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The cubic trigonometric B-spline collocation method for the time-fractional stochastic Advection-Diffusion equation

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Abstract

The ultimate goal of this performance study is to provide a proposed scheme for solving the time-fractional stochastic advection-diffusion equation (TFSADE) of order $\alpha(0 \le \alpha < 1)$. In this proposed scheme, we utilize an approach based on cubic trigonometric B-spline collocation methods (CTBSCM). In this study, we replace the existing fractional derivative with the fractional Caputo derivative for time discretization and then replace the first and second derivatives of the equation using cubic trigonometric B-spline functions for spatial discretization. Applying this proposed scheme to TFSADE causes the equation to reduce to the linear system. In the end, the examples show that the order of convergence of the proposed method is $O(\tau^{2-\alpha}+h^2)$ where h and τ are the spatial and time step lengths, respectively.

Keywords: Fractional stochastic equation, Cubic trigonometric B-spline, Brownian motion
2020 MSC: 34K50, 65L60

1 Introduction

Recently, finding a solution for a class of fractional differential equations involving Brownian motion is highly important, because this type of equation is rarely be solved due to randomness, and the analysis of differential equations involving random coefficients gives us more details of the phenomenon behavior. Mathematical models play a key role in the fields of science and industry. As a result, most scientists deal with stochastic differential equations. Mathematical models in most fields include coefficients that are not completely known and have types of random environmental disturbances and noise [1]. We originally intend to obtain the numerical solution of the following stochastic equation:

$$D_t^{\alpha}u(x,t)+\sigma_1\frac{\partial u(x,t)}{\partial x}-(\sigma_2+\sigma_3\dot{B}(t))\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t), \quad x\in(a,b), \quad t\in(0,T], \tag{1.1}$$

with the following initial and boundary conditions

$$u(x,0)=g(x), \quad x\in[a,b], \tag{1.2}$$
$$u(a,t)=u(b,t)=0, \quad t\in(0,T], \tag{1.3}$$

where σ_1 is the coefficient of advection, and σ_2 and σ_3 are the coefficients of diffusion terms. $g(x)$ is a continuous function. The source function $f(x,t)$ is a sufficiently smooth function. Also, the phrase $\dot{B}(t)=\frac{dB(t)}{dt}$ is white noise where $B(t)$ is a Brownian motion. For discretization of $B(t)$, we set $t=t_j$ and let $B_j=B(t_j)$.

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2 Numerical Scheme

First we consider two arbitrary constants $M, N \in \mathbb{N}$. We assume

$$a = x_0 < x_1 < \cdots < x_M = b, \quad x_i = a + i\left(\frac{b-a}{M}\right), (i = 0, 1, 2, \cdots, M)$$

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_k = k\left(\frac{T}{N}\right), (k = 0, 1, 2, \cdots, N).$$

are uniform partition in the solution domain $[a, b]$ and $[0, T]$, respectively.

Now let $TB_m^3(x)$ for $m = -1, \cdots, M+1$ be the cubic trigonometric B-spline function in the uniform partition on $[a, b]$ that can be defined as follows

$$TB_m^3(x) = \frac{1}{w}\begin{cases} p^3(x_m), & x \in [x_{m-2}, x_{m-1}], \\ p(x_m)[p(x_m)q(x_{m+2}) + q(x_{m+3})p(x_{m+1})] + q(x_{m+4})p^2(x_{m+1}), & x \in [x_{m-1}, x_m], \\ q(x_{m+4})[p(x_{m+1})q(x_{m+3}) + q(x_{m+4})p(x_{m+2})] + p(x_m)q^2(x_{m+3}), & x \in [x_m, x_{m+1}], \\ q^3(x_{m+4}), & x \in [x_{m+1}, x_{m+2}], \\ 0, & o.w. \end{cases} \tag{2.1}$$

where

$$p(x_m) = \sin\left(\frac{x - x_m}{2}\right), \quad q(x_m) = \sin\left(\frac{x_m - x}{2}\right), \quad w = \sin\left(\frac{h}{2}\right).\sin(h).\sin\left(\frac{3h}{2}\right),$$

where $h = \frac{b-a}{M}$.

It is obvious that the support of the cubic trigonometric B-spline $TB_m^3(x)$ and its derivative is $[x_{m-2}, x_{m+2}]$.

Let $u(x, t)$ and $U(x, t)$ are the analytical and numerical solutions of the differential equation (1.1), respectively. According to the collocation method, the numerical solution can be approximated as

$$u(x,t) \simeq U(x,t) = \sum_{m=-1}^{M+1} \boldsymbol{\Upsilon}_m(t).TB_m^3(x), \tag{2.2}$$

and the coefficients $\boldsymbol{\Upsilon}_m(t)$ are to be determined by the numerical scheme proposed in this paper.

Given the bases of cubic trigonometric B-spline (2.1) and the numerical solution in (2.2), we present an approximation for the discretization of the first and second derivatives of equation (1.1) as follows

$$\begin{cases} U(x_m, t) = a_1\boldsymbol{\Upsilon}_{m-1}(t) + a_2\boldsymbol{\Upsilon}_m(t) + a_1\boldsymbol{\Upsilon}_{m+1}(t), \\ \frac{\partial U(x_m,t)}{\partial x} = -a_3\boldsymbol{\Upsilon}_{m-1}(t) + a_3\boldsymbol{\Upsilon}_{m+1}(t), \\ \frac{\partial^2 U(x_m,t)}{\partial x^2} = a_4\boldsymbol{\Upsilon}_{m-1}(t) - a_5\boldsymbol{\Upsilon}_m(t) + a_4\boldsymbol{\Upsilon}_{m+1}(t), \end{cases} \tag{2.3}$$

where

$$a_1 = \csc(h).\csc\left(\frac{3h}{2}\right).\sin^2\left(\frac{h}{2}\right), \quad a_2 = \frac{2}{1 + 2\cos(h)}, \quad a_3 = \frac{3}{4}\csc\left(\frac{3h}{2}\right),$$

$$a_4 = \frac{3 + 9\cos(h)}{4\cos\left(\frac{h}{2}\right) - 4\cos\left(\frac{5h}{2}\right)}, \quad a_5 = -\frac{3\cot^2\left(\frac{h}{2}\right)}{2 + 4\cos(h)}.$$

The Caputo's time fractional derivative of order $\alpha \in (0, 1]$ is given by [2]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u}{\partial \tau}(x,\tau)\frac{d\tau}{(t-\tau)^\alpha}. \tag{2.4}$$

where Γ is the Gamma function. Using forward difference formulation, for $k = 1, \cdots, N-1$, we have:

$$\begin{aligned}
D_t^{\alpha} u(x, t_k) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - \tau)^{-\alpha} u_\tau(x, \tau) d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \frac{u^{j+1} - u^j}{\tau} \int_{t_j}^{t_{j+1}} (t_k - \tau)^{-\alpha} d\tau + O(\tau^{1-\alpha}) \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} [(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}](u^{j+1} - u^j) + O(\tau^{2-\alpha}) \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} b_{\alpha,j} (u^{k-j+1} - u^{k-j}) + O(\tau^{2-\alpha}), \qquad (2.5)
\end{aligned}$$

where $\tau = \frac{T}{N}$ and $b_{\alpha,j} = (j+1)^{1-\alpha} - j^{1-\alpha}$. By applying the difference form of the time derivative in (2.5), the fractional advection-diffusion equation (1.1), for $i = 1, \cdots, M-1,\ k = 0, \cdots N-1$ can be written as

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_{\alpha,j} \left[U_i^{k-j+1} - U_i^{k-j} \right] + \sigma_1 (U_x)_i^{k+1} = \theta_j (U_{xx})_i^{k+1} + f_i^{k+1} \qquad (2.6)$$

where $U(x_j, t_k) := U_j^k$, and $f(x_j, t_k) := f_j^k$, and $\dot{B} \simeq \frac{B(t_j) - B(t_{j-1})}{\tau} := \zeta_j$ for $j = 1, \cdots, N$, and $\sigma_2 + \sigma_3 \zeta_j := \theta_j$. Then using the collocation method and substituting (2.3) in (2.6) for $i = 1, \cdots, M-1$ and $k = 1, \cdots, N$, leads to the following recurrence difference formula corresponding to the parameters $\boldsymbol{\Upsilon}_m^k$,

$$Z_1 \boldsymbol{\Upsilon}_{i-1}^{k+1} + Z_2 \boldsymbol{\Upsilon}_i^{k+1} + Z_3 \boldsymbol{\Upsilon}_{i+1}^{k+1} = f_i^{k+1} + r(\boldsymbol{\Upsilon}_i^k - \sum_{j=0}^{k-1} b_{\alpha,j} [\boldsymbol{\Upsilon}_i^{k-j+1} - \boldsymbol{\Upsilon}_i^{k-j}]), \qquad (2.7)$$

where $\boldsymbol{\Upsilon}_m^k = \boldsymbol{\Upsilon}_m(t_k)$ and

$$r = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}, \quad Z_1 = -a_3\sigma_1 - a_4\theta_j, \quad Z_2 = a_5\theta_j + r b_{\alpha,0}, \quad Z_3 = a_3\sigma_1 - a_4\theta_j.$$

and the coefficients matrices Z is

$$Z = \begin{pmatrix}
Z_2 & Z_3 & & & & \\
Z_1 & Z_2 & Z_3 & & & \\
 & Z_1 & Z_2 & Z_3 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & Z_1 & Z_2 & Z_3 & \\
 & & & Z_1 & Z_2 & Z_3 \\
 & & & & Z_1 & Z_2
\end{pmatrix}.$$

The matrix form of Equation (2.7) is as follows

$$Z\boldsymbol{\Upsilon}^{k+1} = f_i^{k+1} + r(b_{\alpha,0}\boldsymbol{\Upsilon}_i^k - \sum_{j=0}^{k-1} b_{\alpha,j} [\boldsymbol{\Upsilon}_i^{k-j+1} - \boldsymbol{\Upsilon}_i^{k-j}]), \quad k = 0, \cdots, N. \qquad (2.8)$$

Note that $\boldsymbol{\Upsilon}^k = [\boldsymbol{\Upsilon}_{-1}^k, \boldsymbol{\Upsilon}_0^k, \boldsymbol{\Upsilon}_1^k, \cdots, \boldsymbol{\Upsilon}_M^k, \boldsymbol{\Upsilon}_{N+1}^k]^T$ is the unknown parameters, and $\boldsymbol{f}^k = [f_0^k, \cdots, f_M^k]^T$. For solve the system (2.7) with matrix in $(N+1) \times (N+3)$ dimensions, by using the boundary conditions of problem (1.2), the unknown parameters $\boldsymbol{\Upsilon}_{-1}^k$ and $\boldsymbol{\Upsilon}_{M+1}^k$ may be eliminated from the system as follows:

Let $i = 0$ and $i = M$, by using the conditions (1.3) and relation (2.3) we have

$$\begin{cases}
U(x_0 = a, t) = a_1\boldsymbol{\Upsilon}_{-1}(t) + a_2\boldsymbol{\Upsilon}_0(t) + a_1\boldsymbol{\Upsilon}_1(t) = 0, \\
U(x_M = b, t) = a_1\boldsymbol{\Upsilon}_{M-1}(t) + a_2\boldsymbol{\Upsilon}_M(t) + a_1\boldsymbol{\Upsilon}_{M+1}(t) = 0.
\end{cases}$$

Thus, for every k:

$$\begin{cases} \mathbf{\Upsilon}^k_{-1} = -a_2\mathbf{\Upsilon}^k_0 - a_1\mathbf{\Upsilon}^k_1, \\ \mathbf{\Upsilon}^k_{M+1} = -a_2\mathbf{\Upsilon}^k_M - a_1\mathbf{\Upsilon}^k_{M-1}. \end{cases} \tag{2.9}$$

Having the initial vector $\mathbf{\Upsilon}^0$, the system (2.9) has a unique solution. The starting vector $\mathbf{\Upsilon}^0 = [\mathbf{\Upsilon}^0_{-1}, \mathbf{\Upsilon}^0_0, \cdots, \mathbf{\Upsilon}^0_{M+1}]^T$ can be determined by (2.3) and initial conditions of the problem, as the following forms

$$U(x_i, 0) = a_1\mathbf{\Upsilon}^0_{i-1} + a_2\mathbf{\Upsilon}^0_i + a_1\mathbf{\Upsilon}^0_{i+1} = g(x_i), \quad i = 0, 1, \cdots, M.$$

Therefore, the initial vector $\mathbf{\Upsilon}^0$ is determined from the following matrix equation;

$$\begin{pmatrix} a_1 & a_2 & a_1 & & & & & & \\ & a_1 & a_2 & a_1 & & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & & a_1 & a_2 & a_1 & \\ & & & & & & a_1 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} \mathbf{\Upsilon}^0_{-1} \\ \mathbf{\Upsilon}^0_0 \\ \vdots \\ \mathbf{\Upsilon}^0_M \\ \mathbf{\Upsilon}^0_{M+1} \end{pmatrix} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_{M-1}) \\ g(x_M) \end{pmatrix}$$

Corresponding to the time fractional derivative discretization, and as regards the matrix Z is positive definite, then the proposed numerical scheme (2.7) is consistent to the differential equation (1.1)-(1.2).

3 Numerical examples

To check the accuracy of the present scheme (2.7), numerical study of test examples is presented. The error norm L_2 and rate are calculated. The numerical results obtained from CBSCM are compared with given exact solutions and the numerical methods available in literature. Here, an example have been considered to verify the validity of proposed numerical algorithm (2.7).

Example 1: Consider the following time-fractional stochastic advection-diffusion equation:

$$\begin{cases} D^{(\alpha)}_t(x,t) + \sigma_1 u_x(x,t) = (\sigma_2 + \sigma_3\dot{B}(t))u_{xx}(x,t) + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \end{cases} \tag{3.1}$$

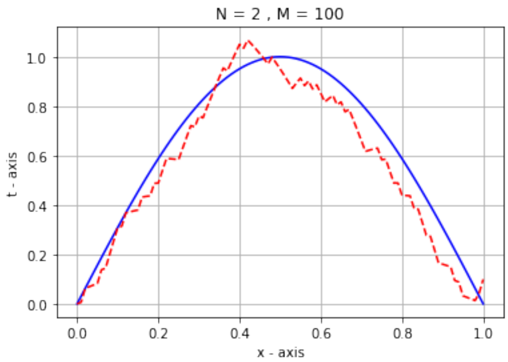
where $0 < \alpha \leq 1$. Let $\sigma_3 = \sigma_1 = 1$ and $\sigma = \frac{1}{\pi^2}$ the exact solution is $u(x,t) = t^2\sin(\pi x)$, is the exact solution of the equation (3.1). So we have

$$f(x,t) = \frac{2t^{2-\alpha}\sin(\pi x)}{\Gamma(3-\alpha)} + \Big(\frac{1}{\pi^2} + 1\Big)\pi^2 t^2 sin(\pi x) - \pi t^2\cos(\pi x).$$

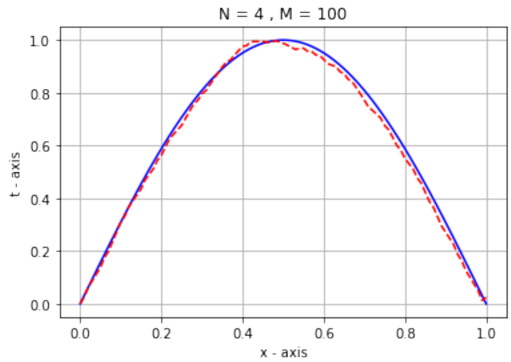
In Table 3 and Figure 1 we show the result of applying scheme (2.7) for solving the equation (3.1)

Table 1: Absolute error L_2 and experimental order of convergence of TFSAD equations for Example 1 and $M = N = 1000$

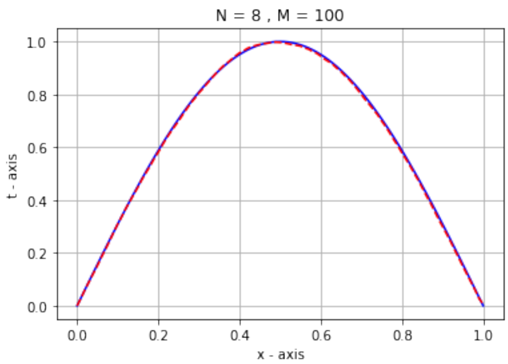
	Exact solution	Scheme (2.7)		
(x_i, t_i)		$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
$(0.0, 0.0)$	0.000000000	0.000000000	0.000000000	0.000000000
$(0.1, 0.1)$	0.003090169	0.003083545	0.003057546	0.002911341
$(0.2, 0.2)$	0.023511410	0.023504785	0.023478786	0.023332581
$(0.3, 0.3)$	0.072811529	0.072804905	0.072778905	0.072632700
$(0.4, 0.4)$	0.152169042	0.152162416	0.152136418	0.151990213
$(0.5, 0.5)$	0.250000000	0.249993375	0.249967376	0.249821171
$(0.6, 0.6)$	0.342380345	0.342373721	0.342347722	0.342201517
$(0.7, 0.7)$	0.396418327	0.396411702	0.396385703	0.396239498
$(0.8, 0.8)$	0.376182561	0.376175937	0.376149937	0.376003732
$(0.9, 0.9)$	0.250303765	0.250297141	0.250271141	0.250124936
$(1.0, 1.0)$	0.000000000	0.000000000	0.000000000	0.0000000000



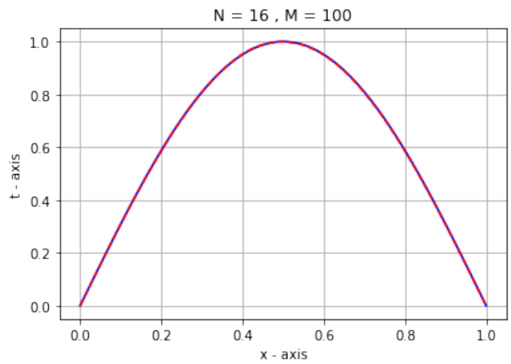
(a) $M = N = 2$ and $t = 0.4$



(b) $M = N = 4$ and $t = 0.6$



(c) $M = N = 8$ and $t = 0.4$



(d) $M = N = 16$ and $t = 0.6$

Figure 1: Exact and numerical solutions in example 1 for $\alpha = 1.05$.

Example 2: Consider the following time-fractional stochastic advection-diffusion equation:

$$\begin{cases} D_t^{(\alpha)}(x,t) + \sigma_1 u_x(x,t) = (\sigma_2 + \sigma_3 \dot{B}(t))u_{xx}(x,t) + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \le x \le 1, \end{cases} \tag{3.2}$$

where $0 < \alpha \le 1$ Let $\sigma_3 = 1, \sigma_1 = -3$ and $\sigma_2 = 1$ the exact solution is $u(x,t) = e^{2x-2t}$, is the exact solution of the equation (3.1). So

$$f(x,t) = -2e^{2x-2t}\big[1 + 2\big(1 + \dot{B}(t)\big) - 3\big]. \tag{3.3}$$

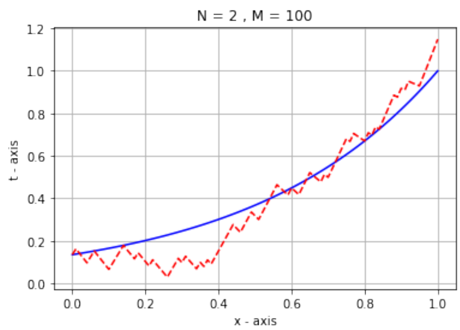
In Table 2, 3 and Figure 2 we show the result of applying scheme (2.7) for solving the equation (3.2)

Table 2: Comparison of the errors of approximate solutionsand rate when $h = 0.01$ and $t = 1$.

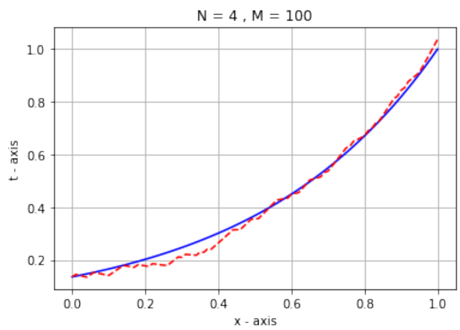
α	N	τ	$L_2 - norm$	$L_\infty - norm$	$Rate$
$\alpha = 0.25$	2	0.5	0.8582072	0.198029	–
	4	0.25	0.2145514	0.049507	1.34
	8	0.125	0.0536379	0.0123768	1.49
	16	0.0625	0.0134094	0.0030942	1.53
	32	0.03125	0.0033523	0.0007735	1.67
	64	0.015625	0.0008380	0.0008381	1.72
$\alpha = 0.5$	2	0.5	0.9121994	0.1732758	–
	4	0.25	0.2280498	0.0433189	1.28
	8	0.125	0.0570124	0.0108297	1.33
	16	0.0625	0.0014253	0.0027074	1.39
	32	0.03125	0.0035632	0.0006768	1.48
	64	0.015625	0.0008908	0.0001692	1.50
$\alpha = 0.75$	2	0.5	0.9121994	0.1732758	–
	4	0.25	0.2280498	0.0433189	1.05
	8	0.125	0.0570124	0.0108297	1.19
	16	0.0625	0.0014253	0.0027074	1.21
	32	0.03125	0.0035632	0.0006768	1.24
	64	0.015625	0.0008908	0.0001692	1.25

Table 3: Comparison of the errors of approximate solutionsand rate when $\tau = 0.01$ and $t = 1$.

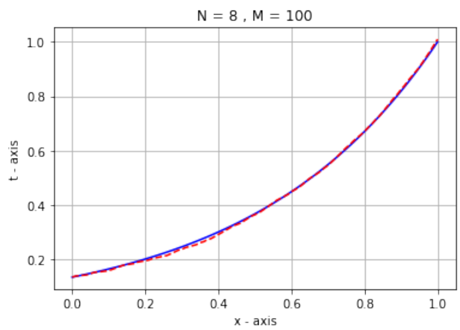
M	h	$L_2 - norm$	$L_\infty - norm$	$Rate$
2	0.5	0.8582072	0.198029	–
4	0.25	0.2145514	0.049507	1.89
8	0.125	0.0536379	0.0123768	1.92
16	0.0625	0.0134094	0.0030942	1.95
32	0.03125	0.0033523	0.0007735	1.98
64	0.015625	0.0008380	0.0008381	2.00



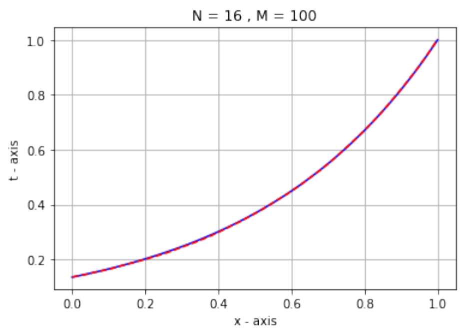
(a) $M = N = 2$ and $t = 0.4$ at different time level



(b) $M = N = 4$ and $t = 0.6$ at different time level



(c) $M = N = 8$ and $t = 0.4$ at different time level



(d) $M = N = 16$ and $t = 0.6$ at different time level

Figure 2: Exact and numerical solutions in example 2 for $\alpha = 1.05$.

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