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An iterative algorithm for a system of generalized quasi-variational inequalities

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Abstract

In this paper, we introduce and study a system of generalized quasi-variational inequalities involving nonlinear, nonconvex and nondifferentiable terms in uniformly smooth Banach space. By means of the retraction mapping technique, we prove the existence of solutions for this system of quasi-variational inequalities. Further, we suggest an iterative algorithm for finding the approximate solution of this system and discuss the convergence criteria of the sequences generated by the iterative algorithm under some suitable conditions.

Keywords: System of generalized quasi-variational inequalities, nonlinear, nonconvex and nondifferentiable term, uniformly smooth Banach space, retraction mapping, iterative algorithm, convergence analysis
2020 MSC: 47H09, 47J20, 49J40

1 Introduction

Many problems in physics, mechanics, elasticity and engineering sciences can be formulated in variational inequalities involving nonlinear, nonconvex and nondifferentiable term, see for example Baiocchi and Capelo [3], Duvaut and Lions [8] and Kikuchi and Oden [11]. The proximal (resolvent) method used to study the convergence analysis of iterative algorithms for variational inclusions, see [10, 16], cannot be adopted for studying such classes of variational inequalities due to the presence of nondifferentiable term.

There are some methods, for example projection method and auxiliary principle method which can be used to study such classes of variational inequalities, see [6, 12, 14, 15] and the relevent references cited therein. It is remarked that most of the work, using projection method and auxiliary principle method, has been done in the setting of Hilbert space. Alber and Yao [2], Chen et al.[5] studied some classes of co-variational inequality and co-complementarity problems in Banach spaces. Therefore, the study of other classes of variational inequalities using projection method and auxiliary principle method in the setting of Banach space remains an interesting problem. Chidume et al. [6] studied some classes of variational inequalities involving nonlinear, convex and nondifferentiable term, using auxiliary principle method in the setting of reflexive Banach space.

Motivated and inspired by the above achievements, in this paper, we study a new system of generalized quasi-variational inequalities involving nonlinear, nonconvex and nondifferentiable term in uniformly smooth Banach space. Using sunny retraction mapping, we establish that SGQVI is equivalent to some relations. Further, using these relations, we suggest an iterative algorithm with errors for approximating the solution of the system and discuss the

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convergence of iterative sequence generated by the iterative algorithm. The results presented in this paper improve and extend many known results in the literature, see for example [7].

2 Preliminaries and formulation of problem

We need the following definitions and results from the literature.

Let X be a real uniformly smooth Banach space equipped with norm $\|.\|$ and $X^\star$ be the topological dual space of X. Let $<.,.>$ be the dual pair between X and $X^\star$. Let $CB(X)$ be the family of all closed and bounded subsets of X, $CC(X)$ be the family of all nonempty, closed and convex subsets of X. Let 2^X be the power set of X.

Definition 2.1. A mapping $J : X \to 2^{X^\star}$ is said to be a normalized duality mapping, if it is defined by

$$J(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|_{X^\star}\}, \quad \text{for all} \quad x \in X.$$

In the sequel, we shall denote a selection of normalized duality mapping J by j. It is well known that if X is smooth, then J is single-valued and if $X \equiv H$, a real Hilbert space, then J is an identity map.

Definition 2.2. [17] A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \sigma \right\}.$$

Definition 2.3. [17] A Banach space X is said to be

(i) uniformly smooth if $\lim\limits_{\sigma\to 0} \dfrac{\rho_X(\sigma)}{\sigma} = 0$,

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that $\rho_X(\sigma) \le c\sigma^q$, $\sigma \in [0,\infty)$.

Note that if X is uniformly smooth, then J_q becomes single-valued.

Lemma 2.4. [16] Let X be a uniformly smooth Banach space and let $J : X \to X^\star$ be the normalized duality mapping. Then for all $x, y \in X$, we have

(a) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle$,
(b) $\langle x - y, Jx - Jy\rangle \le 2d^2\rho_X(4\|x - y\|/d)$, where $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

Definition 2.5. A mapping $h : X \to X$ is said to be

(i) Lipschitz continuous if, there exists a constant $L_h > 0$ such that

$$\|h(x) - h(y)\| \le L_h\|x - y\|, \quad \forall x, y \in X,$$

(ii) ξ-strongly accretive if,

$$\Big\langle h(x) - h(y), J(x - y)\Big\rangle \ge \xi\|x - y\|^2, \quad \forall x, y \in X.$$

Definition 2.6. A mapping $S : X \times X \times X \to X$ is said to be Lipschitz continuous with respect to first argument if, there exists a constant $L_S > 0$ such that

$$\Big\|S(x_1, x_2, x_3) - S(y_1, x_2, x_3)\Big\| \le L_S\|x_1 - y_1\|, \quad \forall x_1, y_1, x_2, x_3 \in X.$$

Similarly, we can define the Lipschitz continuity of S in second and third arguments.

Definition 2.7. [1, 5, 9] Let $K \subset X$ be a nonempty closed convex set. A mapping $G_K : X \to K$ is said to be

(i) retraction if

$$G_K^2 = G_K,$$

(ii) nonexpansive retraction if

$$\|G_K(x) - G_K(y)\| \leq \|x - y\|, \quad \forall x, y \in X,$$

(ii) sunny retraction if

$$G_K(G_K x - t(x - G_K x)) = G_K x, \quad \forall x \in X, t \in R.$$

Lemma 2.8. [5, 9] A retraction G_K is sunny and nonexpansive if and only if

$$\Big\langle x - G_K(x), J(G_K(x) - y)\Big\rangle \geq 0, \quad \forall x, y \in X.$$

Definition 2.9. The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(X)$ is defined by

$$\mathcal{D}(S,T) = \max\left\{\sup_{u\in S}\inf_{v\in T} d(u,v), \ \sup_{v\in T}\inf_{u\in S} d(u,v)\right\}, \ S, T \in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on X.

Definition 2.10. 4 A set-valued mapping $T : X \to CB(X)$ is said to be γ-$\mathcal{D}$-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \gamma\|x - y\|, \quad \forall x, y \in X.$$

Theorem 2.11. [13] Let $T : X \to CB(X)$ be a set-valued mapping on X and (X, d) be a complete metric space.

(i) For any given $\mu > 0$ and for any given $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u,v) \ \leq \ (1+\mu)\mathcal{D}(T(x), T(y)).$$

(ii) If $T : X \to C(X)$, then (i) holds for $\mu = 0$.

Now, we formulate our main problem.

Let X_i be a uniformly smooth Banach space. Let for each $i = \{1,2,3\}$, $T_i : X_1 \times X_2 \times X_3 \to X_i$, $k_i, f_i, g_i : X_i \to X_i$ be single-valued mappings, $B_{i1}, B_{i2}, B_{i3} : X_i \to CB(X_i)$ and $K_i : X_i \to CC(X_i)$ be set-valued mappings. We consider the following system of generalized quasi-variational inequalities (SGQVI): Find $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}$ such that for each $i \in \{1,2,3\}, (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3, u_{i1} \in B_{i1}(x_1), u_{i2} \in B_{i2}(x_2), u_{i3} \in B_{i3}(x_3)$, such that $f_i(x_i) \in K_i(x_i)$ and

$$\left.\begin{aligned}
&\Big\langle k_1(f_1(x_1)), J_1(y_1 - f_1(x_1))\Big\rangle + \rho_1 b_1(x_1, y_1) - \rho_1 b_1(x_1, f_1(x_1))\\
&\qquad \geq \Big\langle k_1(x_1), J_1(y_1 - f_1(x_1))\Big\rangle - \rho_1\Big\langle T_1(u_{11}, u_{12}, u_{13}) - g_1, J_1(y_1 - f_1(x_1))\Big\rangle,\\
&\Big\langle k_2(f_2(x_2)), J_2(y_2 - f_2(x_2))\Big\rangle + \rho_2 b_2(x_2, y_2) - \rho_2 b_2(x_2, f_2(x_2))\\
&\qquad \geq \Big\langle k_2(x_2), J_2(y_2 - f_2(x_2))\Big\rangle - \rho_2\Big\langle T_2(u_{21}, u_{22}, u_{23}) - g_2, J_2(y_2 - f_2(x_2))\Big\rangle,\\
&\Big\langle k_3(f_3(x_3)), J_3(y_3 - f_3(x_3))\Big\rangle + \rho_3 b_3(x_3, y_3) - \rho_3 b_3(x_3, f_3(x_3))\\
&\qquad \geq \Big\langle k_3(x_3), J_3(y_3 - f_3(x_3))\Big\rangle - \rho_3\Big\langle T_3(u_{31}, u_{32}, u_{33}) - g_3, J_3(y_3 - f_3(x_3))\Big\rangle
\end{aligned}\right\} \tag{2.1}$$

for all $y_i \in K_i(x_i)$, where $\rho_i > 0$ are constants, $g_i \in X_i$ and $b_i(.,.) : X_i \times X_i \to R$ are nonlinear, nonconvex and nondifferentiable forms satisfying the following conditions:

Condition 2.12.

(i) $b_i(.,.)$ is linear in the first argument.

(ii) there exists a constant $\mu > 0$ such that
$$b_i(x_i, y_i) \leq \mu_i \|x_i\| \; \|y_i\|, \quad \forall x_i, y_i \in X_i.$$

(iii)
$$b_i(x_i, y_i) - b_i(x_i, y_i') \leq b_i(x_i, y_i - y_i'), \quad \forall x_i, y_i \in X_i.$$

Remark 2.13.

(i) Condition 2.12(i)-(ii) imply that
$$-b_i(x_i, y_i) \leq \mu_i \|x_i\| \; \|y_i\|, \quad \forall x_i, y_i \in X_i.$$

Hence, we have
$$|b_i(x_i, y_i)| \leq \mu_i \|x_i\| \; \|y_i\|, \quad \forall x_i, y_i \in X_i.$$

(ii) Also Condition 2.12(i)-(iii) imply that
$$|b_i(x_i, y_i) - b_i(x_i, y_i')| \leq \mu_i \|x_i\| \; \|y_i - y_i'\|, \quad \forall x_i, y_i, y_i' \in X_i.$$

That is, $b_i(x_i, y_i)$ are continuous with respect to the second argument.

Special cases:

I. If in problem (2.1), $X_i = X$ is a real reflexive Banach space, $X^\star$ is a topological dual space of X, D is a nonempty convex subset of X, $T_i = T : X^\star \times X^\star \to X^\star$, $k_i \equiv 0, g_i = g \in X^\star, f_i = f \in X$, $B_{11}, B_{12} : X \to CB(X^\star), u_{11} \in B_{11}(x), u_{12} \in B_{12}(x)$, then problem (2.1) reduces to the following problem: find $x \in X$, such that

$$\Big\langle T(u_{11}, u_{12}) - g, y - f(x) \Big\rangle + \rho b(x, y) - \rho b(x, f(x)) \geq 0, \quad \forall y \in D. \tag{2.2}$$

This type of problem (2.2) has been considered and studied by Ding and Yao [7].

3 Existence of solution

First, we give the following technical lemma:

Lemma 3.1. Let ρ_i, λ_i be positive parameters and let Condition 2.12 hold. Then the following statements are equivalent:

(a) SGQVI (2.1) has a solution $x_i \in X_i$ with $f_i(x_i) \in K_i(x_i)$,

(b) there exist $x_i \in X_i$ such that $f_i(x_i) \in K_i(x_i)$ and
$$\Big\langle x_i - \Omega_i(x_i), J_i(y_i - f_i(x_i)) \Big\rangle \geq 0 \quad \forall y_i \in K_i(x_i), \tag{3.1}$$

where $\Omega_i : X_i \to X_i$ is defined by

$$\Big\langle \Omega_i(x_i), J_i(y_i) \Big\rangle = \langle x_i, J_i(y_i) \rangle - \Big\langle k_i(f_i(x_i)), J_i(y_i) \Big\rangle + \Big\langle k_i(x_i), J_i(y_i) \Big\rangle$$
$$-\rho_i \Big\langle T_i(u_{i1}, u_{i2}, u_{i3}) - g_i, J_i(y_i) \Big\rangle - \rho_i b_i(x_i, y_i), \quad \forall x_i, y_i \in X_i, \tag{3.2}$$

(c) there exist $x_i \in X_i$ such that $f_i(x_i) \in K_i(x_i)$ and
$$f_i(x_i) = G_{K_i(x_i)} \Big[f_i(x_i) - \lambda_i x_i + \lambda_i \Omega_i(x_i) \Big], \tag{3.3}$$

where the mapping $G_{K_i(x_i)}$ is sunny retraction from $X_i \to K_i(x_i)$.

Proof . $(a) \Rightarrow (b)$. Let (a) hold. That is, there is $x_i \in X_i$ such that $f_i(x_i) \in K_i(x_i)$ and

$$\Big\langle k_i(f_i(x_i)), J_i(y_i - f_i(x_i))\Big\rangle + \rho_i b_i(x_i, y_i) - \rho_i b_i(x_i, f_i(x_i))$$
$$\geq \Big\langle k_i(x_i), J_i(y_i - f_i(x_i))\Big\rangle - \rho_i \Big\langle T_i(u_{i1}, u_{i2}, u_{i3}) - g_i, J_i(y_i - f_i(x_i))\Big\rangle, \tag{3.4}$$

which can be rewritten as

$$\Big\langle x_i, J_i(y_i - f_i(x_i))\Big\rangle \geq \Big\langle x_i, J_i(y_i - f_i(x_i))\Big\rangle - \Big\langle k_i(f_i(x_i)), J_i(y_i - f_i(x_i))\Big\rangle + \Big\langle k_i(x_i), J_i(y_i - f_i(x_i))\Big\rangle$$
$$-\rho_i b_i(x_i, y_i - f_i(x_i)) - \rho_i \Big\langle T_i(u_{i1}, u_{i2}, u_{i3}) - g_i, J_i(y_i - f_i(x_i))\Big\rangle. \tag{3.5}$$

By using (3.2), (3.5) becomes

$$\Big\langle x_i - \Omega_i(x_i), J_i(y_i - f_i(x_i))\Big\rangle \geq 0 \quad \forall y_i \in X_i. \tag{3.6}$$

Hence (b) holds.

$(b) \Rightarrow (a)$. It is immediately followed by retracing the above steps and using Condition 2.12.

So, for $\lambda_i > 0$, we have

$$\lambda_i \Big\langle x_i - \Omega_i(x_i), J_i(y_i - f_i(x_i))\Big\rangle$$
$$= \Big\langle f_i(x_i) - (f_i(x_i) - \lambda_i x_i + \lambda_i \Omega_i(x_i)), J_i(y_i - f_i(x_i))\Big\rangle \quad \forall x_i, y_i \in X_i. \tag{3.7}$$

Therefore, from (3.7) and Lemma 2.8, it follows the statements (b) and (c) are equivalent. This completes the proof. □

4 Iterative algorithm and convergence analysis

Now, using the result of Nadler[13], we give an iterative method with error terms for finding an approximate solution of SGQVI (2.1):

Iterative Algorithm 4.1. For $i = \{1, 2, 3\}$, given $x_i^0 \in X_i$, we can take $u_{i1}^0 \in B_{i1}(x_1^0)$,$u_{i2}^0 \in B_{i2}(x_2^0)$,$u_{i3}^0 \in B_{i3}(x_3^0)$, and let

$$x_i^1 = (1 - \beta_i)x_i^0 + \beta_i\Big[x_i^0 - f_i(x_i^0) + G_{K_i(x_i^0)}\Big(f_i(x_i^0) - \lambda_i x_i^0 + \lambda_i \Omega_i(x_i^0)\Big)\Big] + \beta_i e_i^0.$$

Since $u_{i1}^0 \in B_{i1}(x_1^0)$,$u_{i2}^0 \in B_{i2}(x_2^0)$,$u_{i3}^0 \in B_{i3}(x_3^0)$, by Nadler's Theorem, there exist $u_{i1}^1 \in B_{i1}(x_1^1)$,$u_{i2}^1 \in B_{i2}(x_2^1)$,$u_{i3}^1 \in B_{i3}(x_3^1)$, such that

$$\|u_{i1}^1 - u_{i1}^0\| \leq (1+1)\mathcal{D}_1(B_{i1}(x_1^1), B_{i1}(x_1^0)),$$
$$\|u_{i2}^1 - u_{i2}^0\| \leq (1+1)\mathcal{D}_2(B_{i2}(x_2^1), B_{i2}(x_2^0)),$$
$$\|u_{i3}^1 - u_{i3}^0\| \leq (1+1)\mathcal{D}_3(B_{i3}(x_3^1), B_{i3}(x_3^0)).$$

Again, let

$$x_i^2 = (1 - \beta_i)x_i^1 + \beta_i\Big[x_i^1 - f_i(x_i^1) + G_{K_i(x_i^1)}\Big(f_i(x_i^1) - \lambda_i x_i^1 + \lambda_i \Omega_i(x_i^1)\Big)\Big] + \beta_i e_i^1.$$

By Nadler's Theorem, there exist $u_{i1}^2 \in B_{i1}(x_1^2)$,$u_{i2}^2 \in B_{i2}(x_2^2)$,$u_{i3}^2 \in B_{i3}(x_3^2)$, such that

$$\|u_{i1}^2 - u_{i1}^1\| \leq \Big(1 + \frac{1}{2}\Big)\mathcal{D}_1(B_{i1}(x_1^2), B_{i1}(x_1^1)),$$
$$\|u_{i2}^2 - u_{i2}^1\| \leq \Big(1 + \frac{1}{2}\Big)\mathcal{D}_2(B_{i2}(x_2^2), B_{i2}(x_2^1)),$$
$$\|u_{i3}^1 - u_{i3}^0\| \leq \Big(1 + \frac{1}{2}\Big)\mathcal{D}_3(B_{i3}(x_3^2), B_{i3}(x_3^1)).$$

Continuing the above process inductively, we can obtain the sequences $\{x_i^n\}, \{u_{i1}^n\}, \{u_{i2}^n\}, \{u_{i3}^n\}$, by the following iterative:

$$x_i^{n+1} = (1-\beta_i)x_i^n + \beta_i\Big[x_i^n - f_i(x_i^n) + G_{K_i(x_i^n)}\Big[f_i(x_i^n) - \lambda_i x_i^n + \lambda_i\Omega_i(x_i^n)\Big)\Big] + \beta_i e_i^n.$$

$$\|u_{i1}^{n+1} - u_{i1}^n\| \le \Big(1+\frac{1}{n+1}\Big)\mathcal{D}_1(B_{i1}(x_1^{n+1}), B_{i1}(x_1^n)),$$

$$\|u_{i2}^{n+1} - u_{i2}^n\| \le \Big(1+\frac{1}{n+1}\Big)\mathcal{D}_2(B_{i2}(x_2^{n+1}), B_{i2}(x_2^n)),$$

$$\|u_{i3}^{n+1} - u_{i3}^n\| \le \Big(1+\frac{1}{n+1}\Big)\mathcal{D}_3(B_{i3}(x_3^{n+1}), B_{i3}(x_3^n)),$$

where $n = 0, 1, 2, \ldots$ for $i \in \{1,2,3\}, \beta_i > 0, \lambda_i > 0$ are constants, $e_i^n \in X_i (n \ge 0)$ are errors to take into account a possible inexact computation and $\mathcal{D}_i(.,.)$ are the Hausdorff metrics on $CB(X_i)$.

Now, we give the convergence analysis of the sequences generated by the iterative algorithm 4.1.

Theorem 4.1. Let for each $i = 1, 2, 3$, X_i be a uniformly smooth Banach space with $\rho_{X_i}(t) \le c_i t^2$ for some constant $c_i > 0$. Let f_i be δ_i-strongly accretive and ν_i-Lipschitz continuous, let k_i be L_{k_i}-Lipschitz continuous and τ_i-strongly accretive with respect to f_i and f_i be L_{f_i}-Lipschitz continuous. Let T_i be $L_{T_{i1}}, L_{T_{i2}}$ and $L_{T_{i3}}$-Lipschitz continuous in the first, second and third arguments, respectively, and B_i be $L_{B_{i1}}, L_{B_{i2}}, L_{B_{i3}} - \mathcal{D}-$ Lipschitz continuous in the first, second and third arguments, respectively. Assume that for some constant $\gamma_i > 0$,

$$\|G_{K_i(x_i)}(z_i) - G_{K_i(y_i)}(z_i)\| \le \gamma_i\|x_i - y_i\|, \quad \forall x_i, y_i \in X_i, \tag{4.1}$$

$$0 < r < 1, \tag{4.2}$$

where

$$r = \max\{h_1 + t_1, h_2 + t_2, h_3 + t_3\},$$

$$h_i = \quad (1-\beta_i) + \beta_i\sqrt{1 - 2\delta_i + 64c_i\nu_i^2} + \beta_i\sqrt{\lambda_i^2 - 2\lambda_i\delta_i + 64c_i\nu_i^2}$$

$$+\beta_i\lambda_i\sqrt{1 - 2\tau_i + 64c_i L_{k_i}^2 L_{f_i}^2} + \beta_i\lambda_i L_{k_i} + \rho_i\beta_i\lambda_i\mu_i + \beta_i\gamma_i < 1,$$

$$t_i = \quad \textstyle\sum_{j=1}^3 \beta_j\lambda_j\rho_j L_{T_{ji}} L_{B_{ji}} < 1,$$

$$\text{and } \sum_{d=1}^{\infty}\|e_1^d - e_1^{d-1}\|h^{-d} < \infty, \quad \sum_{d=1}^{\infty}\|e_2^d - e_2^{d-1}\|h^{-d} < \infty, \quad \sum_{d=1}^{\infty}\|e_3^d - e_3^{d-1}\|h^{-d} < \infty,$$

$$\lim_{n\to\infty} e_1^n = \lim_{n\to\infty} e_2^n = \lim_{n\to\infty} e_3^n = 0, \qquad \text{for each } h \in (0,1).$$

Then the SGQVI (2.1) admits a solution $(x_1, x_2, x_3, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$ and the iterative sequences $\{x_i^n\}, \{u_{i1}^n\}, \{u_{i2}^n\}, \{u_{i3}^n\}$ generated by iterative algorithm 4.1 strongly converge to $x_i, u_{i1}, u_{i2}, u_{i3}$, respectively for each $i \in \{1,2,3\}$.

Proof . We have

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| \quad &= \Bigg\|(1-\beta_1)x_1^n + \beta_1\Big[x_1^n - f_1(x_1^n) + G_{K_1(x_1^n)}\Big\{f_1(x_1^n) - \lambda_1 x_1^n + \lambda_1\Omega_1(x_1^n)\Big\}\Big] + \beta_1 e_1^n \\
&-\Bigg[(1-\beta_1)x_1^{n-1} + \beta_1\Big[x_1^{n-1} - f_1(x_1^{n-1}) + G_{K_1(x_1^{n-1})}\Big\{f_1(x_1^{n-1}) - \lambda_1 x_1^{n-1} + \lambda_1\Omega_1(x_1^{n-1})\Big\}\Big] + \beta_1 e_1^{n-1}\Bigg]\Bigg\| \\
&\le (1-\beta_1)\Big\|x_1^n - x_1^{n-1}\Big\| + \beta_1\Big\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big\| \\
&+\beta_1\Big\|G_{K_1(x_1^n)}\Big\{f_1(x_1^n) - \lambda_1 x_1^n + \lambda_1\Omega_1(x_1^n)\Big\} - G_{K_1(x_1^{n-1})}\Big\{f_1(x_1^{n-1}) - \lambda_1 x_1^{n-1} + \lambda_1\Omega_1(x_1^{n-1})\Big\}\Big\| \\
&+\beta_1\|e_1^n - e_1^{n-1}\|
\end{aligned}$$

$$\leq (1-\beta_1)\Big\|x_1^n - x_1^{n-1}\Big\| + \beta_1\Big\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big\|$$

$$+\beta_1\Big\|G_{K_1(x_1^n)}\Big\{f_1(x_1^n) - \lambda_1 x_1^n + \lambda_1\Omega_1(x_1^n)\Big\} - G_{K_1(x_1^n)}\Big\{f_1(x_1^{n-1}) - \lambda_1 x_1^{n-1} + \lambda_1\Omega_1(x_1^{n-1})\Big\}\Big\|$$

$$+\beta_1\Big\|G_{K_1(x_1^n)}\Big\{f_1(x_1^{n-1}) - \lambda_1 x_1^{n-1} + \lambda_1\Omega_1(x_1^{n-1})\Big\} - G_{K_1(x_1^{n-1})}\Big\{f_1(x_1^{n-1}) - \lambda_1 x_1^{n-1} + \lambda_1\Omega_1(x_1^{n-1})\Big\}\Big\| + \beta_1\|e_1^n - e_1^{n-1}\|.$$

Then

$$\|x_1^{n+1} - x_1^n\| \leq (1-\beta_1)\|x_1^n - x_1^{n-1}\| + \beta_1\Big\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big\| + \beta_1\Big\|f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\Big\| + \beta_1\lambda_1\Big\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\Big\| + \beta_1\gamma_1\|x_1^n - x_1^{n-1}\| + \beta_1\|e_1^n - e_1^{n-1}\|. \quad (4.3)$$

Since f_1 is δ_1-strongly accretive and ν_1-Lipschitz continuous, using Lemma 2.4, we have

$$\Big\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big\|^2$$
$$\leq \|x_1^n - x_1^{n-1}\|^2 - 2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1\Big(x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big)\Big\rangle$$
$$\leq \|x_1^n - x_1^{n-1}\|^2 - 2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1(x_1^n - x_1^{n-1})\Big\rangle$$
$$+2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1(x_1^n - x_1^{n-1}) - J_1\Big(x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big)\Big\rangle$$
$$\leq (1 - 2\delta_1 + 64c_1\nu_1^2)\|x_1^n - x_1^{n-1}\|^2.$$

This implies

$$\Big\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\Big\| \leq \sqrt{1 - 2\delta_1 + 64c_1\nu_1^2}\ \|x_1^n - x_1^{n-1}\|. \quad (4.4)$$

Similarly,

$$\Big\|f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\Big\|^2$$
$$\leq \lambda_1^2\|x_1^n - x_1^{n-1}\|^2 - 2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1\Big(f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\Big)\Big\rangle$$
$$= \lambda_1^2\|x_1^n - x_1^{n-1}\|^2 - 2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1\Big(\lambda_1(x_1^n - x_1^{n-1})\Big)\Big\rangle$$
$$+2\Big\langle f_1(x_1^n) - f_1(x_1^{n-1}), J_1\Big(\lambda_1(x_1^n - x_1^{n-1})\Big) - J_1\Big(f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\Big)\Big\rangle$$
$$\leq (\lambda_1^2 - 2\lambda_1\delta_1 + 64c_1\nu_1^2)\ \|x_1^n - x_1^{n-1}\|^2.$$

Then

$$\Big\|f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\Big\| \leq \sqrt{\lambda_1^2 - 2\lambda_1\delta_1 + 64c_1\nu_1^2}\ \|x_1^n - x_1^{n-1}\|. \quad (4.5)$$

So by using (3.2), $\Omega_i : X_i \to X_i$ is defined by

$$\Big\langle\Omega_i(x_i), J_i(y_i)\Big\rangle = \langle x_i, J_i(y_i)\rangle - \Big\langle k_i(f_i(x_i)), J_i(y_i)\Big\rangle + \Big\langle k_i(x_i), J_i(y_i)\Big\rangle - \rho_i\Big\langle T_i(u_{i1}, u_{i2}, u_{i3}) - g_i, J_i(y_i)\Big\rangle - \rho_i b_i(x_i, y_i), \quad \forall x_i, y_i \in X_i. \quad (4.6)$$

Therefore, using Condition 2.12 (ii), we have

$$\Big\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\Big\|^2$$

$$
\begin{aligned}
&= \left|\left\langle \Omega_1(x_1^n) - \Omega_1(x_1^{n-1}), J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle\right| \\
&= \left|\left\langle x_1^n - x_1^{n-1}, J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle - \rho_1 b_1\left(x_1^n - x_1^{n-1}, \Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right. \\
&\quad -\left\langle k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1})), J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle \\
&\quad +\left\langle k_1(x_1^n) - k_1(x_1^{n-1}), J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle \\
&\quad \left.-\rho_1\left\langle T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1}), J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle\right| \\
&= \left|\left\langle x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right.\right. \\
&\quad \left.\left.+\left(k_1(x_1^n) - k_1(x_1^{n-1})\right) - \rho_1\left(T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right), J_1\left(\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right\rangle\right| \\
&\quad +\rho_1\left|b_1\left(x_1^n - x_1^{n-1}, \Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right)\right| \\
&\leq \left[\left\|x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right\| + \left\|k_1(x_1^n) - k_1(x_1^{n-1})\right\|\right. \\
&\quad \left.+\rho_1\left\|T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right\|\right] \left\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right\| \\
&\quad +\rho_1\mu_1\left\|x_1^n - x_1^{n-1}\right\| \left\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right\| \\
&\leq \left[\left\|x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right\| + \left\|k_1(x_1^n) - k_1(x_1^{n-1})\right\|\right. \\
&\quad +\rho_1\left\|T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right\| \\
&\quad \left.+\rho_1\mu_1\left\|x_1^n - x_1^{n-1}\right\|\right] \left\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right\|
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right\| \leq\ & \left[\left\|x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right\| + \left\|k_1(x_1^n) - k_1(x_1^{n-1})\right\|\right. \\
& \left.+\rho_1\left\|T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right\| + \rho_1\mu_1\left\|x_1^n - x_1^{n-1}\right\|\right]. \qquad (4.7)
\end{aligned}
$$

Since k_1 is L_{k_1}-Lipschitz continuous and τ_1-strongly accretive w.r.t f_1 and f_1 is L_{f_1}-Lipschitz continuous, using Lemma 2.4, we have

$$
\begin{aligned}
&\left\|x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right\|^2 \\
&\leq \|x_1^n - x_1^{n-1}\|^2 - 2\left\langle k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}), J_1\left(x_1^n - x_1^{n-1} - \left(k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1}))\right)\right)\right\rangle \\
&= \|x_1^n - x_1^{n-1}\|^2 - 2\left\langle k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1})), J_1(x_1^n - x_1^{n-1})\right\rangle \\
&\quad +2\left\langle k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1})), J_1(x_1^n - x_1^{n-1}) - J_1\left(x_1^n - x_1^{n-1} - (k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1})))\right)\right\rangle \\
&\leq (1 - 2\tau_1 + 64c_1L_{k_1}^2L_{f_1}^2)\|x_1^n - x_1^{n-1}\|^2.
\end{aligned}
$$

This implies

$$\left\|x_1^n - x_1^{n-1} - (k_1(f_1(x_1^n)) - k_1(f_1(x_1^{n-1})))\right\| \leq \sqrt{1 - 2\tau_1 + 64c_1L_{k_1}^2L_{f_1}^2}\ \|x_1^n - x_1^{n-1}\|. \tag{4.8}$$

Since k_1 is L_{k_1}-Lipschitz continuous, we have

$$\left\|k_1(x_1^n) - k_1(x_1^{n-1})\right\| \leq L_{k_1}\left\|x_1^n - x_1^{n-1}\right\|. \tag{4.9}$$

Again, since T_1 is $L_{T_{11}}, L_{T_{12}}$ and $L_{T_{13}}$-Lipschitz continuous in the first, second and third arguments, respectively, B_1 is $L_{B_{11}}, L_{B_{12}}$ and $L_{B_{13}} - \mathcal{D}-$ Lipschitz continuous in the first, second and third arguments, respectively, we have

$$\begin{aligned}
&\left\|T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right\| \\
&\leq \left\|T_1(u_{11}^n, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^n, u_{13}^n)\right\| + \left\|T_1(u_{11}^{n-1}, u_{12}^n, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^n)\right\| \\
&\quad + \left\|T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^n) - T_1(u_{11}^{n-1}, u_{12}^{n-1}, u_{13}^{n-1})\right\| \\
&\leq L_{T_{11}}\left\|u_{11}^n - u_{11}^{n-1}\right\| + L_{T_{12}}\left\|u_{12}^n - u_{12}^{n-1}\right\| + L_{T_{13}}\left\|u_{13}^n - u_{13}^{n-1}\right\| \\
&\leq L_{T_{11}}L_{B_{11}}\left(1 + \tfrac{1}{n}\right)\left\|x_1^n - x_1^{n-1}\right\| + L_{T_{12}}L_{B_{12}}\left(1 + \tfrac{1}{n}\right)\left\|x_2^n - x_2^{n-1}\right\| \\
&\quad + L_{T_{13}}L_{B_{13}}\left(1 + \frac{1}{n}\right)\left\|x_3^n - x_3^{n-1}\right\|.
\end{aligned} \tag{4.10}$$

Combining (4.3)-(4.10), we have

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| &\leq (1 - \beta_1)\left\|x_1^n - x_1^{n-1}\right\| + \beta_1\left\|x_1^n - x_1^{n-1} - (f_1(x_1^n) - f_1(x_1^{n-1}))\right\| \\
&\quad + \beta_1\left\|f_1(x_1^n) - f_1(x_1^{n-1}) - \lambda_1(x_1^n - x_1^{n-1})\right\| + \beta_1\lambda_1\left\|\Omega_1(x_1^n) - \Omega_1(x_1^{n-1})\right\| \\
&\quad + \beta_1\gamma_1\|x_1^n - x_1^{n-1}\| + \beta_1\|e_1^n - e_1^{n-1}\| \\
&\leq (1 - \beta_1)\left\|x_1^n - x_1^{n-1}\right\| + \beta_1\sqrt{1 - 2\delta_1 + 64c_1\nu_1^2}\ \|x_1^n - x_1^{n-1}\| \\
&\quad + \beta_1\sqrt{\lambda_1^2 - 2\lambda_1\delta_1 + 64c_1\nu_1^2}\ \|x_1^n - x_1^{n-1}\| + \beta_1\lambda_1\Big\{\sqrt{1 - 2\tau_1 + 64c_1L_{k_1}^2L_{f_1}^2}\ \|x_1^n - x_1^{n-1}\| \\
&\quad + L_{k_1}\left\|x_1^n - x_1^{n-1}\right\| + \rho_1\Big(L_{T_{11}}L_{B_{11}}\left(1 + \tfrac{1}{n}\right)\left\|x_1^n - x_1^{n-1}\right\| + L_{T_{12}}L_{B_{12}}\left(1 + \tfrac{1}{n}\right)\left\|x_2^n - x_2^{n-1}\right\| \\
&\quad + L_{T_{13}}L_{B_{13}}\left(1 + \tfrac{1}{n}\right)\left\|x_3^n - x_3^{n-1}\right\|\Big) + \rho_1\mu_1\left\|x_1^n - x_1^{n-1}\right\|\Big\} \\
&\quad + \beta_1\gamma_1\|x_1^n - x_1^{n-1}\| + \beta_1\|e_1^n - e_1^{n-1}\| \\
&\leq \Big[(1 - \beta_1) + \beta_1\sqrt{1 - 2\delta_1 + 64c_1\nu_1^2} + \beta_1\sqrt{\lambda_1^2 - 2\lambda_1\delta_1 + 64c_1\nu_1^2} + \beta_1\lambda_1\sqrt{1 - 2\tau_1 + 64c_1L_{k_1}^2L_{f_1}^2} \\
&\quad + \beta_1\lambda_1L_{k_1} + \rho_1\beta_1\lambda_1\mu_1 + \beta_1\gamma_1 + \beta_1\lambda_1\rho_1L_{T_{11}}L_{B_{11}}\left(1 + \tfrac{1}{n}\right)\Big]\left\|x_1^n - x_1^{n-1}\right\| \\
&\quad + \rho_1\beta_1\lambda_1L_{T_{12}}L_{B_{12}}\left(1 + \tfrac{1}{n}\right)\left\|x_2^n - x_2^{n-1}\right\| + \rho_1\beta_1\lambda_1L_{T_{13}}L_{B_{13}}\left(1 + \tfrac{1}{n}\right)\left\|x_3^n - x_3^{n-1}\right\| \\
&\quad + \beta_1\|e_1^n - e_1^{n-1}\|.
\end{aligned} \tag{4.11}$$

Similarly, following the same procedure as in (4.3)-(4.10), it follows that

$$\begin{aligned}\|x_2^{n+1}-x_2^n\| \quad\leq\quad & \rho_2\beta_2\lambda_2 L_{T_{21}}L_{B_{21}}\left(1+\tfrac{1}{n}\right)\left\|x_1^n-x_1^{n-1}\right\| \\ & +\Bigg[(1-\beta_2)+\beta_2\sqrt{1-2\delta_2+64c_2\nu_2^2}+\beta_2\sqrt{\lambda_2^2-2\lambda_2\delta_2+64c_2\nu_2^2}+\beta_2\lambda_2\sqrt{1-2\tau_2+64c_2L_{k_2}^2L_{f_2}^2} \\ & +\beta_2\lambda_2L_{k_2}+\rho_2\beta_2\lambda_2\mu_2+\beta_2\gamma_2+\beta_2\lambda_2\rho_2L_{T_{22}}L_{B_{22}}\left(1+\tfrac{1}{n}\right)\Bigg]\left\|x_2^n-x_2^{n-1}\right\| \\ & +\rho_2\beta_2\lambda_2L_{T_{23}}L_{B_{23}}\left(1+\frac{1}{n}\right)\left\|x_3^n-x_3^{n-1}\right\|+\beta_2\|e_2^n-e_2^{n-1}\| \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\|x_3^{n+1}-x_3^n\| \quad\leq\quad & \rho_3\beta_3\lambda_3 L_{31}L_{B_{31}}\left(1+\tfrac{1}{n}\right)\left\|x_1^n-x_1^{n-1}\right\|+\rho_3\beta_3\lambda_3L_{32}L_{B_{32}}\left(1+\tfrac{1}{n}\right)\left\|x_2^n-x_2^{n-1}\right\| \\ & +\Bigg[(1-\beta_3)+\beta_3\sqrt{1-2\delta_3+64c_3\nu_3^2}+\beta_3\sqrt{\lambda_3^2-2\lambda_3\delta_3+64c_3\nu_3^2}+\beta_3\lambda_3\sqrt{1-2\tau_3+64c_3L_{k_3}^2L_{f_3}^2} \\ & +\beta_3\lambda_3L_{k_3}+\rho_3\beta_3\lambda_3\mu_3+\beta_3\gamma_3+\beta_3\lambda_3\rho_3L_{T_{33}}L_{B_{33}}\left(1+\tfrac{1}{n}\right)\Bigg]\left\|x_3^n-x_3^{n-1}\right\| \\ & +\beta_3\|e_3^n-e_3^{n-1}\|. \end{aligned} \tag{4.13}$$

Therefore, combining (4.11)-(4.13), we have

$$\begin{aligned} & \left\|x_1^{n+1}-x_1^n\right\|+\left\|x_2^{n+1}-x_2^n\right\|+\left\|x_3^{n+1}-x_3^n\right\|=\sum_{i=1}^{3}\left\|x_i^{n+1}-x_i^n\right\| \\ & \leq \sum_{i=1}^{3}\Bigg[(1-\beta_i)+\beta_i\sqrt{1-2\delta_i+64c_i\nu_i^2}+\beta_i\sqrt{\lambda_i^2-2\lambda_i\delta_i+64c_i\nu_i^2} \\ & +\beta_i\lambda_i\sqrt{1-2\tau_i+64c_iL_{k_i}^2L_{f_i}^2}+\beta_i\lambda_iL_{k_i}+\rho_i\beta_i\lambda_i\mu_i+\beta_i\gamma_i \\ & +\sum_{j=1}^{3}\beta_j\lambda_j\rho_jL_{T_{ji}}L_{B_{ji}}\left(1+\tfrac{1}{n}\right)\Bigg]\left\|x_i^n-x_i^{n-1}\right\|+\sum_{i=1}^{3}\beta_i\|e_i^n-e_i^{n-1}\| \\ & \leq\sum_{i=1}^{3}(h_i+t_i^n)\left\|x_i^n-x_i^{n-1}\right\|+\sum_{i=1}^{3}\beta_i\|e_i^n-e_i^{n-1}\|, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} h_i = \quad & (1-\beta_i)+\beta_i\sqrt{1-2\delta_i+64c_i\nu_i^2}+\beta_i\sqrt{\lambda_i^2-2\lambda_i\delta_i+64c_i\nu_i^2} \\ & +\beta_i\lambda_i\sqrt{1-2\tau_i+64c_iL_{k_i}^2L_{f_i}^2}+\beta_i\lambda_iL_{k_i}+\rho_i\beta_i\lambda_i\mu_i+\beta_i\gamma_i \\ t_i^n = \quad & \sum_{j=1}^{3}\beta_j\lambda_j\rho_jL_{T_{ji}}L_{B_{ji}}\left(1+\tfrac{1}{n}\right). \end{aligned}$$

Then

$$\sum_{i=1}^{3}\left\|x_i^{n+1}-x_i^n\right\| \leq \sum_{i=1}^{3} r^n\left\|x_i^n-x_i^{n-1}\right\| + \sum_{i=1}^{3}\beta_i\|e_i^n-e_i^{n-1}\| \tag{4.15}$$

where $r^n = \max\{h_1+t_1^n, h_2+t_2^n, h_3+t_3^n\}$, for all $n=1,2,3,\cdots$. Letting $r=\max\{h_1+t_1, h_2+t_2, h_3+t_3\}$, where $t_i=\sum_{j=1}^{3}\beta_j\lambda_j\rho_j L_{T_{ji}}L_{B_{ji}}, \quad \forall i\in\{1,2,3\}$, we get $r^n\to r, t_i^n\to t_i$ as $n\to\infty, i\in\{1,2,3\}$.

From (4.2), since $0<r<1$, there exist $n_0\in N$ and $r_0\in(r,1)$ such that $r^n\leq r_0$ for all $n\geq n_0$. This implies from (4.15) that

$$\sum_{i=1}^{3}\left\|x_i^{n+1}-x_i^n\right\| \leq \sum_{i=1}^{3} r_0\left\|x_i^n-x_i^{n-1}\right\| + \sum_{i=1}^{3}\beta_i\|e_i^n-e_i^{n-1}\| \tag{4.16}$$

$$\leq \sum_{i=1}^{3} r_0^{n-n_0}\left\|x_i^{n_0+1}-x_i^{n_0}\right\| + \sum_{p=1}^{n-n_0}\sum_{i=1}^{3}\beta_i r_0^{p-1}s_i^{n-(p-1)}, \quad \forall n\geq n_0, \tag{4.17}$$

where $s_i^n=\|e_i^n-e_i^{n-1}\|, \quad \forall n\geq n_0$. Hence for any $m\geq n>n_0$, we have

$$\sum_{i=1}^{3}\left\|x_i^m-x_i^n\right\| \leq \sum_{d=n}^{m-1}\sum_{i=1}^{3}\left\|x_i^{d+1}-x_i^d\right\|$$

$$\leq \sum_{d=n}^{m-1}\sum_{i=1}^{3} r_0^{d-n_0}\left\|x_i^{n_0+1}-x_i^{n_0}\right\| + \sum_{d=n}^{m}\sum_{p=1}^{d-n_0}\sum_{i=1}^{3}\beta_i r_0^{p-1}s_i^{d-(p-1)} \tag{4.18}$$

$$\leq \sum_{d=n}^{m-1}\sum_{i=1}^{3} r_0^{d-n_0}\left\|x_i^{n_0+1}-x_i^{n_0}\right\| + \sum_{d=n}^{m}\sum_{p=1}^{d-n_0}\sum_{i=1}^{3}\beta_i r_0^{d}\frac{s_i^{d-(p-1)}}{r_0^{d-(p-1)}}. \tag{4.19}$$

Since $\sum_{d=1}^{\infty}s_1^d h^{-d}<\infty, \sum_{d=1}^{\infty}s_2^d h^{-d}<\infty$ and $\sum_{d=1}^{\infty}s_3^d h^{-d}<\infty, \quad \forall h\in(0,1)$ and $r_0<1$.

Therefore, (4.19) implies that $\left\|x_1^m-x_1^n\right\|\to 0$, $\left\|x_2^m-x_2^n\right\|\to 0$, $\left\|x_3^m-x_3^n\right\|\to 0$ as $n\to\infty$, so $\{x_1^n\},\{x_2^n\},\{x_3^n\}$ are Cauchy sequences in X_1,X_2,X_3 respectively. Thus, there exist $x_1\in X_1, x_2\in X_2, x_3\in X_3$ such that $x_1^n\to x_1, x_2^n\to x_2, x_3^n\to x_3$ as $n\to\infty$.

Now we prove that $u_{i1}^n\to u_{i1}\in B_{i1}(x_1), u_{i2}^n\to u_{i2}\in B_{i2}(x_2), u_{i3}^n\to u_{i3}\in B_{i3}(x_3)$, for each $i\in\{1,2,3\}$. In fact, it follows from the Lipschitz continuity of B_{i1},B_{i2},B_{i3} and from above iterative algorithm 4.1, that

$$\|u_{i1}^{n+1}-u_{i1}^n\|\leq\left(1+\frac{1}{n+1}\right)\mathcal{D}_1(B_{i1}(x_1^{n+1}),B_{i1}(x_1^n)), \tag{4.20}$$

$$\|u_{i2}^{n+1}-u_{i2}^n\|\leq\left(1+\frac{1}{n+1}\right)\mathcal{D}_2(B_{i2}(x_2^{n+1}),B_{i2}(x_2^n)), \tag{4.21}$$

$$\|u_{i3}^{n+1}-u_{i3}^n\|\leq\left(1+\frac{1}{n+1}\right)\mathcal{D}_3(B_{i3}(x_3^{n+1}),B_{i3}(x_3^n)). \tag{4.22}$$

From (4.20)-(4.22), it follows that $\{u_{i1}^n\},\{u_{i2}^n\}$ and $\{u_{i3}^n\}$ are also Cauchy sequences. Therefore, there exists $u_{i1}\in X_1, u_{i2}\in X_2$ and $u_{i3}\in X_3$ such that $u_{i1}^n\to u_{i1}, u_{i2}^n\to u_{i2}, u_{i3}^n\to u_{i3}$, as $n\to\infty$.

Further, for each $i\in\{1,2,3\}$,

$$\begin{aligned} d(u_{i1},B_{i1}(x_1)) &\leq \|u_{i1}-u_{i1}^n\|+d(u_{i1}^n,B_{i1}(x_1))\\ &\leq \|u_{i1}-u_{i1}^n\|+\mathcal{D}_1(B_{i1}(x_1^n),B_{i1}(x_1))\\ &\leq \|u_{i1}-u_{i1}^n\|+L_{B_{i1}}\|x_1^n-x_1\|\\ &\to 0 \text{ as } n\to\infty. \end{aligned}$$

Since B_{i1} is closed, we have $u_{i1} \in B_{i1}(x_1)$. Similarly $u_{i2} \in B_{i2}(x_2)$, $u_{i3} \in B_{i3}(x_3)$, respectively. Thus the approximate solution $\{x_i^n\}, \{u_{i1}^n\}, \{u_{i2}^n\}, \{u_{i3}^n\}$ generated by iterative algorithm 4.1 converge strongly to $x_i, u_{i1}, u_{i2}, u_{i3}$, respectively for each $i \in \{1, 2, 3\}$. $\square$

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