Int. J. Nonlinear Anal. Appl. 14 (2023) 7, 255–260
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27492.3639

Change in the form of fourth order two-point boundary value problem for solving by Adomian decomposition and homotopy perturbation methods

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(Communicated by Asadollah Aghajani)

Abstract

In this paper, we convert the fourth-order differential equations with two-point boundary conditions into a differential equation with homogeneous boundary conditions. Because the decomposition methods are closely related to the McLaren series, the McLaren series has a higher accuracy for points close to zero. Then we use Adomian decomposition and homotopy perturbation methods to solve three linear and nonlinear examples.

Keywords: Boundary value problems, Homotopy perturbation method, Adomian decomposition method, Modification, Stability
2020 MSC: 65L10

1 Introduction

A perturbation method is widely used in the analysis of nonlinear engineering problems [14]. To solve this problem, other techniques such as homotopy analysis method [3, 9, 12, 16] and Adomian decomposition method [1, 2, 19] were studied by researchers. Fourth order boundary value problems occur in various fields of applied mathematics such as solid mechanics, chemical kinetics, quantum mechanics, engineering, physical sciences, etc. Two-point and multi-point boundary value problems for fourth order ordinary differential equations have attracted a lot of attention.

In this paper, we consider the fourth-order boundary value problems of the type:

$$y^{(4)} = f(x, y, y', y'', y''') \tag{1.1}$$

with two-point boundary conditions:

$$y(a) = A_1, \quad y'(a) = A_2, \quad y(b) = B_1, \quad y'(b) = B_2,$$

where f is continuous function on [a, b] and the parameters $A_i, i = 1, 2$ and $B_j, j = 1, 2$ are real constants. Determined solution of a system of fourth order boundary value problem using cubic non-polynomial spline method [17]. used

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Received: May 2022 *Accepted:* July 2022

RKM for the solution of fourth order singularly perturbed boundary value problem. And also used RKM for the solution of fifth order boundary value problem [8]. Adomian decomposition method has been used to solve linear and nonlinear ordinary differential equations [13]. This method provides the solution in a rapid convergent series with computable terms. However, for the solution of boundary value problems using ADM, it is necessary to determine some unknown parameters and therefore, it is required to solve nonlinear algebraic differential equations. Geng and Cui proposed a method for solving nonlinear second order two-point BVP by the combination of ADM and RKM [7]. Geng and Cui [6] presented a method for solving nonlinear multi-point boundary value problems by combining homotopy perturbation and variational iteration methods. Dehghan and Tatari [18] used the Adomian decomposition method for solving multi-point boundary value problems. Recently, Cui and Geng [5, 4] have presented RKM for singular linear two-point boundary value problem, singular nonlinear two-point periodic boundary value problem. However, for numerical methods for solving singular multi-point BVPs, few works are available. In this paper, we convert the fourth-order differential equations with two-point boundary value into a simpler equation by changing the variable, and then solve the problem using the methods of Adomian decomposition and homotopy perturbation.

2 The idea of the method [11]

We first rewrite equation 1.1 with the given boundary values as follows.

$$y^{(4)}(x) = f(x, y(x), y'(x), y''(x), y'''(x)), \tag{2.1}$$

assuming variable change $x = a + (b-a)t$, and that $y(x) = w(t)$ and $h = b - a$, we have from relation 2.1:

$$w^{(4)}(t) = g(t, w(t), w'(t), w''(t), w'''(t)), \tag{2.2}$$

$$w(0) = A_1, \quad w'(0) = A_2, \quad w(1) = B_1, \quad w'(1) = B_2,$$

where in

$$g(t, w(t), w'(t), w''(t), w'''(t)) = h^4 f(a + (b-a)t, w(t), h^{-1}w'(t), h^{-2}w''(t), h^{-3}w'''(t)).$$

If w(x) is the answer to problem 2.2, then the answer to problem 2.1 is as follows:

$$y(x) = w(\frac{x-a}{b-a}).$$

Now to adjust the problem 2.2 with the assumption:

$$w(t) = z(t) + A_1 + (B_1 - A_1)t,$$

from its placements and derivatives in relation 2.2 we have:

$$z^{(4)}(t) = h(t, z(t), z'(t), z''(t), z'''(t)), \tag{2.3}$$

$$z(0) = 0, z'(0) = A_2 - B_1 + A_1, z(1) = 0, z'(1) = B_2 - B_1 + A_1,$$

where in

$$h(t, z(t), z'(t), z''(t), z'''(t)) = g(t, z(t) + A_1 + (B_1 - A_1)t, z'(t) + B_1 - A_1, z''(t), z'''(t)).$$

That problem 2.3 is more suitable for solving the proposed methods.

3 Numerical examples of the present study

In this section, we consider three linear and nonlinear examples for the application of the ADM and HPM, and we compare the results at the end of each example.

Example 3.1. Consider the fourth-order linear differential equation [17]:

$$y^{(4)} = y + y'' + e^x(x-3), \quad 1 \le x \le 2, \tag{3.1}$$

with the boundary conditions:

$$y(1) = 0, \quad y'(1) = -e, \quad y(2) = -e^2, \quad y'(2) = -2e^2,$$

the analytic solution is $y = (1-x)e^x$. Assuming

$$x = a + (b-a)t = 1 + t, \quad y(x) = z(t),$$

then by changing the variable $z = w + A_1 + (B_1 - A_1)t = w - e^2 t$ and the relation 3.1, we obtain the equation with the following homogeneous boundary conditions:

$$w^{(4)} = w + w'' + e^{t+1}(t-2) - e^2 t, \quad 0 \le x \le 1, \tag{3.2}$$

$$w(0) = 0, \quad w(1) = 0, \quad w'(0) = -e + e^2, \quad w'(1) = -e^2,$$

the exact solution is $w = -te^{t+1} + e^2 t$. Now we use methods ADM and HPM to solve problem 3.2. The results are shown in Table 1.
1.1. ADM method

$$w = 521.9101110 - 192e^{x+1} + 15xe^{x+1} + 485.8066579x + 217.4625450x^2 + 65.23876201x^3 +$$
$$14.49751423x^4 + 2.537064492x^5 + .3624214310x^6 + \cdots.$$

1.2. HPM method

$$w = 521.9101110 - 192e^{x+1} + 15xe^{x+1} + 485.8066578x + 217.4625469x^2 + 65.23876379x^3 +$$
$$14.49750256x^4 + 2.537062972x^5 + .3624383132x^6 + \cdots.$$

Table 1: Example 3.1 erorr estimates

x_i	y(exact)	erorr(ADM)	erorr(HPM)
0.0	0.0000000000	3.300000e-07	0.000000e+00
0.1	0.4384890075	4.687000e-07	7.110000e-08
0.2	0.8137878354	1.224800e-06	9.620000e-08
0.3	1.1159278300	6.090000e-07	2.100000e-07
0.4	1.3335424530	6.960000e-07	1.160000e-07
0.5	1.4536835150	8.160000e-07	9.300000e-08
0.6	1.4616142050	9.720000e-07	1.210000e-07
0.7	1.3405760950	1.390000e-07	3.190000e-07
0.8	1.0715269080	2.900000e-08	2.330000e-07
0.9	0.6328454910	1.210000e-07	6.990000e-08
1.0	0.0000000000	1.220729e-06	1.895773e-07

Note: To solve examples 3.2 and 3.3, we use polynomials He for nonlinear terms [18]. The polynomial relation He for its nonlinear part is as:

$$Nu = \sum_{i=0}^{\infty} A_i(u_0, u_1, \cdots, u_i),$$

which polynomials A_n were defined by

$$A_n(u_0, u_1, \cdots, u_n) = [\frac{1}{n!}\frac{d^n}{dp^n} N(\sum_{i=0}^{\infty} p^i u_i)]_{p=0}, \quad n = 0, 1, 2, \cdots.$$

Example 3.2. Consider the fourth-order nonlinear differential equation [17]:

$$y^{(4)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \tag{3.3}$$

with the boundary conditions

$$y(1) = 1, \quad y'(1) = 1, \quad y(2) = 8, \quad y'(2) = 24,$$

the analytic solution is $y = x^5 - 2x^4 + 2x^2$. Assuming

$$x = a + (b - a)t = 1 + t, \quad y(x) = z(t),$$

then by changing the variable $z = w + A_1 + (B_1 - A_1)t = w + 1 + 7t$ and the relation 3.3, we obtain the equation with the following homogeneous boundary conditions.

$$w^{(4)} = (w + 1 + 7t)^2 - (1 + t)^{10} + 4(1 + t)^9 - 4(1 + t)^8 - 4(1 + t)^7 + 8(1 + t)^6 -$$
$$4(1 + t)^4 + 120(1 + t) - 48,$$
$$w(0) = 0, \quad w(1) = 0, \quad w'(0) = -6, \quad w'(1) = 17, \tag{3.4}$$

the exact solution is $w = (1 + t)^5 - 2(1 + t)^4 + 2(1 + t)^2 - 1 - 7t$. The results of solving equation 3.4 are shown in Table 2.
2.1. ADM method

$$w = -6.000000001x + 1.1222 \times 10^{-8}x^2 + 1.999999990x^3 + 3x^4 + x^5 - 1.635 \times 10^{-9}x^6 +$$
$$3.03 \times 10^{-10}x^7 - 6.225 \times 10^{-9}x^8 + 1.0214 \times 10^{-8}x^9 - 4.941 \times 10^{-9}x^{10} + \cdots.$$

2.2. HPM method

$$w = -6x - 1.31926866 \times 10^{-7}x^2 + 2.000000191x^3 + 3x^4 + x^5 - 2.27169166 \times 10^{-8}x^6 +$$
$$+4.39188126 \times 10^{-9}x^7 - 4.6964831 \times 10^{-7}x^8 + 8.3015109 \times 10^{-7}x^9 - 4.6475871 \times 10^{-7}x^{10} + \cdots.$$

Table 2: Example 3.2 erorr estimates

x_i	y(exact)	erorr(ADM)	erorr(HPM)
0.0	0.0000000000	0.000000e+00	0.000000e+00
0.1	-0.5976900000	2.000000e-10	1.100000e-09
0.2	-1.1788800000	1.000000e-09	3.000000e-09
0.3	-1.7192700000	1.000000e-09	7.000000e-09
0.4	-2.1849600000	2.000000e-09	9.000000e-09
0.5	-2.5312500000	1.000000e-09	9.000000e-09
0.6	-2.7014400000	1.000000e-09	1.000000e-08
0.7	-2.6256300000	1.000000e-09	1.000000e-08
0.8	-2.2195200000	1.000000e-09	4.000000e-09
0.9	-1.3832100000	1.000000e-09	0.000000e+00
1.0	0.0000000000	5.224692e-10	2.019360e-10

Example 3.3. Consider the fourth-order nonlinear differential equation [17]:

$$u^{(4)} = sinx + sin^2x - (u'')^2, \tag{3.5}$$

with the boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u(1) = sin(1), \quad u'(1) = cos(1),$$

the analytic solution is $u = sinx$. Assuming

$$x = a + (b - a)t = t, \quad u(x) = z(t),$$

then by changing the variable $z = w + A_1 + (B_1 - A_1)t = w + sin(1)t$ and the relation 3.5, we obtain the equation with the following homogeneous boundary conditions:

$$w^{(4)} = sint + sin^2t - (w'')^2, \tag{3.6}$$

$$w(0) = 0, \quad w'(0) = 1 - sin(1), \quad w(1) = 0, \quad w'(1) = cos(1) - sin(1),$$

the exact solution is $w = sint - sin(1)t$. The results of solving equation 3.6 are shown in Table 3.
3.1. ADM method

$$w = -9.345397sin(x) - .479451cos(x) - 0.001872sin(2x) + 0.021253cos(2x) + 0.001543sin(3x)-$$

$$0.000030cos(4x) - .119862xsin(x) + 4xcos(x) + 0.000936xcos(2x) + 0.015625xsin(2x)-$$

$$0.00390cos(2x)x^2 + .5sin(x)x^2 + .45822 + 5.50210x - .10513x^2 - .21768x^3 - 0.00083x^4 - \cdots.$$

3.2. HPM method

$$w = -6.846703cos(x) + .936442sin(x) - 0.026744sin(2x) + 0.000496cos(2x) + 0.0625sin^2(x)-$$

$$1.7116xsin(x) + 0.0133xcos(2x) + 6.8462 - .7377x - 1.7731x^2 - 0.0195x^3 + 0.2054x^4 - \cdots.$$

Table 3: Example 3.3 erorr estimates

x_i	y(exact)	erorr(ADM)	erorr(HPM)
0.0	0.0000000000	8.532000e-10	0.000000e+00
0.1	0.0156863182	5.320000e-08	5.110000e-09
0.2	0.0303751338	2.120500e-07	3.053000e-08
0.3	0.0430789113	7.981000e-07	7.182000e-08
0.4	0.0528299484	2.260000e-07	1.073100e-07
0.5	0.0586900462	6.694000e-07	1.226600e-07
0.6	0.0597598825	7.041000e-07	1.061600e-07
0.7	0.0551879978	3.890000e-07	6.083000e-08
0.8	0.0441793031	2.782000e-07	1.319000e-08
0.9	0.0260030233	1.263000e-07	6.350000e-09
1.0	0.0000000000	1.178000e-07	5.173000e-11

4 Conclusions

In this paper, we convert the fourth-order differential equations with two-point boundary value problem into a special case of with homogeneous boundary conditions. Because Adomian decomposition and homotopy perturbation methods are closely related to the McLaren series, solving a new problem with these methods is more accurate and stable than shown in the examples.

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