it is clear that $\{\phi(a_n)\}$ is decreasing sequence, and as $\phi(a_n)$ is bounded below, so it converges to some non-negative real number r. Hence $\{\phi(a_n) - \phi(a_{n+1})\} \to 0$, as $n \to \infty$. Accordingly, for $l \in (0, 1)$, then there exists $j \in \mathbb{N}$ such that for each $n \geq j$, we can say $\{\phi(a_n) - \phi(a_{n+1})\} \leq l$.
Now for all $i, j \in \mathbb{N}$, we have

$$\varrho(a_{n+i}, a_{n+j}) \leq (\phi(a_{n+i-1}) - \phi(a_{n+i}))N(a_{n+i-1}, a_{n+j-1}).$$

So we get

$$\varrho(a_{n+i}, a_{n+j}) \leq l.N(a_{n+i-1}, a_{n+j-1}),$$

$$\varrho(f^{n+i}a_0, f^{n+j}a_0) \leq l.max\left\{\varrho(f^{n+i-1}a_0, f^{n+j-1}a_0), \varrho(f^{n+i}a_0, f^{n+j-1}a_0),\right.$$
$$\left.\varrho(f^{n+i}a_0, f^{n+i-1}a_0), \varrho(f^{n+j}a_0, f^{n+j-1}a_0), \varrho(f^{n+j}a_0, f^{n+i-1}a_0)\right\},$$

which implies that,

$$\delta(\varrho, f, f^n a_0) \leq l\delta(\varrho, f, f^{n-1}a_0),$$

implies

$$\delta(\varrho, f, f^n a_0) \leq l^n\delta(\varrho, f, fa_0), \text{ for } n \geq 1.$$

Using the inequality stated above,

$$\varrho(f^n a_0, f^{n+m}a_0) \leq \delta(\varrho, f, f^n a_0) \leq l^n\delta(\varrho, f, a_0).$$

Since

$$\delta(\varrho, f, a_0) < +\infty \,\text{and}\, l \in (0,1),$$

we obtain

$$\lim_{n,m\to\infty} \varrho(f^n a_0, f^{n+m}a_0) = 0.$$

This suggests that the sequence $\{f^n a_0\}$ must be a Cauchy. As (X, ϱ) is complete, there must be some $\omega \in X$, such that $\{f^n a_0\}$ is convergent to $\omega \in X$. Also,

$$\varrho(f^n a_0, f^{n+m}a_0) \leq l^n\delta(\varrho, f, a_0), n, m \in \mathbb{N},$$

again by property D_3 of Definition 2.9,

$$\varrho(\omega, f^n a_0) \leq K \limsup_{n\to\infty} \varrho(f^n a_0, f^{n+m}a_0) \leq Kl^n\delta(\varrho, f, a_0) \text{ for } n \in \mathbb{N},$$

and

$$\varrho(fa_0, f\omega) \leq \max\{\varrho(a_0,\omega), \varrho(a_0, fa_0), \varrho(\omega, f\omega), \varrho(fa_0,\omega), \varrho(a_0, f\omega)\},$$

from above inequalities, we get

$$\varrho(a_0,\omega) \leq K\delta(D, f, a_0),$$
$$\varrho(a_0, fa_0) \leq \delta(\varrho, f, a_0),$$
$$\varrho(fa_0,\omega) \leq K.l\delta(\varrho, f, a_0).$$

Hence, we get

$$\varrho(fa_0, f\omega) \leq \max\{lK\delta(\varrho, f, a_0), l\delta(\varrho, f, a_0), l\varrho(\omega, f\omega), l\varrho(a_0, f\omega)\},$$

from above inequality we obtain,

$$\varrho(f^2a_0, f\omega) \leq \max\{l^2K\delta(\varrho, f, a_0), l^2\delta(\varrho, f, a_0), l\varrho(\omega, f\omega), l^2\varrho(a_0, f\omega)\},$$

By induction we get,

$$\varrho(f^n a_0, f\omega) \leq \max\{l^nK\delta(\varrho, f, a_0), l^n\delta(\varrho, f, a_0), l\varrho(\omega, f\omega), l^n\varrho(a_0, f\omega)\} \text{ for } n \geq 1.$$

Therefore, we have

$$\limsup_{n\to\infty} \varrho(f^n a_0, f\omega) \leq l\varrho(\omega, f\omega).$$

Now, using the property D_3 we have,

$$\varrho(f\omega,\omega) \leq K \limsup_{n\to\infty} \varrho(f^n a_0, f\omega) \leq lK\varrho(\omega, f\omega),$$

hence $\varrho(f\omega,\omega) = 0$ and so, $f\omega = \omega$. □

Every b-metric space is a pseudo non-triangular metric space, so we can define the corollary as:

Corollary 4.5. Let a self-map f on the set Ω and ϱ be a complete b-metric. If there exists $a \in \Omega$ such that $\delta(\varrho, f, a) < \infty$ and let a function $\phi : X \to \mathbb{R}^+$ such that for every

$$a, c \in \Omega, \varrho(a, f(a)) > 0 \Rightarrow \varrho(f(a), f(c)) \leq (\phi(a) - \phi(f(a)))N(a, c).$$

Then there exists some $x \in \Omega$, such that x is a fixed point of f.

Example 4.6. Let (Ω, ϱ) be a pseudo non-triangular metric space as define in Example 2.16. Let $f : \Omega \to \Omega$ be define as $f(a) = \begin{cases} \frac{1}{4} \text{ if } a \in A. \\ \frac{1}{5} \text{ if } a \in B. \end{cases}$.

Here, as the ϱ is finite so for each $a \in \Omega$, $\delta(\varrho, f, a) < \infty$. If we let $\phi : \Omega \to \mathbb{R}^+$, define as $\phi(a) = \begin{cases} 0 \text{ if } a = \frac{1}{4} \text{ or } a = \frac{1}{5}, \\ 1 \text{ if otherwise,} \end{cases}$ and if $a \in A \setminus \{\frac{1}{4}\}$ then $\varrho(a, f(a)) > 0$ and for $a \in B \implies \varrho(a, f(a)) > 0$. Then the map f possesses every condition of Theorem 4.4.

So f has a fixed point at $a = \frac{1}{4}$.

Remark 4.7. In the context of Example 4.6, Theorem 4.3 given by E. Karapinar is not applicable since the metric considered in Example 4.6 is not a b-metric.

5 Conclusion

The motive of this paper is to provide an understanding of the role of generalizations of metric spaces in a fixed point perspective. For this purpose, the idea of pseudo non-triangular metric space is introduced, which becomes a minimal required metric structure to develop a new fixed point theorem for contractive type mappings. Then the basic topological properties of this new space are investigated, and a new fixed point theorem is proved for the self maps of a complete pseudo non-triangular metric space with Hausdorfness property.

However, some interesting topics for further research remain. It would be of interest if the readers could utilize pseudo non-triangular metrics to study and investigate some new fixed point results for various emerging contractive type mappings as well as non-contractive type mappings e. g. [9, 19, 21]. We shall investigate these questions in subsequent papers.

Acknowledgement

The authors are grateful to the Editor-in-Chief/Area Editors and referee(s) for their insightful comments and suggestions, which helped significantly improve the paper.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), no. 1-2, 31–37.

[3] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[4] L. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[5] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostraviensis **1** (1993), no. 1, 5—11.

[6] A. Deshmukh and D. Gopal, *Topology of non-triangular metric spaces and related fixed point results*, Filomat **3** (2021), no. 11, 3557–3570.

[7] L. Gajić and N. Ralević, *A common fixed point result in strong JS-metric space*, Bull. Cl. Sci. Math. Nat. Sci. Math. **43** (2018), 113–121.

[8] R. George, S. Radenovic, K. P. Reshma and S. Shukla, *Rectangular b-metric space and contraction principles*, J. Nonlinear Sci. Appl. **8** (2015), no. 6, 1005–1013.

[9] D. Gopal, M. Abbas, D.K. Patel and C. Vetro, *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*, Acta Math. Sci. **36** (2016), no. 3, 957–970.

[10] D. Gopal, P. Kumam and P. Agarwal, *Metric Structures and Fixed Point Theory*, CRC Press, Florida, 2021.

[11] G.E. Hardy and T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), no. 2, 201—206.

[12] P. Hitzler and A.K. Seda, *Dislocated topologies*, J. Electr. Eng. **51** (2000), no. 12, 3-–7.

[13] J.R. Jachymski, *Equivalence of some contractivity properties over metrical structure*, Proc. Amer. Math. Soc. **125** (1997), 2327–2335.

[14] M. Jleli and B. Samet, *A generalized metric space and related fixed point theorems*, J. Fixed Point Theory Appl. **2015** (2015), no. 1, 1–14.

[15] E. Karapinar, F.Khojasteh and Z.D. Mitrović, *A proposal for revisiting Banach and Caristi type theorems in b-metric Spaces*, Math. **7** (2019), no. 4, 308–311.

[16] M.A. Khamsi and W.A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley & Sons, New York, 2015.

[17] F. Khojasteh and H. Khandani, *Scrutiny of some fixed point results by S-operators without triangular inequality*, Math. Slovaca **70** (2020), no. 2, 467–476.

[18] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Proc. Am. Math. Soc. **226** (1977), 257–290.

[19] A. F. Roldán López de Hierro and N. Shahzad, *New fixed point theorem under R-contractions*, Fixed Point Theory Appl. **2015** (2015), 1—18.

[20] V. M. Sehgal and J.W. Thomas, *A fixed point theorem for semigroups of mappings with a contractive iterate*, Proc. Amer. Math. Soc. **29** (1971), no. 3, 565—570.

[21] N. Shahzad, A. F. Roldán López de Hierro and F. Khojasteh, *Some new fixed point theorems under $(\mathcal{A},\mathcal{S})$-contractivity conditions*, RACSAM **111** (2017), 307—324.