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Convergence of the S-iteration process for the multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings in Banach spaces

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Abstract

In this work, we first introduce the modified S-iteration method to approximate common fixed points of three multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings in the setting of Banach spaces. Then we prove some convergence theorems for these mappings by using it. Also, a numerical example is given to illustrate the main results. Furthermore, we introduce a one-step iterative process for generalized $(\alpha-\beta)$-nonexpansive mappings and we prove strong convergence results for the one-step iterative process involving such mappings. Thus our results extend and generalise corresponding results in the contemporary literature.

Keywords: fixed point, multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings, uniformly convex Banach space
2020 MSC: 47H10, 54H25

1 Introduction and Preliminaries

The theory of multivalued mappings has an important place in the field of pure and applied mathematics. The importance of this theory has increased over the years and many problems involving such mappings have become the focus of discussion. In fact, among the various approaches used to develop this theory, one of the most interesting is based on methods of fixed point theory. Some of these works are [5], [6], [9], [11], [13], [14], [15] and [20].

Let $CB(\mathcal{C})$ be a collection of all nonempty closed and bounded subsets of $\mathcal{C}$ and let $KC(\mathcal{C})$ be a collection of all nonempty compact convex subsets of $\mathcal{C}$. For $\mathcal{A},\mathcal{B}\in CB(\mathcal{C})$, define the mapping $\mathcal{H}: CB(\mathcal{C})\to CB(\mathcal{C})$ by

$$\mathcal{H}(\mathcal{A},\mathcal{B})=\max\left\{\sup_{\varkappa\in\mathcal{A}}\mathfrak{d}(\varkappa,\mathcal{B}),\sup_{\varkappa\in\mathcal{B}}\mathfrak{d}(\varkappa,\mathcal{A})\right\}, \tag{1.1}$$

with $\mathfrak{d}(\varkappa,\mathcal{C})=\inf_{\mathfrak{q}\in\mathcal{C}}\|\varkappa-\mathfrak{q}\|$. We know that $\mathcal{H}$ is called Hausdorff metric.

A multivalued mapping $\mathcal{Q}:\mathcal{C}\to CB(\mathcal{C})$ is said to be a *nonexpansive* if

$$\mathcal{H}(\mathcal{Q}\varkappa,\mathcal{Q}\mathfrak{q})\le\|\varkappa-\mathfrak{q}\|$$

for all $\varkappa,\mathfrak{q}\in\mathcal{C}$.

Many generalizations of nonexpansive mapping have been introduced in the literature (see [2], [4], [10], [12]). In 2008, Suzuki [19] gave a new class of nonexpansive mapping as follows:

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Definition 1.1. Let $\mathcal{C}$ be a nonempty subset of a Banach space E. A mapping $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\varkappa - \mathcal{Q}\varkappa\| \leq \|\varkappa - \mathfrak{q}\| \implies \|\mathcal{Q}\varkappa - \mathcal{Q}\mathfrak{q}\| \leq \|\varkappa - \mathfrak{q}\|,$$

for all $\varkappa, \mathfrak{q} \in \mathcal{C}$.

He showed that the mappings satisfying this condition include nonexpansive mappings. He also proved some fixed point theorems for such mappings.

In 2017, Pant et al. [16] introduced generalized α-nonexpansive mapping which is another new class of nonexpansive single-valued mapping. And, they gave some fixed point results for such mappings in Banach spaces.

Definition 1.2. A mapping $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is said to be a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ such that

$$\begin{aligned} \frac{1}{2}\|\varkappa - \mathcal{Q}\varkappa\| &\leq \|\varkappa - \mathfrak{q}\| \Rightarrow \\ \|\mathcal{Q}\varkappa - \mathcal{Q}\mathfrak{q}\| &\leq \alpha\|\varkappa - \mathcal{Q}\mathfrak{q}\| + \alpha\|\mathfrak{q} - \mathcal{Q}\varkappa\| + (1-2\alpha)\|\varkappa - \mathfrak{q}\|, \end{aligned}$$

for all $\varkappa, \mathfrak{q} \in \mathcal{C}$.

In 2020, Ullah et al. [21] suggested a two parametric class of nonlinear mapppings. And, they proved existence of fixed point and showed convergence for this class of mappings in uniformly convex Banach spaces and compare several well known iterative algorithms in literature.

Definition 1.3. A mapping $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is said to be a generalized $(\alpha - \beta)$-nonexpansive mapping if there exists real numbers $\alpha, \beta \in [0, 1)$ satisfying $\alpha + \beta < 1$ such that

$$\begin{aligned} \frac{1}{2}\|\varkappa - \mathcal{Q}\varkappa\| &\leq \|\varkappa - \mathfrak{q}\| \Rightarrow \\ \|\mathcal{Q}\varkappa - \mathcal{Q}\mathfrak{q}\| &\leq \alpha\|\varkappa - \mathcal{Q}\mathfrak{q}\| + \alpha\|\mathfrak{q} - \mathcal{Q}\varkappa\| + \beta\|\varkappa - \mathcal{Q}\varkappa\| \\ &\quad + \beta\|q - \mathcal{Q}\mathfrak{q}\| + (1-2\alpha-2\beta)\|\varkappa - \mathfrak{q}\|, \end{aligned}$$

for all $\varkappa, \mathfrak{q} \in \mathcal{C}$.

Now, we will give the multivalued versions of the mappings given above for single-valued.

Definition 1.4. Let $\mathcal{C}$ be a nonempty subset of a Banach space E. A mapping $\mathcal{Q} : \mathcal{C} \to CB(\mathcal{C})$ is said to be

a) nonexpansive if

$$\mathcal{H}(\mathcal{Q}\varkappa, \mathcal{Q}\mathfrak{q}) \leq \|\varkappa - \mathfrak{q}\|, \forall \varkappa, \mathfrak{q} \in \mathcal{C}.$$

b) quasi nonexpansive if $\mathbb{F}(\mathcal{Q}) \neq \emptyset$ and

$$\mathcal{H}(\mathcal{Q}\varkappa, \mathcal{Q}p) \leq \|\varkappa - p\|, \forall \varkappa \in \mathcal{C} \text{ and } \forall p \in \mathbb{F}(\mathcal{Q}).$$

c) Suzuki generalized nonexpansive or endowed with condition $(\mathcal{C})$ if for $\varkappa, \mathfrak{q} \in \mathcal{C}$,

$$\frac{1}{2}\mathfrak{d}(\varkappa, \mathcal{Q}\varkappa) \leq \|\varkappa - \mathfrak{q}\| \implies \mathcal{H}(\mathcal{Q}\varkappa, \mathcal{Q}\mathfrak{q}) \leq \|\varkappa - \mathfrak{q}\|.$$

d) generalized α-nonexpansive if if there exists an $\alpha \in [0, 1)$ such that

$$\begin{aligned} \frac{1}{2}\mathfrak{d}(\varkappa, \mathcal{Q}\varkappa) &\leq \|\varkappa - \mathfrak{q}\| \implies \\ \mathcal{H}(\mathcal{Q}\varkappa, \mathcal{Q}\mathfrak{q}) &\leq \alpha\mathfrak{d}(\varkappa, \mathcal{Q}\mathfrak{q}) + \alpha\mathfrak{d}(\mathfrak{q}, \mathcal{Q}\varkappa) + (1-2\alpha)\|\varkappa - \mathfrak{q}\| \end{aligned}$$

for $\varkappa, \mathfrak{q} \in \mathcal{C}$,

e) generalized $(\alpha-\beta)$-nonexpansive if there exists real numbers $\alpha,\beta\in[0,1)$ satisfying $\alpha+\beta<1$ such that

$$\begin{aligned}\frac{1}{2}\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa) &\leq \|\varkappa-\mathfrak{q}\| \text{ implies}\\ \mathcal{H}(\mathcal{Q}\varkappa,\mathcal{Q}\mathfrak{q}) &\leq \alpha\mathfrak{d}(\varkappa,\mathcal{Q}\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}\varkappa)+\beta\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa)\\ &\quad+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}\mathfrak{q})+(1-2\alpha-2\beta)\|\varkappa-\mathfrak{q}\|.\end{aligned}$$

for $\varkappa,\mathfrak{q}\in\mathcal{C}$.

In 2021, Ullah et al. [22] introduced the class of multi-valued generalized $(\alpha-\beta)$-nonexpansive and they showed some main properties of these mappigs. They also gave some weak and strong convergence theorems by using a multi-valued version of the M-iterative process.

In this work, we divide our results into two sections. At the first section, we introduce multi-valued version of the S-iteration process for the multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings in Banach spaces. We also obtain some convergence results for such mappings and we exemplify the results numerically. At the second section, we also introduce one step iteration process for the multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings in Banach spaces. And, we obtain some strong convergence theorems for such mappings by using this iteration process. Our main theorems extend the study of [16] and [17] to the multivalued setting and prove a few important results. Furthermore, our results extend and generalize the existing literature of the nonexpansive multivalued mappings.

Proposition 1.5. [23] Let $\mathcal{Q}:E\to CB(E)$ be a multivalued mapping. Then the followings hold.

(i) If $\mathcal{Q}$ is Suzuki generalized nonexpansive then $\mathcal{Q}$ is a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping for some $\alpha,\beta\in[0,1)$.

(ii) If $\mathcal{Q}$ is generalized α-nonexpansive then $\mathcal{Q}$ is a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping for some $\alpha,\beta\in[0,1)$.

(iii) If $\mathcal{Q}$ is a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping and $\mathbb{F}(\mathcal{Q})$ is nonempty then $\mathcal{Q}$ is quasi nonexpansive.

Remark 1.6. [23]In the above proposition, the converse of (i) is not true in general, i.e., if $\mathcal{Q}$ is a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping, it does not necessarily imply that the mapping satisfy condition $(\mathcal{C})$. Similarly, if $\mathcal{Q}$ is a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping, it does not necessarily imply that the mapping is generalized α-nonexpansive.

The set of all fixed points of $\mathcal{Q}$ is denoted by $\mathbb{F}(\mathcal{Q})$. A multivalued mapping $\mathcal{Q}:\mathcal{C}\to CB(\mathcal{C})$ is said to have a fixed point if there exists an element $p\in\mathcal{C}$ such that $p\in\mathcal{Q}(p)$.

Proposition 1.7. [23] Let $\mathcal{C}$ be a nonempty subset of a Banach space E and $\mathcal{Q}:\mathcal{C}\to CB(\mathcal{C})$ be a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping for some $\alpha,\beta\in[0,1)$. Then for each $\varkappa,\mathfrak{q}\in\mathcal{C}$

(i) $\mathcal{H}(\mathcal{Q}\varkappa,\mathcal{Q}z)\leq\|\varkappa-z\|$, $\forall z\in\mathcal{Q}\varkappa$.

(ii) Either $\frac{1}{2}\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa)\leq\|\varkappa-\mathfrak{q}\|$ or $\frac{1}{2}\mathfrak{d}(z,\mathcal{Q}z)\leq\|z-\mathfrak{q}\|$, $\forall z\in\mathcal{Q}\varkappa$.

(iii) Either $\mathcal{H}(\mathcal{Q}\varkappa,\mathcal{Q}\mathfrak{q})\leq\alpha\mathfrak{d}(\varkappa,\mathcal{Q}\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}\varkappa)+\beta\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa)+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}\mathfrak{q})+(1-2\alpha-2\beta)\|\varkappa-\mathfrak{q}\|$ or $\mathcal{H}(\mathcal{Q}z,\mathcal{Q}\mathfrak{q})\leq\alpha\mathfrak{d}(z,\mathcal{Q}\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}z)+\beta\mathfrak{d}(z,\mathcal{Q}z)+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}\mathfrak{q})+(1-2\alpha-2\beta)\|z-\mathfrak{q}\|$, $\forall z\in\mathcal{Q}\varkappa$.

Proposition 1.8. [23] Let $\mathcal{C}$ be a nonempty subset of a Banach space E and $\mathcal{Q}:\mathcal{C}\to P(\mathcal{C})$ be a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping, then

$$\mathcal{H}(\mathcal{Q}\varkappa,\mathcal{Q}\mathfrak{q})\leq 2\frac{(1+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa)+\|\varkappa-\mathfrak{q}\|,$$

for all $\varkappa,\mathfrak{q}\in\mathcal{C}$.

Lemma 1.9. [22] Let $\mathcal{C}$ be a nonempty subset of a Banach space E and $\mathcal{Q}:\mathcal{C}\to CB(\mathcal{C})$ be a multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping for some $\alpha,\beta\in[0,1)$. Then for all $\varkappa,\mathfrak{q}\in\mathcal{C}$,

$$\mathfrak{d}(\varkappa,\mathcal{Q}\mathfrak{q})\leq\frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa,\mathcal{Q}\varkappa)+\|\varkappa-\mathfrak{q}\|.$$

Lemma 1.10. [8] Let E be a uniformly convex Banach space. Then for any $\xi\in[0,1]$ and for all $\varkappa,\mathfrak{q}\in E$, we have

$$\|\xi\varkappa+(1-\xi)\mathfrak{q}\|^2\leq\xi\|\varkappa\|^2+(1-\xi)\|\mathfrak{q}\|^2-\xi(1-\xi)\|\varkappa-\mathfrak{q}\|^2.$$

Lemma 1.11. [18] Let E be a uniformly convex Banach space. Let $\{\xi_n\}$ be a sequence of real numbers such that $0<s\leq\xi_n\leq t<1$ for all $n\geq1$, and $\{\varkappa_n\}$ and $\{\mathfrak{q}_n\}$ be sequences of E such that $\limsup_{n\to\infty}\|\varkappa_n\|\leq k$, $\limsup_{n\to\infty}\|\mathfrak{q}_n\|\leq k$ and $\lim_{n\to\infty}\|\xi_n\varkappa_n+(1-\xi_n)\mathfrak{q}_n\|=k$ for some $k\geq0$. Then, $\lim_{n\to\infty}\|\varkappa_n-\mathfrak{q}_n\|=0$.

2 Convergence of Modified S-iterative method

Recently, Agarwal et al. [3] have introduced the S-iteration process to find the fixed point of a single-valued nonexpansive mapping as follows:

Let $\mathcal{C}$ be a nonempty subset of a Banach space E, and let $t:\mathcal{C}\to\mathcal{C}$ be a mapping. For $\varkappa_0\in\mathcal{C}$, the sequence $\{\varkappa_n\}$ is defined by

$$\begin{cases}\mathfrak{q}_n=(1-\eta_n)\varkappa_n+\eta_n t\varkappa_n,\\ \varkappa_{n+1}=(1-\mu_n)t\varkappa_n+\mu_n t\mathfrak{q}_n,\end{cases}$$

where $\{\mu_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$.

In 2020, Abdeljawad et al. [1] defined the S-iterative process $\{\varkappa_n\}$ for multivalued mapping as follows: Let $\mathcal{C}$ be a nonempty convex subset of a Banach space E, and let $\mathcal{Q}:\mathcal{C}\to CB(\mathcal{C})$ be a multivalued mapping. The S-iteration process $\{\varkappa_n\}$ is defined by

$$\begin{cases}\mathfrak{q}_n=(1-\eta_n)\varkappa_n+\eta_n z_n,\\ \varkappa_{n+1}=(1-\mu_n)z_n+\mu_n z_n',\end{cases}$$

where $\varkappa_0\in\mathcal{C}$, $z_n\in\mathcal{Q}\varkappa_n$ and $z_n'\in\mathcal{Q}\mathfrak{q}_n$ and μ_n , $\eta_n\in[0,1]$.

Keeping above in mind, we introduce the modified S-iteration method for three multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings in the following way: Let $\mathcal{C}$ be a nonempty closed bounded convex subset of a Banach space E, and $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be three multi-valued nonexpansive mappings. The modified S-iteration method $\{\varkappa_n\}$ is defined by

$$\begin{cases}\mathfrak{q}_n=(1-\eta_n)\varkappa_n+\eta_n z_n,\\ \varkappa_{n+1}=(1-\mu_n)u_n+\mu_n v_n,\end{cases}\tag{2.1}$$

where $\varkappa_0\in\mathcal{C}$, $u_n\in\mathcal{Q}_1\varkappa_n, v_n\in\mathcal{Q}_2\mathfrak{q}_n, z_n\in\mathcal{Q}_3\varkappa_n$, and $0<s\leq\mu_n$, $\eta_n\leq t<1$.

Lemma 2.1. Let $\mathcal{C}$ be a nonempty compact convex subset of a uniformly convex Banach space E, and let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping. Also suppose that $\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)\neq\emptyset$ and $\mathcal{Q}_1\varpi=\mathcal{Q}_2\varpi=\mathcal{Q}_3\varpi=\{\varpi\}$ for all $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$. Let $\{\varkappa_n\}$ be the sequence generated by the S-iteration method defined by (2.1). Then $\lim_{n\to\infty}\|\varkappa_n-\varpi\|$ exists for all $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$.

Proof . As $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping and $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$, by Proposition 1.5, we have

$$\begin{aligned}\mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi)&\leq\|\varkappa_n-\varpi\|,\\ \mathcal{H}(\mathcal{Q}_2\varkappa_n,\mathcal{Q}_2\varpi)&\leq\|\varkappa_n-\varpi\|\end{aligned}\tag{2.2}$$

and

$$\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi)\leq\|\varkappa_n-\varpi\|.$$

Now,

$$\begin{aligned}
\|\varkappa_{n+1}-\varpi\| &= \|(1-\mu_n)u_n+\mu_n v_n-\varpi\| \qquad (2.3)\\
&\le (1-\mu_n)\|u_n-\varpi\|+\mu_n\|v_n-\varpi\|\\
&= (1-\mu_n)\mathfrak{d}(u_n,\mathcal{Q}_1\varpi)+\mu_n\mathfrak{d}(v_n,\mathcal{Q}_2\varpi)\\
&\le (1-\mu_n)\mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi)+\mu_n\mathcal{H}(\mathcal{Q}_2\mathfrak{q}_n,\mathcal{Q}_2\varpi)\\
&\le (1-\mu_n)\|\varkappa_n-\varpi\|+\mu_n\|\mathfrak{q}_n-\varpi\|\\
&= (1-\mu_n)\|\varkappa_n-\varpi\|+\mu_n\|(1-\eta_n)\varkappa_n+\eta_n z_n-\varpi\|\\
&\le (1-\mu_n)\|\varkappa_n-\varpi\|+\mu_n(1-\eta_n)\|\varkappa_n-\varpi\|+\mu_n\eta_n\|z_n-\varpi\|\\
&= (1-\mu_n\eta_n)\|\varkappa_n-\varpi\|+\mu_n\eta_n\mathfrak{d}(z_n,\mathcal{Q}_3\varpi)\\
&\le (1-\mu_n\eta_n)\|\varkappa_n-\varpi\|+\mu_n\eta_n\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi)\\
&\le (1-\mu_n\eta_n)\|\varkappa_n-\varpi\|+\mu_n\eta_n\|\varkappa_n-\varpi\|\\
&= \|\varkappa_n-\varpi\|.
\end{aligned}$$

Since the sequence $\{\|\varkappa_n-\varpi\|\}$ is bounded and decreasing, we conclude that the limit of the sequence exists. □

Lemma 2.2. Let $\mathcal{C}$ be a nonempty compact convex subset of a uniformly convex Banach space E,and and let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping. Also suppose that $\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)\neq\emptyset$ and $\mathcal{Q}_1\varpi=\mathcal{Q}_2\varpi=\mathcal{Q}_3\varpi=\{\varpi\}$ for all $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$. Let $\{\varkappa_n\}$ be the sequence generated by the modified S-iteration defined by (2.1). If $0<s\le\mu_n,\eta_n\le t<1$ for some $s,t\in\mathbb{R}$, then,

$$\lim_{n\to\infty}\mathfrak{d}(\varkappa_n,\mathcal{Q}_1\varkappa_n)=\lim_{n\to\infty}\mathfrak{d}(\varkappa_n,\mathcal{Q}_2\varkappa_n)=\lim_{n\to\infty}\mathfrak{d}(\varkappa_n,\mathcal{Q}_3\varkappa_n)=0.$$

Proof . As $\mathcal{Q}_1,\mathcal{Q}_2$ and $\mathcal{Q}_3$ are multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping and $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$, by Proposition 1.5, we have

$$\begin{aligned}
\mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi) &\le \|\varkappa_n-\varpi\|,\\
\mathcal{H}(\mathcal{Q}_2\varkappa_n,\mathcal{Q}_2\varpi) &\le \|\varkappa_n-\varpi\|
\end{aligned}$$

and

$$\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi)\le\|\varkappa_n-\varpi\|.$$

By Lemma 2.1, it follows that $\lim_{n\to\infty}\|\varkappa_n-\varpi\|$ exists. Let us assume that $\lim_{n\to\infty}\|\varkappa_n-\varpi\|=c$.

Since

$$\begin{aligned}
\|u_n-\varpi\| &\le \mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi)\le\|\varkappa_n-\varpi\|,\\
\|v_n-\varpi\| &\le \mathcal{H}(\mathcal{Q}_2\mathfrak{q}_n,\mathcal{Q}_2\varpi)\le\|\mathfrak{q}_n-\varpi\|,
\end{aligned}$$

and

$$\|z_n-\varpi\|\le\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi)\le\|\varkappa_n-\varpi\|,$$

so,

$$\limsup_{n\to\infty}\|u_n-\varpi\|\le c \qquad (2.4)$$

and

$$\limsup_{n\to\infty}\|z_n-\varpi\|\le c.$$

From (2.2) in Lemma 2.1, we also have the following inequality

$$\begin{aligned}
\|\mathfrak{q}_n-\varpi\| &= \|(1-\eta_n)\varkappa_n+\eta_n z_n-\varpi\| \qquad (2.5)\\
&\le (1-\eta_n)\|\varkappa_n-\varpi\|+\eta_n\|z_n-\varpi\|\\
&\le (1-\eta_n)\|\varkappa_n-\varpi\|+\eta_n\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi)\\
&\le (1-\eta_n)\|\varkappa_n-\varpi\|+\eta_n\|\varkappa_n-\varpi\|\\
&= \|\varkappa_n-\varpi\|.
\end{aligned}$$

This implies that $\limsup_{n\to\infty} \|v_n - \varpi\| \leq c$. From (2.3), we know that

$$\lim_{n\to\infty} \|(1-\mu_n)u_n + \mu_n v_n - \varpi\| = c.$$

So, using Lemma 1.11, we obtain

$$\lim_{n\to\infty} \|u_n - v_n\| = 0.$$

Also,

$$\begin{aligned} \|\varkappa_{n+1} - \varpi\| &= \|(1-\mu_n)u_n + \mu_n v_n - \varpi\| \\ &= \|(u_n - \varpi) + \mu_n (u_n - v_n)\| \\ &\leq \|u_n - \varpi\| + \|u_n - v_n\| \end{aligned}$$

implies that

$$c \leq \liminf_{n\to\infty} \|u_n - \varpi\|. \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$\lim_{n\to\infty} \|u_n - \varpi\| = c.$$

Taking lim sup on both sides of (2.5), we obtain

$$\limsup_{n\to\infty} \|\mathfrak{q}_n - \varpi\| \leq c. \tag{2.7}$$

Thus

$$\begin{aligned} \|u_n - \varpi\| &\leq \|u_n - v_n\| + \|v_n - \varpi\| \\ &\leq \|u_n - v_n\| + \mathcal{H}(\mathcal{Q}_2\mathfrak{q}_n, \mathcal{Q}_2\varpi) \\ &\leq \|u_n - v_n\| + \|\mathfrak{q}_n - \varpi\|, \end{aligned}$$

gives

$$c \leq \liminf_{n\to\infty} \|\mathfrak{q}_n - \varpi\|,$$

and, in turn, by (2.7), we have

$$\lim_{n\to\infty} \|\mathfrak{q}_n - \varpi\| = c.$$

Applying Lemma 1.11 once again,

$$\lim_{n\to\infty} \|\varkappa_n - u_n\| = 0.$$

Since $\mathfrak{d}(\varkappa_n, \mathcal{Q}_1\varkappa_n) \leq \|\varkappa_n - u_n\|$, we have

$$\lim_{n\to\infty} \mathfrak{d}(\varkappa_n, \mathcal{Q}_1\varkappa_n) = 0.$$

In a similar way, we can show that

$$\lim_{n\to\infty} \mathfrak{d}(\varkappa_n, \mathcal{Q}_2\varkappa_n) = 0 \text{ and } \lim_{n\to\infty} \mathfrak{d}(\varkappa_n, \mathcal{Q}_3\varkappa_n) = 0.$$

□

Now, we prove that the sequence generated by the modified S-iteration method (2.1) converges strongly to a common fixed point of three multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings.

Theorem 2.3. Let $\mathcal{C}$ be a nonempty compact convex subset of a uniformly convex Banach space E,and and let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3 : \mathcal{C} \to CB(\mathcal{C})$ be multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping. Also suppose that $\mathbb{F}(\mathcal{Q}_1) \cap \mathbb{F}(\mathcal{Q}_2) \cap \mathbb{F}(\mathcal{Q}_3) \neq \emptyset$ and $\mathcal{Q}_1\varpi = \mathcal{Q}_2\varpi = \mathcal{Q}_3\varpi = \{\varpi\}$ for all $\varpi \in \mathbb{F}(\mathcal{Q}_1) \cap \mathbb{F}(\mathcal{Q}_2) \cap \mathbb{F}(\mathcal{Q}_3)$. Let $\{\varkappa_n\}$ be the sequence generated by the modified S-iteration defined by (2.1). If $0 < s \leq \mu_n, \eta_n \leq t < 1$ for some $s, t \in \mathbb{R}$ and for any convergent subsequence $\{\varkappa_{n_i}\}$of $\{\varkappa_n\}$ with $\varkappa_{n_i} \to \mathfrak{q}$, then $\mathfrak{q} \in \mathbb{F}(\mathcal{Q}_1) \cap \mathbb{F}(\mathcal{Q}_2) \cap \mathbb{F}(\mathcal{Q}_3)$.

Proof . Let $\lim_{n\to\infty}\|\varkappa_{n_i}-\mathfrak{q}\|=0$.Then by Lemmas 1.9 and 2.2, we have

$$\begin{aligned}
\mathfrak{d}(\mathfrak{q},\mathcal{Q}_1\mathfrak{q}) &\leq \|\mathfrak{q}-\varkappa_{n_i}\|+\mathfrak{d}(\varkappa_{n_i},\mathcal{Q}_1\mathfrak{q})\\
&\leq \|\mathfrak{q}-\varkappa_{n_i}\|+\frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa_{n_i},\mathcal{Q}_1\varkappa_{n_i})+\|\mathfrak{q}-\varkappa_{n_i}\|\\
&\leq 2\|\mathfrak{q}-\varkappa_{n_i}\|+\frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa_{n_i},\mathcal{Q}_1\varkappa_{n_i}).
\end{aligned}$$

Thus, it follows that $\mathfrak{q}\in\mathcal{Q}_1\mathfrak{q}$ i.e., $\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_1)$.

Similarly, we can show that $\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_2)$ and $\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_3)$. Hence, we get

$$\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3).$$

□

We now present the final theorem which proves the strong convergence of the modified S-iteration.

Theorem 2.4. Let $\mathcal{C}$ be a nonempty compact convex subset of a uniformly convex Banach space E, and let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be three multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping.Let us also suppose $\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)\neq\emptyset$ and $\mathcal{Q}_1\varpi=\mathcal{Q}_2\varpi=\mathcal{Q}_3\varpi=\{\varpi\}$ for all $\varpi\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$.Let $\varkappa_n$ be the sequence of the modified S-iteration defined by (2.1).If $0<s\leq\mu_n,\eta_n\leq t<1$ for some $s,t\in\mathbb{R}$.Then $\{\varkappa_n\}$ converges strongly to a common fixed point of $\mathcal{Q}_1,\mathcal{Q}_2$ and $\mathcal{Q}_3$.

Proof . Since $\mathcal{C}$ is compact, there exists a subsequence $\{\varkappa_{n_i}\}$ of $\{\varkappa_n\}$ which converges strongly to some point $\mathfrak{q}\in\mathcal{C}$, that is, $\lim_{n\to\infty}\|\varkappa_{n_i}-\mathfrak{q}\|=0$. By Theorem 2.3, we get $\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$, and hence by Lemma 2.1, we conclude that $\lim_{n\to\infty}\|\varkappa_n-\mathfrak{q}\|$ exists and it must be same as $\lim_{n\to\infty}\|\varkappa_{n_i}-\mathfrak{q}\|$ .Therefore,

$$\lim_{n\to\infty}\|\varkappa_n-\mathfrak{q}\|=0.$$

Thus, $\{\varkappa_n\}$ converges strongly to $\mathfrak{q}\in\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$. □

Example 2.5. Let $E=\mathbb{R}$ and $C=[0,\infty)$ and $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be three multi-valued mapping defined by $\mathcal{Q}_1=[0,\frac{2\varkappa}{5}]$, $\mathcal{Q}_2=[0,\frac{\varkappa}{3}]$, $\mathcal{Q}_3=[0,\frac{3\varkappa}{7}]$.

Then $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are generalized $(\alpha-\beta)$-nonexpansive mappings for $\alpha=\beta=\frac{1}{4}$ and have a common fixed point at $\varpi=0$.

Indeed, for $\alpha=\beta=\frac{1}{4}$, we get

$$\begin{aligned}
&\alpha\mathfrak{d}(\varkappa,\mathcal{Q}_1\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}_1\varkappa)+\beta\mathfrak{d}(\varkappa,\mathcal{Q}_1\varkappa)+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}_1\mathfrak{q})+(1-2\alpha-2\beta)\|\varkappa-\mathfrak{q}\|\\
=\;&\frac{1}{4}\left|\varkappa-\frac{2\mathfrak{q}}{5}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{2\varkappa}{5}\right|+\frac{1}{4}\left|\varkappa-\frac{2\varkappa}{5}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{2\mathfrak{q}}{5}\right|+(1-\frac{2}{4}-\frac{2}{4})\|\varkappa-\mathfrak{q}\|\\
=\;&\frac{1}{4}\left|\frac{5\varkappa-2\mathfrak{q}}{5}\right|+\frac{1}{4}\left|\frac{5\mathfrak{q}-2\varkappa}{5}\right|+\frac{1}{4}\left|\frac{3\varkappa}{5}\right|+\frac{1}{4}\left|\frac{3\mathfrak{q}}{5}\right|\\
\geq\;&\frac{1}{4}\left|\frac{7\varkappa-7\mathfrak{q}}{5}\right|+\frac{1}{4}\left|\frac{3\varkappa-3\mathfrak{q}}{5}\right|\\
=\;&\frac{1}{2}|\varkappa-\mathfrak{q}|\\
\geq\;&\frac{2}{5}|\varkappa-\mathfrak{q}|=\mathcal{H}(\mathcal{Q}_1\varkappa,\mathcal{Q}_1\mathfrak{q}).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \alpha\mathfrak{d}(\varkappa,\mathcal{Q}_2\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}_2\varkappa)+\beta\mathfrak{d}(\varkappa,\mathcal{Q}_2\varkappa)+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}_2\mathfrak{q})+(1-2\alpha-2\beta)\left\|\varkappa-\mathfrak{q}\right\| \\
= & \frac{1}{4}\left|\varkappa-\frac{\mathfrak{q}}{3}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{\varkappa}{3}\right|+\frac{1}{4}\left|\varkappa-\frac{\varkappa}{3}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{\mathfrak{q}}{3}\right|+(1-\frac{2}{4}-\frac{2}{4})\left\|\varkappa-\mathfrak{q}\right\| \\
= & \frac{1}{4}\left|\frac{3\varkappa-\mathfrak{q}}{3}\right|+\frac{1}{4}\left|\frac{3\mathfrak{q}-\varkappa}{3}\right|+\frac{1}{4}\left|\frac{2\varkappa}{3}\right|+\frac{1}{4}\left|\frac{2\mathfrak{q}}{3}\right| \\
\geq & \frac{1}{4}\left|\frac{4\varkappa-4\mathfrak{q}}{3}\right|+\frac{1}{4}\left|\frac{2\varkappa-2\mathfrak{q}}{3}\right| \\
= & \frac{1}{2}\left|\varkappa-\mathfrak{q}\right| \\
\geq & \frac{1}{3}\left|\varkappa-\mathfrak{q}\right|=\mathcal{H}(\mathcal{Q}_2\varkappa,\mathcal{Q}_2\mathfrak{q})
\end{aligned}$$

and

$$\begin{aligned}
& \alpha\mathfrak{d}(\varkappa,\mathcal{Q}_3\mathfrak{q})+\alpha\mathfrak{d}(\mathfrak{q},\mathcal{Q}_3\varkappa)+\beta\mathfrak{d}(\varkappa,\mathcal{Q}_3\varkappa)+\beta\mathfrak{d}(\mathfrak{q},\mathcal{Q}_3\mathfrak{q})+(1-2\alpha-2\beta)\left\|\varkappa-\mathfrak{q}\right\| \\
= & \frac{1}{4}\left|\varkappa-\frac{3\mathfrak{q}}{7}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{3\varkappa}{7}\right|+\frac{1}{4}\left|\varkappa-\frac{3\varkappa}{7}\right|+\frac{1}{4}\left|\mathfrak{q}-\frac{3\mathfrak{q}}{7}\right|+(1-\frac{2}{4}-\frac{2}{4})\left\|\varkappa-\mathfrak{q}\right\| \\
= & \frac{1}{4}\left|\frac{7\varkappa-3\mathfrak{q}}{7}\right|+\frac{1}{4}\left|\frac{7\mathfrak{q}-3\varkappa}{7}\right|+\frac{1}{4}\left|\frac{4\varkappa}{7}\right|+\frac{1}{4}\left|\frac{4\mathfrak{q}}{7}\right| \\
\geq & \frac{1}{4}\left|\frac{10\varkappa-10\mathfrak{q}}{7}\right|+\frac{1}{4}\left|\frac{4\varkappa-4\mathfrak{q}}{7}\right| \\
= & \frac{1}{2}\left|\varkappa-\mathfrak{q}\right| \\
\geq & \frac{3}{7}\left|\varkappa-\mathfrak{q}\right|=\mathcal{H}(\mathcal{Q}_3\varkappa,\mathcal{Q}_3\mathfrak{q}).
\end{aligned}$$

Hence, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are generalized $(\alpha-\beta)$-nonexpansive mappings. Also, these three mappins have a common fixed point at $\{0\}$, since we have $\varpi=0$ for $\mathcal{Q}_1\varpi=\varpi$, $\mathcal{Q}_2\varpi=\varpi$ and $\mathcal{Q}_3\varpi=\varpi$.

Moreover, since conditions of the Theorem 2.4 are satisfied, sequence $\{\varkappa_n\}$ defined by (2.1) converges strongly to the common fixed point $\varpi=0$.

3 Convergence of a one-step iterative method

This section is devoted to approximate a common fixed point of three multivalued mapping satisfying generalized $\alpha-\beta$-nonexpansiveness.

In 2011, Eslamian and Abkar [7] introduced a one-step iterative method for finding the common fixed points of a finite family of nonexpansive multivalued mappings satisfying condition $(\mathcal{C})$.Here, we extend the above one-step iterative method for three multi-valued generalized $(\alpha-\beta)$-nonexpansive mappings.Let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to CB(\mathcal{C})$ be three multivalued mappings, then the iterative scheme is defined as follows:

$$\begin{cases} \varkappa_1\in\mathcal{C}, \\ \varkappa_{n+1}=a_n\varkappa_n+b_nu_n+c_nv_n+d_nz_n,\ n\in\mathbb{N}, \end{cases} \tag{3.1}$$

where $u_n\in\mathcal{Q}_1\varkappa_n$, $v_n\in\mathcal{Q}_2\varkappa_n$, $z_n\in\mathcal{Q}_3\varkappa_n$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences of real numbers in $[0,1)$ such that for every natural number n, $a_n+b_n+c_n+d_n=1$.

Now, we prove some weak and strong convergence results of the sequence generated by (3.1) in a uniformly convex Banach spaces. In the context, $\digamma=\mathbb{F}(\mathcal{Q}_1)\cap\mathbb{F}(\mathcal{Q}_2)\cap\mathbb{F}(\mathcal{Q}_3)$ is the set of all common fixed points of the mappings $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$.We begin with following result.

Lemma 3.1. Let E be a uniformly convex Banach space and $\mathcal{C}$ be a nonempty closed convex subset of E.Let $\mathcal{Q}_i:\mathcal{C}\to CB(\mathcal{C})$ for $i=1,2,3$ be multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping with $\digamma$ nonempty. Also, let us suppose that for $i=1,2,3$, $\mathcal{Q}_i\varpi=\{\varpi\}$ for all $\varpi\in\digamma$.Then for the sequence $\{\varkappa_n\}$ generated by (3.1), we have $\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\mathcal{Q}_i\varkappa_n\right)=0$ for $i=1,2,3$.

Proof . By Proposition 1.5 we conclude that $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are quasi-nonexpansive.Thus for any $\varpi \in F$ and $\varkappa \in \mathcal{C}$, we have for $i=1,2,3$.

$$\mathcal{H}(\mathcal{Q}_i\varkappa, \mathcal{Q}_i\varpi) \leq \|\varkappa-\varpi\|.$$

Now, by the definition of $\{\varkappa_n\}$ in (3.1), it follows that

$$\begin{aligned}
\|\varkappa_{n+1}-\varpi\| &= \|a_n\varkappa_n+b_nu_n+c_nv_n+d_nz_n-\varpi\| \\
&= \|a_n(\varkappa_n-\varpi)+b_n(u_n-\varpi)+c_n(v_n-\varpi)+d_n(z_n-\varpi)\| \\
&\leq a_n\|\varkappa_n-\varpi\|+b_n\|u_n-\varpi\|+c_n\|v_n-\varpi\|+d_n\|z_n-\varpi\| \\
&\leq a_n\|\varkappa_n-\varpi\|+b_n\mathfrak{d}(u_n,\mathcal{Q}_1\varpi)+c_n\mathfrak{d}(v_n,\mathcal{Q}_2\varpi)+d_n\mathfrak{d}(z_n,\mathcal{Q}_3\varpi) \\
&\leq a_n\|\varkappa_n-\varpi\|+b_n\mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi)+c_n\mathcal{H}(\mathcal{Q}_2\varkappa_n,\mathcal{Q}_2\varpi)+d_n\mathcal{H}(\mathcal{Q}_3\varkappa_n,\mathcal{Q}_3\varpi) \\
&\leq a_n\|\varkappa_n-\varpi\|+b_n\|\varkappa_n-\varpi\|+c_n\|\varkappa_n-\varpi\|+d_n\|\varkappa_n-\varpi\| \\
&= (a_n+b_n+c_n+d_n)\|\varkappa_n-\varpi\| \\
&= \|\varkappa_n-\varpi\|.
\end{aligned}$$

Thus for each $\varpi \in F$, $\lim_{n\to\infty}\|\varkappa_n-\varpi\|$ exists and let us suppose that $\lim_{n\to\infty}\|\varkappa_n-\varpi\|=c$ for some $c>0$.Then,

$$\|u_n-\varpi\| = \mathfrak{d}(u_n,\mathcal{Q}_1\varpi) \leq \mathcal{H}(\mathcal{Q}_1\varkappa_n,\mathcal{Q}_1\varpi) \leq \|\varkappa_n-\varpi\|.$$

Thus, $\lim_{n\to\infty}\sup\|u_n-\varpi\| \leq c$ and in a similar way, we can show that $\lim_{n\to\infty}\sup\|v_n-\varpi\| \leq c$ and

$$\lim_{n\to\infty}\sup\|z_n-\varpi\| \leq c.$$

Now

$$\begin{aligned}
&\lim_{n\to\infty}\sup\left\|\frac{a_n}{1-d_n}(\varkappa_n-\varpi)+\frac{b_n}{1-d_n}(u_n-\varpi)+\frac{c_n}{1-d_n}(v_n-\varpi)\right\| \\
\leq\ &\lim_{n\to\infty}\sup\frac{a_n}{1-d_n}\|\varkappa_n-\varpi\|+\frac{b_n}{1-d_n}\|u_n-\varpi\|+\frac{c_n}{1-d_n}\|v_n-\varpi\| \\
\leq\ &\lim_{n\to\infty}\sup\left(\frac{a_n}{1-d_n}+\frac{b_n}{1-d_n}+\frac{c_n}{1-d_n}\right)\|\varkappa_n-\varpi\| \\
\leq\ &\lim_{n\to\infty}\sup\|\varkappa_n-\varpi\| = c.
\end{aligned}$$

Also,

$$\begin{aligned}
&\lim_{n\to\infty}\left\|(1-d_n)\left[\frac{a_n}{1-d_n}(\varkappa_n-\varpi)+\frac{b_n}{1-d_n}(u_n-\varpi)+\frac{c_n}{1-d_n}(v_n-\varpi)\right]+d_n(z_n-\varpi)\right\| \\
=\ &\lim_{n\to\infty}\|a_n(\varkappa_n-\varpi)+b_n(u_n-\varpi)+c_n(v_n-\varpi)+d_n(z_n-\varpi)\| \\
=\ &\lim_{n\to\infty}\|a_n\varkappa_n+b_nu_n+c_nv_n+d_nz_n-\varpi\| \\
=\ &\lim_{n\to\infty}\|\varkappa_{n+1}-\varpi\| = c.
\end{aligned}$$

Thus, using the above relations and Lemma 1.11, we get

$$\lim_{n\to\infty}\left\|\frac{a_n}{1-d_n}(\varkappa_n-\varpi)+\frac{b_n}{1-d_n}(u_n-\varpi)+\frac{c_n}{1-d_n}(v_n-\varpi)-(z_n-\varpi)\right\| = 0.$$

Therefore,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left\|\frac{a_n}{1-d_n}\varkappa_n+\frac{b_n}{1-d_n}u_n+\frac{c_n}{1-d_n}v_n-\frac{a_n+b_n+c_n}{1-d_n}\varpi-(z_n-\varpi)\right\| \\
&= \lim_{n\to\infty}\left\|\frac{a_n\varkappa_n+b_nu_n+c_nv_n}{1-d_n}-z_n+(1-\frac{a_n+b_n+c_n}{1-d_n})\varpi\right\| \\
&= \lim_{n\to\infty}\left\|\frac{a_n\varkappa_n+b_nu_n+c_nv_n+d_nz_n-z_n}{1-d_n}\right\| \\
&= \lim_{n\to\infty}\left\|\frac{\varkappa_{n+1}-z_n}{1-d_n}\right\| \\
& \lim_{n\to\infty}\frac{1}{1-d_n}\left\|\varkappa_{n+1}-z_n\right\|.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|\varkappa_{n+1}-z_n\|=0.$$

Similarly, we can prove that $\lim_{n\to\infty}\|\varkappa_{n+1}-v_n\|=0$, $\lim_{n\to\infty}\|\varkappa_{n+1}-u_n\|=0$ and $\lim_{n\to\infty}\|\varkappa_{n+1}-\varkappa_n\|=0$.

Hence, from the above deduction, we have

$$\lim_{n\to\infty}\|\varkappa_n-z_n\|\le\lim_{n\to\infty}\|\varkappa_{n+1}-\varkappa_n\|+\lim_{n\to\infty}\|\varkappa_{n+1}-z_n\|=0,$$

$$\lim_{n\to\infty}\|\varkappa_n-v_n\|\le\lim_{n\to\infty}\|\varkappa_{n+1}-\varkappa_n\|+\lim_{n\to\infty}\|\varkappa_{n+1}-v_n\|=0,$$

and

$$\lim_{n\to\infty}\|\varkappa_n-u_n\|\le\lim_{n\to\infty}\|\varkappa_{n+1}-\varkappa_n\|+\lim_{n\to\infty}\|\varkappa_{n+1}-u_n\|=0.$$

Now,

$$\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\mathcal{Q}_1\varkappa_n\right)\le\lim_{n\to\infty}\|\varkappa_n-u_n\|=0,$$
$$\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\mathcal{Q}_2\varkappa_n\right)\le\lim_{n\to\infty}\|\varkappa_n-v_n\|=0,$$

and

$$\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\mathcal{Q}_3\varkappa_n\right)\le\lim_{n\to\infty}\|\varkappa_n-z_n\|=0.$$

Hence, for $i=1,2,3$, we conclude that $\lim_{n\to\infty}\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n)=0$. □

Theorem 3.2. Let $\mathcal{C}$ be a closed convex subset of a real Banach space E, and let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3:\mathcal{C}\to B(\mathcal{C})$ be multi-valued generalized $(\alpha-\beta)$-nonexpansive mapping.Also suppose that for $i=1,2,3$, $\mathcal{Q}_i\varpi=\{\varpi\}\ \forall\varpi\in\digamma$.Let $\{\varkappa_n\}$ be the sequence generated by (3.1).Then the sequence $\{\varkappa_n\}$ converges strongly to a common fixed point of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ if and only if $\lim_{n\to\infty}\inf\mathfrak{d}\left(\varkappa_n,\digamma\right)=0$.

Proof . Let $\{\varkappa_n\}$ converges to a common fixed point $\varkappa$ of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ i.e. $\varkappa\in\digamma$.Thus it obviously follows that $\lim_{n\to\infty}\inf dis\mathcal{Q}\left(\varkappa_n,\digamma\right)=0$.

Conversely let us suppose that $\lim_{n\to\infty}\inf\mathfrak{d}\left(\varkappa_n,\digamma\right)=0$, then from Lemma 3.1, for each $\varpi\in\digamma$ we have $\|\varkappa_{n+1}-\varpi\|\le\|\varkappa_n-\varpi\|$ which implies

$$\mathfrak{d}\left(\varkappa_{n+1},\digamma\right)\le\mathfrak{d}\left(\varkappa_n,\digamma\right)$$

Hence $\{\mathfrak{d}\left(\varkappa_n,\digamma\right)\}$ is a decreasing sequence of real numbers which is bounded below and we also have $\lim_{n\to\infty}\inf dis\mathcal{Q}\left(\varkappa_n,\digamma\right)=$ 0, together we conclude $\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\digamma\right)=0$. We claim that, $\{\varkappa_n\}$ is a Cauchy sequence in $\mathcal{C}$. Let $\epsilon>0$ be arbitrarily chosen. Since $\lim_{n\to\infty}\mathfrak{d}\left(\varkappa_n,\digamma\right)=0$, there exists $p\in\mathbb{N}$ such that for all $n\ge p$, we have

$$\mathfrak{d}\left(\varkappa_n,\digamma\right)<\frac{\epsilon}{2}$$

In particular, $\inf\{\|\varkappa_p-\varpi\|:\varpi\in F\}<\frac{\epsilon}{2}$, so there exist some $\bar{\varpi}\in F$ such that,

$$\left\|\varkappa_p-\bar{\varpi}\right\|<\frac{\epsilon}{2}$$

Now for $m,n\geq p$, we have

$$\|\varkappa_{n+m}-\varkappa_n\|\leq\left\|\varkappa_{n+m}-\bar{\varpi}\right\|+\left\|\varkappa_n-\bar{\varpi}\right\|<2\left\|\varkappa_p-\bar{\varpi}\right\|<2\frac{\epsilon}{2}=\epsilon$$

Hence $\{\varkappa_n\}$ is a Cauchy sequence. Now $\mathcal{C}$ being a closed subset of E and $\{\varkappa_n\}$ is a Cauchy sequnce in $\mathcal{C}$, it must converge in $\mathcal{C}$. Let $\lim_{n\to\infty}\varkappa_n=z$. Now for $i=1,2,3$ we have,

$$\mathfrak{d}(z,\mathcal{Q}_iz)\leq\|z-\varkappa_n\|+\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n)+\mathcal{H}(\mathcal{Q}_i\varkappa_n,\mathcal{Q}_iz)$$

applying Proposition 1.8 we obtain,

$$\begin{aligned}\mathfrak{d}(z,\mathcal{Q}_iz) &\leq \|z-\varkappa_n\|+\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n)+2\frac{(1+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n)+\|\varkappa_n-z\|\\ &\leq 2\|z-\varkappa_n\|+\frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n).\end{aligned}$$

Since $\lim_{n\to\infty}\varkappa_n=z$ and $\lim_{n\to\infty}\mathfrak{d}(\varkappa_n,\mathcal{Q}_i\varkappa_n)=0$ for $i\in\{1,2,3\}$ (by Lemma 3.1) we conclude $\mathfrak{d}(z,\ \mathcal{Q}_iz)=0$. Hence $z\in\mathcal{Q}_iz$ for each $i\in\{1,2,3\}$, therefore $z\in F$. This completes the proof. □

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