Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 2123–2128
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.7173

Derivation of some statistics for beta exponential distribution

Mujtaba Zuhair Ali*, Ahmed A.H. Alkhalidi

Computer Technical Engineering Department, College of Technical Engineering, The Islamic University, Najaf, Iraq

(Communicated by Javad Vahidi)

Abstract

We propose a beta exponential distribution that is formed from the logarithm of a random variable with a beta value, as well as a thorough investigation of the distribution's mathematical characteristics. Because we dealt with the beta exponential distribution, we have a clear and understandable way to comprehend the equation and apply it to the actual problem. To do this, we gathered the questions that statisticians find interesting and studied the most significant properties and statistics related to the distribution. Future research will utilize this data to identify specific industrial flaws or inefficiencies as a function of survival. Included are system statistics, BE distributions for graphs, generation functions, moments, and Hazard functions.

Keywords: Hazard function, Moment, Generating Function, shape and Order statistics of BE Distribution
2020 MSC: Primary 82C23; Secondary 62C25, 62E17

1 Introduction

Nadarajah and Kotz [2] introduced the beta-exponential (BE) distribution. They presented an equation for the sth moment, Hazard function characteristics, distribution findings for the sum of BE random variables, maximum likelihood estimation, and some asymptotic results in their article.

Let $G(x) = 1 - e^{-\lambda x}$ be the cdf of the exponential distribution, with $\lambda > 0$ and $x > 0$. The density of the BE distribution is given by:

$$f(x) = \frac{\lambda}{B(a,b)} e^{-b\lambda x} \left(1 - e^{-\lambda x}\right)^{a-1}, \quad a > 0,\ b > 0,\ \lambda > 0,\ x > 0. \tag{1.1}$$

As a specific example for $b = 1$, An exponential distribution has been included in this distribution (Gupta and Kundu, [1]).The BE distribution is identical to the exponential distribution with parameter b when $a = 1$. $X \sim BE(a,\ b,\ \lambda)$ denotes a random variable that follows a beta exponential distribution $b\lambda$.

With $b = \lambda = 1$, Figure 1.1 depicts several densities of the BE distribution.

*Corresponding author

Email addresses: mujtaba.z.albohidari@iunajaf.edu.iq (Mujtaba Zuhair Ali), ahmedalkhalidi8618@iunajaf.edu.iq (Ahmed A.H. Alkhalidi), ahmedalkhalidi8618@gmail.com (Ahmed A.H. Alkhalidi)

Received: August 2022 *Accepted:* November 2022

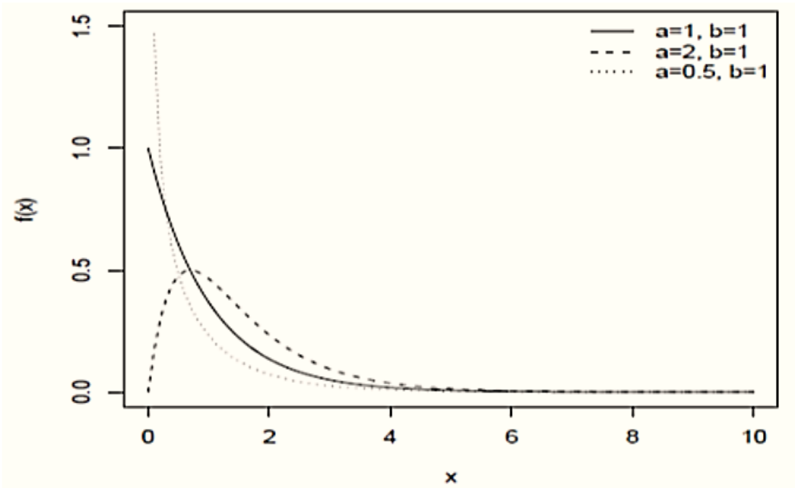

Figure 1.1: The BE pdf for several a values when $b = \lambda = 1$.

2 Hazard function

The Hazard function is defined by

$$h(x) = \frac{\lambda e^{-b\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}}{B_{e^{-\lambda x}}(b, a)}, \qquad a > 0,\ b > 0,\ \lambda > 0,\ x > 0. \tag{2.1}$$

where $B_{e^{-\lambda x}}(b, a)$ is the beta function that isn't complete.

The Hazard function's form is determined solely by the parameter a. When $a < 1$, $h(x)$ decreases monotonically with x, and when $a > 1$, $h(x)$ increasing monotonically with x. When the hazard function is constant, it means that

$$a = 1.$$

Figure 2.1 show the hazard function for $b = \lambda = 1$ and $a = \{0.5,\ 1,\ 2\}$.

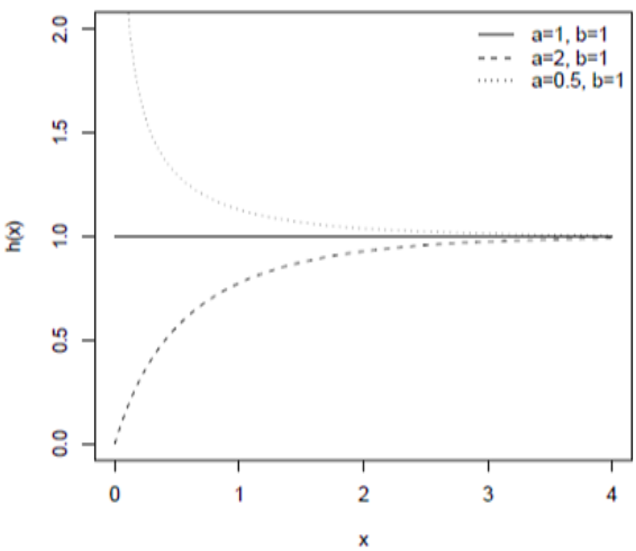

Figure 2.1: The BE Hazard function for some values of a with $b = \lambda = 1$.

3 Moment

We now let $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_{(n)}$ denote order statistics to the BE distribution with density function in (1.1). We calculate the general single moment

$$\begin{aligned}
E\left(X_{r:n}^k\right) &= \int_0^\infty x^k f_{r:n}(x)\,dx = \frac{n!}{(r-1)!\,(n-r)!}\int_0^\infty x^k [F(x)]^{r-1}[1-F(x)]^{n-r} f(x)\,dx \\
&= \frac{n!\lambda}{B(a,b)\,(r-1)!\,(n-r)!}\int_0^\infty x^k \left[1-e^{-\lambda x}\right]^{r-1}\left[1-\left[1-e^{-\lambda x}\right]\right]^{n-r} e^{-b\lambda x}\left\{1-e^{-\lambda x}\right\}^{a-1} dx \\
&= \frac{n!\lambda}{B(a,b)\,(r-1)!\,(n-r)!}\int_0^\infty x^k \left[1-e^{-\lambda \boldsymbol{x}}\right]^{r-2+a} e^{\lambda x(r-n-b)}\, dx \\
&= \frac{n!\lambda}{B(a,b)\,(r-1)!\,(n-r)!}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j}\int_0^\infty x^k e^{\lambda x(r-n-b+j)}\, dx.
\end{aligned}$$

But

$$\begin{aligned}
\int_0^\infty x^k e^{\lambda x(r-n-b+j)}\, dx &= (r-n-b+j)^{k+1}\int_0^\infty t^k e^{-t} dt \\
&= \frac{\Gamma(k+1)}{(r-n-b+j)^{k+1}}.
\end{aligned}$$

Therefore

$$E\left(X_{r:n}^k\right) = \frac{n!\lambda}{B(a,b)\,(r-1)!\,(n-r)!}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j}\frac{\Gamma(k+1)}{(r-n-b+j)^{k+1}} \quad , r=1,2,3,\ldots, k \geq 1. \tag{3.1}$$

4 Generating Function

For a random variable X with the density function BE, we calculate the rth moment around the origin and the moment generating function in this section

$$f(x) = \frac{\lambda}{\mathrm{B}(a,b)} e^{-b\lambda x}\left\{1-e^{-\lambda x}\right\}^{a-1}$$

and

$$S_{d,b,a} = \int_0^\infty x^{d-1} e^{-b\lambda x}\left\{1-e^{-\lambda x}\right\}^{a-1} dx.$$

Let $u = e^{-\lambda x}$, then $\ln u = -\lambda x$ and so, $\lambda x = |\ln u|$,

$$\frac{\lambda dx}{du} = \frac{1}{u} \Rightarrow dx = \frac{du}{u\lambda}$$

$$S_{d,b,a} = \int_0^\infty |\ln u|^{d-1} u^b \{1-u\}^{a-1}\frac{du}{u\lambda} = \frac{1}{\lambda}\int_0^\infty |\ln u|^{d-1} u^{b-1}\{1-u\}^{a-1}\, du.$$

If a > 0 and it is not an integer we have

$$S_{d,b,a} = \frac{\Gamma a}{\lambda}\sum_{j=0}^{\infty}\frac{(-1)^j}{(a-j)\ j!}\int_0^1 |\ln u|^{d-1} u^{b+j-1}\{1-u\}^{a-1}\, du.$$

Also, for p> -1 and our real q we get

$$\int_0^1 x^p |\ln x|^q\, dx = \frac{\Gamma(1+q)}{(1+p)^{q+1}}.$$

Then

$$\int_0^1 |\ln u|^{d-1} u^{b+j-1}\ du = \frac{\Gamma(1+d-1)}{(b+j-1)^{d-1+1}} = \frac{\Gamma d}{(b+j-1)^d}$$

$$\int_0^\infty x^{d-1} f(x)\, dx = \Gamma a \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma d}{\Gamma(a-j)\, j!(b+j-1)^d}$$

Finally, there is

$$\int_0^\infty x^{d-1} e^{-b\lambda x}\{1-e^{-\lambda x}\}^{a-1} dx = \Gamma a\ \Gamma b \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(a-j)\ \ j!(b+j-1)^d}.$$

If we let $a > 0$ be an integer then

$$\int_0^\infty x^{d-1} f(x)\, dx = \sum_{j=0}^{a-1} \binom{a-1}{j} (-1)^j \int_0^1 |\ln u|^{d-1} u^{b+j-1}\ du.$$

By using we have

$$\int_0^\infty x^{d-1} f(x)\, dx = \sum_{j=0}^{a-1} \frac{\binom{a-1}{j} (-1)^j\ \Gamma a}{(b+j-1)^d},$$

which simplifies to

$$\int_0^\infty x^{d-1} e^{-b\lambda x}\{1-e^{-\lambda x}\}^{a-1} dx = \Gamma d \sum_{j=0}^{a-1} \frac{\binom{a-1}{j} (-1)^j}{(b+j-1)^d}.$$

Then MGF of X can be written

$$M_{(t)} = \frac{\lambda}{\mathrm{B}(a,b)} \int_0^\infty e^{tx} e^{-b\lambda x}\{1-e^{-\lambda x}\}^{a-1} dx$$

$$= \frac{\lambda}{B(a,b)} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r e^{-b\lambda x}(1-e^{-\lambda x})^{a-1}\ \ dx.$$

Considering

$$\int_0^\infty x^r e^{-b\lambda x}(1-e^{-\lambda x})^{a-1}\ \ dx.$$

Let $w = e^{-\lambda x} \Rightarrow \ \ln w\ = -\lambda x\ \ \Rightarrow x = \frac{1}{\lambda} |\ln w|\ \ \Rightarrow\ \frac{dx}{dw} = \frac{1}{\lambda w}$

$$\Rightarrow dx = \frac{dw}{\lambda w}$$

$$\int_0^\infty \left(\frac{1}{\lambda} |\ln w|\right)^r w^b (1-w)^{a-1}\ \ \frac{dw}{\lambda w} = \frac{1}{\lambda^{r+1}} \int_0^\infty\ (|\ln w|)^r\ w^{b-1}(1-w)^{a-1}\ \ dw.$$

If a > 0 is a real and not an integer

$$(1-w)^{a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\mathrm{a})}{\Gamma(a-j)\, j!}\ w^j$$

Then

$$\frac{1}{\lambda^{r+1}} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\mathrm{a})}{\Gamma(a-j)\, j!} \int_0^\infty\ (|\ln w|)^r\ w^{b+j-1}\ \ dw.$$

Using the fact that

$$\int_0^1 x^p |\ln x\ |^q \quad dx = \frac{\Gamma(1+q)}{(1+p)^{q+1}}.$$

We have

$$\frac{1}{\lambda^{r+1}} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\mathrm{a})}{\Gamma\left(a-j\right) j!} \frac{\Gamma(r+1)}{\left(b+j-1\right)^{r+1}}.$$

Therefore mgf of X,

$$\begin{aligned} M_{(t)} =& \frac{\lambda}{B\left(a,b\right)} \sum_{r=0}^{\infty} \frac{t^r}{r!} \quad \frac{1}{\lambda^{r+1}} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\mathrm{a}\right)}{\Gamma\left(a-j\right) j!} \frac{\Gamma\left(r+1\right)}{\left(b+j-1\right)^{r+1}} \\ =& \frac{1}{\lambda^r \ B\left(a,b\right)} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r \left(-1\right)^j \Gamma\left(\mathrm{a}\right) \Gamma\left(r+1\right)}{r! \Gamma\left(a-j\right) j! (b+j-1)^{r+1}} \\ =& \frac{\Gamma\left(a+b\right)}{\lambda^r \Gamma\left(a\right) \Gamma\left(b\right)} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r \left(-1\right)^j \Gamma\left(\mathrm{a}\right) \Gamma\left(r+1\right)}{r! \Gamma\left(a-j\right) j! (b+j-1)^{r+1}}. \end{aligned}$$

Hence,

$$M_{(t)} = \frac{\Gamma\left(a+b\right)}{\lambda^r \Gamma\left(b\right)} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r \left(-1\right)^j \Gamma\left(r+1\right)}{r! \Gamma\left(a-j\right) j! (b+j-1)^{r+1}},$$

and for integer $a > 0$ we have

$$M_{(t)} = \frac{1}{\lambda^r \mathrm{B}\left(\mathrm{a},\mathrm{b}\right)} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{j=0}^{\infty} \binom{a-1}{j} \frac{(-1)^j}{\left(b+j-1\right)^{r+1}}. \tag{4.1}$$

Therefore using the fact that $M_{(t)} = \ \sum_{r=0}^{\infty} \frac{t^r}{r!}\ \mu'_r$, then for non-integer a , the rth moment of X is

$$E\left(x^r\right) = \frac{\Gamma\left(a+b\right)}{\lambda^r \mathrm{B}\left(\mathrm{a}\ ,\ \mathrm{b}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(r+1\right)}{\Gamma\left(a-j\right) \left(b+j-1\right)^{r+1}}. \tag{4.2}$$

When $a > 0$ is integer, we have

$$E\left(x^r\right) = \frac{\Gamma\left(r+1\right)}{\lambda^r \mathrm{B}\left(\mathrm{a}\ ,\ \mathrm{b}\right)} \sum_{j=0}^{a-1} \frac{(-1)^j \binom{a-1}{j}}{\left(b+j-1\right)^{r+1}} \tag{4.3}$$

5 Shape

Let $f\left(x\right) = \frac{\lambda}{B(a,b)}\ e^{-b\lambda x} \left(1-e^{-\lambda x}\right)^{a-1}$ and take logarithm natural to $f\left(x\right)$ obtain

$$\ln f(x) \ = \ln\left(\lambda\right) - b\lambda x + (a-1)(1-e^{-\lambda x})\ . \tag{5.1}$$

Now take the first and second derivative of $\ln f(x)$ in equation (5.1)

$$\frac{d \ln f(x)}{dx} = \frac{\left(a-1\right) \lambda e^{-\lambda x}}{1-e^{-\lambda x}} - b\lambda$$

$$\frac{d^2 \ln f(x)}{dx^2} = \frac{\left(a-1\right) \lambda (\lambda e^{-\lambda x} \left(-1+e^{-\lambda x}\right) - \left(e^{-\lambda x}\right)^2 \lambda)}{\left(1-e^{-\lambda x}\right)^2} = \frac{(1-a) \lambda^2 e^{-\lambda x}}{\left(1-e^{-\lambda x}\right)^2}. \tag{5.2}$$

6 Order statistics of BE distribution

Given that $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n$ denotes the order statistic of a random sample X_1 , X_2 , X_3 , ..., X_n from BE distribution with cdf F (x) and pdf f (x) of X_r can be written as

$$f_r\left(x_{(r)}\right) = \frac{n!}{(r-1)!\,(n-r)!}\; f\left(x\right)\left[F\left(x\right)\right]^{r-1}\left[1-F\left(x\right)\right]^{n-r} \tag{6.1}$$

such that

$$f\left(x\right) = \frac{\lambda}{B(a,b)} e^{-b\lambda x}\left\{1-e^{-\lambda x}\right\}^{a-1} \tag{6.2}$$

and

$$F\left(x\right) = 1-e^{-\lambda x}. \tag{6.3}$$

By using the pdf in equation (6.2) and the cdf in equation (6.3), we get

$$f_r\left(x_{(r)}\right) = \frac{n!}{(r-1)!\,(n-r)!}\;\frac{\lambda}{B\left(a,b\right)} e^{-b\lambda x_{(r)}}\left\{1-e^{-\lambda x_{(r)}}\right\}^{a-1}\left[1-e^{-\lambda x_{(r)}}\right]^{r-1}\left[1-\left[1-e^{-\lambda x_{(r)}}\right]\right]^{n-r}.$$

Hence,

$$f_r\left(x_{(r)}\right) = \frac{\lambda n!}{B\left(a,b\right)(r-1)!\,(n-r)!}\; e^{-\lambda x_{(r)}(b+n-r)}\;\left[1-e^{-\lambda x_{(r)}}\right]^{a-2+r},\; x_{(r)} > 0.$$

The pdf of the largest statistical order x_n is given by

$$f_n\left(x_n\right) = \frac{n\lambda}{B\left(a,b\right)}\; e^{-b\lambda x_n}\;\left[1-e^{-\lambda x_n}\right]^{a-2+n},\quad x_n > 0 \tag{6.4}$$

and the pdf of smallest statistical order x_1 is given by

$$f_1\left(x_1\right) = \frac{n\lambda}{B\left(a,b\right)}\; e^{-\lambda x_1(b+n-1)}\;\left[1-e^{-\lambda x_1}\right]^{a-1},\; x_1 > 0. \tag{6.5}$$

7 Conclusions

We researched and studied a class of distribution, These are known as the beta exponential distributions, and these are the ones I'll be using in this paper. Which result from considering $G(x)$ to be exponential. In this paper, we derive some properties of the beta exponential distribution, They are all of the following properties: Hazard function, generating function, Moment, shape, Order statistics of BE Distribution. We demonstrated that beta exponential distributions are the most traceable of any known distribution. This statement can also be used as a model for data on failure times. The results reported in this work can be used as a starting point for obtaining similar results for other distributions in the same family.

References

[1] R.D. Gupta and D. Kundu, *Generalized exponential distribution: different method of estimations*, J. Statist. Comput. Simul. **69** (2001), 315–337.

[2] S. Nadarajah and A.K. Gupta, *On the moments of the exponentiated Weibull distribution*. Commun. Statist. Theory Meth. **34** (2005), 253–256.