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The results of the superposition operator on sequence space bv_p

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Abstract

In this paper, the conditions for the superposition operators were provided to map the space bv_p into bv_q, where $1 \leq p,\, q < \infty$. Additionally, we presented the necessary and sufficient conditions under which superposition operators become bounded, continuous and uniformly continuous on the sequence space bv_p.

Keywords: Bounded, Continuity modulus, Locally Bounded, Sequence spaces, Superposition operator
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1 Introduction and preliminaries

Superposition operators on sequence spaces have not studied widely, while there are lots of studies have been focused on spaces of functions [2, 3, 5]-[8, 12]. Dedagic and Zabrejko [9] have investigated the continuity of superposition operators on the sequence spaces ℓ_p for $1 \leq p < \infty$. Płuciennik [13] characterized continuous superposition operators from ω_0 into ℓ_1, where ω_0 is the space of all sequences or all functions Cesaro strongly summable to zero. In some other sequence spaces, the continuity of superposition operators, including Orlicz sequence spaces, was studied in [14, 15].

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all-natural numbers and the set of all real numbers, respectively. Let ω be the vector space of all real sequences $x = (x_s) = (x_s)_{s\in\mathbb{N}}$. By the term sequence space, we shall mean any linear subspace of ω.

Let λ and μ be two sequence spaces and let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a real function with $f(s, 0) = 0$ for $s \in \mathbb{N}$. A superposition operator $F_f : \lambda \to \mu$ is defined by

$$F_f(x) = f(s, x(s)) = f(s, x_s), \quad x = x(s) = (x_s) \in \lambda. \tag{1}$$

Sequence spaces have various applications in several branches of functional analysis, in particular, the theory of functions, the theory of locally convex spaces, matrix transformations, as well as the theory of summability invariably depends upon the study of sequences and series. We recall here some of the familiar sequence spaces.

Let us recall some definitions and results. Let x_1 and x_2 be the functions of the sequence space ω, then x_1 and x_2 are called difference disjoint, if $\big(x_1(s) - x_1(s-1)\big)\big(x_2(s) - x_2(s-1)\big) = 0$, for each $s \in \mathbb{N}$.

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We shall denote by ℓ_p, for $1 \le p < \infty$, the space of functions $x : \mathbb{N} \to \mathbb{R}$ (real sequences), for which the following norm makes sense and is finite

$$\|x\|_p := \Big(\sum_{s=1}^{\infty} |x(s)|^p\Big)^{\frac{1}{p}}.$$

For $1 \le p < \infty$, we shall denote by bv_p the space of functions $x : \mathbb{N} \to \mathbb{R}$ (real sequences) of all functions (sequences) of p-bounded variation, for which

$$bv_p = \Big\{x = x(s) \in w : \sum_{s=1}^{\infty} |x(s) - x(s-1)|^p < \infty\Big\},$$

where $x(0) = 0$, bv_p is a Banach space with the following norm:

$$\|x\|_{bv_p} := \Big(\sum_{s=1}^{\infty} |x(s) - x(s-1)|^p\Big)^{\frac{1}{p}}.$$

It was proved that bv_p is linearly isomorphic to the space ℓ_p and the inclusion $bv_p \supset \ell_p$ strictly holds (see [4, 11]).

The operator P_D denotes the multiplication operator which is defined by characteristic function χ_D of the set $D \subset \mathbb{N}$, i.e.,

$$P_D x(s) = \chi_D(s) x(s), \quad s \in \mathbb{N}.$$

We denote by τ the set of all $x \in bv_p$ which satisfy

$$|x(s) - x(s-1)| \le 1, \quad s \in \mathbb{N}.$$

In many situations, the investigation of the basic properties of the superposition operator (1) does not involve any particular difficulties. But this is not always so. In fact, at the beginning of nonlinear analysis it was often tacitly assumed that "nice"properties of a function carry over to the corresponding superposition operator; this turned out to be false even in well-known classical function spaces. A typical example of this phenomenon is the behaviour of the superposition operator in Lebesgue spaces. For instance, the smoothness (and even the analyticity) of a function does not imply the smoothness of corresponding superposition operator, considered as an operator between two Lebesgue spaces [2]. These facts are rather surprising; they show that many of the important properties of a function do not imply analogous properties of the corresponding superposition operator, or vice versa.

Classical mathematical analysis mainly dealt with spaces of continuous or differentiable functions already Lebesgue spaces arose only in special fields, e.g. Fourier series, approximation theory, probability theory. In modern nonlinear analysis, however, the arsenal of available function spaces has been considerably enlarged. In this connection, one should mention Sobolev spaces and their generalizations which are simply indispensable for the study of partial differential equations [1, 12], Orlicz spaces which are the natural tool in the theory of both linear and nonlinear integral equations [14, 16], Holder spaces and their generalizations which are basic for the investigation of singular integral equations [5, 10], and special classes of spaces of differentiable or smooth functions which frequently occur in the theory of ordinary or partial differential equations and variational calculus [2]. The usefulness of all these spaces in various fields of mathematical analysis emphasizes the need for a systematic study of the superposition operator (1), considered as an operator from one such space into another.

In this study, for every $1 \le p, q < \infty$, we present necessary and sufficient conditions under which superposition operator maps the space bv_p into bv_q. In addition, we provide the necessary and sufficient conditions under which superposition operators become bounded, continuous and uniformly continuous on the bounded variation sequence spaces bv_p for $1 \le p < \infty$.

The present paper was organized with the following sections. In section 2 we provided necessary and sufficient conditions under which superposition operator to map the space bv_p into bv_q, where $1 \le p, q < \infty$. In section 3 we presented the necessary and sufficient conditions under which superposition operators become bounded, continuous and uniformly continuous on the sequence space bv_p.

2 Superposition operators on the sequence spaces bv_p

In this section, we present necessary and sufficient conditions under which superposition operator maps the space bv_p into bv_q, where $1 \le p, q < \infty$.

Theorem 2.1. Let $1 \leq p, q < \infty$, and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a real function. Then the following conditions are equivalent:

1) the superposition operator F_f, generated by function f, maps bv_p into bv_q,

2) for $u \in bv_p$ there exists a function $a \in bv_q$ and constants $\delta > 0, n \in \mathbb{N}$ and $b \geq 0$ such that

$$|f(s, u(s)) - f(s-1, u(s-1))| \leq (a(s) - a(s-1)) + b|u(s) - u(s-1)|^{\frac{p}{q}}, \quad (s \geq n, |u(s) - u(s-1)| < \delta),$$

3) for $u \in bv_p$ and each $\varepsilon > 0$ there exist a function $a_\varepsilon \in bv_q$ and constants $\delta_\varepsilon > 0, n_\varepsilon \in \mathbb{N}$ and $b_\varepsilon \geq 0$ such that $\|a_\varepsilon\|_{bv_q} < \varepsilon$ and

$$|f(s, u(s)) - f(s-1, u(s-1))| \leq (a_\varepsilon(s) - a_\varepsilon(s-1)) + b_\varepsilon|u(s) - u(s-1)|^{\frac{p}{q}}, \tag{3}$$

holds, where $s \geq n_\varepsilon$ and $|u(s) - u(s-1)| < \delta_\varepsilon$.

We need the following technical lemma which is used to prove Theorem 2.1.

Lemma 2.2. Let $x \in bv_p$. Then $x \in \tau$ if and only if function x can be represented in the form

$$x = x_1 + \ldots + x_m \tag{2}$$

where $x_1, \ldots, x_m$ are pairwise difference disjoint functions from the unit sphere in bv_p and $m \leq 2\|x\|^p_{bv_p} + 1$.

Proof . At first, we suppose that for pairwise difference disjoint functions x_i, where $i = 1, \ldots, m$, in the unit sphere of the space bv_p, and for each $1 \leq i, j \leq m$ that $i \neq j$ and $s \in \mathbb{N}$, we have

$$x_i \neq x_j \quad and \quad \big(x_i(s) - x_i(s-1)\big)\big(x_j(s) - x_j(s-1)\big) = 0,$$

and the function x be written in the form

$$x = x_1 + \ldots + x_m.$$

We assert that $x \in \tau$. Since x_i are pairwise difference disjoint functions and $\|x_i\|_{bv_p} \leq 1$, for each $1 \leq i \leq m$, $s \in \mathbb{N}$ and $1 \leq p < \infty$, then only for one j,

$$\begin{aligned}|x(s) - x(s-1)|^p &= |(x_1(s) - x_1(s-1))) + \ldots + (x_i(s) - x_i(s-1)) + \ldots + (x_m(s) - x_m(s-1)))|^p \\ &= |x_j(s) - x_j(s-1)|^p \\ &\leq \|x_j\|^p_{bv_p} \leq 1,\end{aligned}$$

therefore $|x(s) - x(s-1)| \leq 1$.

Conversely, let $x \in \tau$. We can produce a finite partition $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ of the set $\mathbb{N}$ such that the functions $P_{\Omega_j} : bv_p \longrightarrow bv_p$ be defined by $P_{\Omega_j}x(s) = \chi_{\Omega_j}(s)x(s)$, where $P_{\Omega_j}x$, for $1 \leq j \leq m$, are pairwise difference disjoint functions. Therefore, we have

$$\begin{aligned}\|P_{\Omega_j}x\|^p_{bv_p} &= \sum_{s=1}^{\infty} |P_{\Omega_j}x(s) - P_{\Omega_j}x(s-1)|^p \\ &= \sum_{s=1}^{\infty} |\chi_{\Omega_j}(s)x(s) - \chi_{\Omega_j}(s-1)x(s-1)|^p \leq 1,\end{aligned}$$

where $1 \leq j \leq m$ and so $P_{\Omega_j}x \in bv_p$. Now for each $1 \leq j \leq m$, put $x_j := P_{\Omega_j}x$. From the sum of x_j, (2) is satisfied. Moreover, without loss of generality we claim that for each $1 \leq j \leq m$ except for one of j we have $2||x_j||^p_{bv_p} > 1$, otherwise, for example, if for $j = 1, 2$ we have

$$2||x_1||^p_{bv_p} \leq 1 \qquad \text{and} \qquad 2||x_2||^p_{bv_p} \leq 1,$$

then

$$||x_1 + x_2||^p_{bv_p} \leq ||x_1||^p_{bv_p} + ||x_2||^p_{bv_p} \leq 1,$$

so we can replace x_1 and x_2 with $x_1 + x_2$ in (2), which is disjoint from the other elements and this is in contradiction with the representation of x in the form m of the distinct element. So if in (2) we had for all the terms except one the inequality $2||x_j||^p_{bv_p} > 1$, then

$$\|x\|^p_{bv_p} = \sum_{j=1}^{m} \|x_j\|^p_{bv_p} \geq \frac{1}{2} \sum_{j=1}^{m-1} 1 = (m-1)2^{-1},$$

therefore we have

$$m \leq 2\|x\|^p_{bv_p} + 1.$$

□

We now present the proof of Theorem 2.1.

Proof . [Proof of Theorem 2.1] Proof of state 3) $\longrightarrow$ 2) is obvious. We present the proofs of states 2) $\longrightarrow$ 1) and 1) $\longrightarrow$ 3).

$\big(2) \longrightarrow 1)\big)$: Assume that condition 2) holds. We show that the operator $F_f : bv_p \longrightarrow bv_q$ is well defined. For this purpose, for $u \in bv_p$ and $s \in \mathbb{N}$, we have

$$\begin{aligned} ||F_f(u)||^q_{bv_q} &= \sum_{s=1}^{\infty} |F_f(u)(s) - F_f(u)(s-1)|^q \\ &\leq \sum_{s=1}^{n-1} |f(s,u(s)) - f(s-1,u(s-1))|^q + \sum_{s=n}^{\infty} |f(s,u(s)) - f(s-1,u(s-1))|^q \\ &\leq \sum_{s=1}^{n-1} |f(s,u(s)) - f(s-1,u(s-1))|^q + \sum_{s=n}^{\infty} |a(s) - a(s-1)|^q + b^q \sum_{s=n}^{\infty} |u(s) - u(s-1)|^p \\ &\leq \sum_{s=1}^{n-1} |f(s,u(s)) - f(s-1,u(s-1))|^q + \|a\|^q_{bv_q} + b^q \|u\|^p_{bv_p} < \infty, \end{aligned}$$

therefore $F_f(u) \in bv_q$, and the proof 2) $\longrightarrow$ 1) is completed.

Now, it suffices to prove that 1) $\longrightarrow$ 3). for this reason, assume that $x \in bv_p$ and condition 1) holds. Let us first prove that for each $\varepsilon > 0$ there exist $\delta_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$, such that $\|x\|_{bv_p} \leq \delta_\varepsilon$ then $\|F_f(P_{n_\varepsilon} x)\|_{bv_q} \leq \varepsilon$, where $P_n := P_{\{n+1,n+2,\ldots\}}$.

Indeed, if we assume that the contrary of the above relation is established, that is, for some $\varepsilon > 0$ and any $n \in \mathbb{N}$, we can find $x_n \in bv_p$ such that $\|x_n\|_{bv_p} < 2^{-n}$ and $\|F_f(P_n x_n)\|_{bv_q} > \varepsilon$. Since for $m > n$ we have

$$\begin{aligned} ||F_f((P_n - P_m)x_n)||_{bv_q} &= \Big(\sum_{s=1}^{\infty} |F_f((P_n - P_m)x_n)(s) - F_f((P_n - P_m)x_n)(s-1)|^q\Big)^{\frac{1}{q}} \\ &= \Big(\sum_{s=n+1}^{m} |F_f(x_n)(s) - F_f(x_n)(s-1)|^q\Big)^{\frac{1}{q}}, \end{aligned}$$

therefore

$$\begin{aligned} \lim_{m\to\infty} ||F_f((P_n - P_m)x_n)||_{bv_q} &= \lim_{m\to\infty} \Big(\sum_{s=n+1}^{m} |F_f(x_n)(s) - F_f(x_n)(s-1)|^q\Big)^{\frac{1}{q}} \\ &= ||F_f(P_n x_n)||_{bv_q}. \end{aligned}$$

Then for every $n \in \mathbb{N}$ there exists $n^{'} > n$ such that $||F_f((P_n - P_{n'})x_n)||_{bv_q} > \varepsilon$. By induction, we produce the sequence n_k of natural numbers such that $n_1 = 1$ and $n_{k+1} = (n_k)^{'}$ for $k = 1, \ldots, m$. Then we have

$$||(P_{n_k} - P_{n_{k+1}})x_{n_k}||_{bv_p} \leq 2^{-k},$$

and

$$||F_f((P_{n_k} - P_{n_{k+1}})x_{n_k})||_{bv_q} > \varepsilon.$$

Put $\tilde{x} := \sum_{k=1}^{\infty}(P_{n_k} - P_{n_{k+1}})x_{n_k}$. Hence, we have

$$\begin{aligned}
||\tilde{x}||_{bv_p} &= \Big(\sum_{s=1}^{\infty} |\tilde{x}(s) - \tilde{x}(s-1)|^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{s=1}^{\infty} \Big| \sum_{k=1}^{\infty}(P_{n_k} - P_{n_{k+1}})x_{n_k}(s) - \sum_{k=1}^{\infty}(P_{n_k} - P_{n_{k+1}})x_{n_k}(s-1)\Big|^p\Big)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty}\Big(\sum_{s=n_k+1}^{n_{k+1}} |x_{n_k}(s) - x_{n_k}(s-1)|^p\Big)^{\frac{1}{p}} \\
&= \sum_{k=1}^{\infty} ||(P_{n_k} - P_{n_{k+1}})x_{n_k}||_{bv_p} \\
&\leq \sum_{k=1}^{\infty} 2^{-k} < \infty,
\end{aligned}$$

so $\tilde{x} \in bv_p$. On the other hand,

$$\begin{aligned}
\|F_f(\tilde{x})\|_{bv_q} &= \Big(\sum_{s=1}^{\infty} |F_f(\tilde{x})(s) - F_f(\tilde{x})(s-1)|^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\infty} |f(s, \tilde{x}(s)) - f(s-1, \tilde{x}(s-1))|^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\infty} |f(s, \sum_{k=1}^{\infty}(P_{n_k} - P_{n_{k+1}})x_{n_k}(s)) - f(s-1, \sum_{k=1}^{\infty}(P_{n_k} - P_{n_{k+1}})x_{n_k}(s-1))|^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{k=1}^{\infty}\sum_{s=n_k+1}^{n_{k+1}} |f(s, x_{n_k}(s)) - f(s-1, x_{n_k}(s-1))|^q\Big)^{\frac{1}{q}} \\
&\geq \Big(\sum_{k=1}^{\infty} \varepsilon^q\Big)^{\frac{1}{q}} = \infty,
\end{aligned}$$

and thus $F_f(\tilde{x}) \notin bv_q$, which contradicts the assumption. Now, for each $\varepsilon > 0$ and $u \in bv_p$, we consider

$$|f_\varepsilon(s, u(s)) - f_\varepsilon(s-1, u(s-1))| = \max\{0, |f(s, u(s)) - f(s-1, u(s-1))| - 2^{\frac{1}{q}}\delta_\varepsilon^{-\frac{p}{q}}\varepsilon|u(s) - u(s-1)|^{\frac{p}{q}}\}. \tag{4}$$

Furthermore, for every function $x \in bv_p$, which is satisfied in condition $|P_{n_\varepsilon}x(s) - P_{n_\varepsilon}x(s-1)| < \delta_\varepsilon$, we put

$$D(x) := \{s > n_\varepsilon; |f(s, x(s)) - f(s-1, x(s-1))| > 2^{\frac{1}{q}}\delta_\varepsilon^{-\frac{p}{q}}\varepsilon|x(s) - x(s-1)|^{\frac{p}{q}}\}, \tag{5}$$

and

$$y := P_{D(x)}x.$$

Hence, for every $s \in \mathbb{N}$ we have

$$|y(s) - y(s-1)| = |P_{D(x)}x(s) - P_{D(x)}x(s-1)| \leq \delta_\varepsilon.$$

According to Lemma 2.2, the function y can be represented in the form of pairwise difference disjoint terms $y_1, ..., y_m$, such that

$$m \leq 2\delta_\varepsilon^{-p}\|y\|_{bv_p}^p + 1 \quad \textit{and} \quad \|y_j\|_{bv_p} < \delta_\varepsilon,$$

where $j = 1, \dots, m$. Then

$$\begin{aligned}\sum_{s=n_\varepsilon+1}^{\infty} |f_\varepsilon(s, x(s)) - f_\varepsilon(s-1, x(s-1))|^q &= \sum_{s=n_\varepsilon+1}^{\infty} |f_\varepsilon(s, y(s)) - f_\varepsilon(s-1, y(s-1))|^q \\ &= \sum_{j=1}^{m} \sum_{s=n_\varepsilon+1}^{\infty} |f_\varepsilon(s, y_j(s)) - f_\varepsilon(s-1, y_j(s-1))|^q \\ &\leq \sum_{j=1}^{m} \Big(\sum_{s=n_\varepsilon+1}^{\infty} |f(s, y_j(s)) - f(s-1, y_j(s-1))|^q \\ &- \sum_{s=n_\varepsilon+1}^{\infty} 2\delta_\varepsilon^{-p} \varepsilon^q |y_j(s) - y_j(s-1)|^q \Big) \\ &\leq m\varepsilon^q - 2\delta_\varepsilon^{-p}\varepsilon^q \|y\|_{bv_p}^p \\ &\leq 2\delta_\varepsilon^{-p}\varepsilon^q \|y\|_{bv_p}^p + \varepsilon^q - 2\delta_\varepsilon^{-p}\varepsilon^q \|y\|_{bv_p}^p = \varepsilon^q.\end{aligned}$$

So it follows from $|x(s) - x(s-1)| \leq \delta_\varepsilon$ that

$$\sum_{s=n_\varepsilon+1}^{\infty} |f_\varepsilon(s, x(s)) - f_\varepsilon(s-1, x(s-1))|^q \leq \varepsilon^q.$$

Now, we put

$$(a_\varepsilon(s) - a_\varepsilon(s-1)) = \begin{cases} 0 & s \leq n_\varepsilon \\ \sup_{|u(s)-u(s-1)| \leq \delta_\varepsilon} |f_\varepsilon(s, u(s)) - f_\varepsilon(s-1, u(s-1))| & s > n_\varepsilon. \end{cases}$$

Then we obtain

$$\begin{aligned}||a_\varepsilon||_{bv_q}^q &= \sum_{s=n_\varepsilon+1}^{\infty} |a_\varepsilon(s) - a_\varepsilon(s-1)|^q \\ &= \sup_{|u(s)-u(s-1)| \leq \delta_\varepsilon} \Big(\sum_{s=n_\varepsilon+1}^{\infty} |f_\varepsilon(s, u(s)) - f_\varepsilon(s-1, u(s-1))|^q \Big) \leq \varepsilon^q.\end{aligned}$$

But from (4) and (5) it follows that

$$\begin{aligned}(a_\varepsilon(s) - a_\varepsilon(s-1)) &\geq |f_\varepsilon(s, u(s)) - f_\varepsilon(s-1, u(s-1))| \\ &\geq |f(s, u(s)) - f(s-1, u(s-1))| - 2^{\frac{1}{q}} \delta_\varepsilon^{-\frac{p}{q}} \varepsilon |u(s) - u(s-1)|^{\frac{p}{q}}, \quad (s > n_\varepsilon,\ |u(s) - u(s-1)| \leq \delta_\varepsilon),\end{aligned}$$

then

$$|f(s, u(s)) - f(s-1, u(s-1))| \leq (a_\varepsilon(s) - a_\varepsilon(s-1)) + 2^{\frac{1}{q}} \delta_\varepsilon^{-\frac{p}{q}} \varepsilon |u(s) - u(s-1)|^{\frac{p}{q}}.$$

If we put $b_\varepsilon = 2^{\frac{1}{q}} \delta_\varepsilon^{-\frac{p}{q}} \varepsilon$, the proof is complete. □

3 Continuity and boundedness of the superposition operators

In this section, we provide the necessary and sufficient conditions under which superposition operators become bounded, continuous and uniformly continuous on the sequence spaces bv_p, where $1 \leq p < \infty$.

By Theorem 2.1, the operator $F_f : bv_p \longrightarrow bv_q$ is neither locally bounded nor continuous function. Therefore, in the following theorem, we provided conditions for continuity of the superposition operators on the sequence spaces bv_p.

Theorem 3.1. Let $1 \leq p, q < \infty$ and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a real function. Then superposition operator $F_f : bv_p \longrightarrow bv_q$, generated by function f, is continuous if and only if for each $s \in \mathbb{N}$ all functions $f(s, \cdot)$ are continuous.

Proof . Let for each $s \in \mathbb{N}$, the functions $f(s,\cdot)$ be continuous and $x_0 \in bv_p$. Furthermore, let $\varepsilon > 0$ be an arbitrary number and δ_ε and n_ε are numbers corresponding to a_ε ($||a_\varepsilon||_{bv_q} \leq \varepsilon$) and $b_\varepsilon \geq 0$, then we have inequality (3). Put $\gamma = \dfrac{\delta_\varepsilon}{2}$. We show that the operator superposition F_f is continuous on the sphere with center x_0 and radius γ. Indeed, let there exists a natural number $\tilde{n}$ such that $\tilde{n} \geq n_\varepsilon$ and $||P_{\tilde{n}} x_0||_{bv_p} \leq \gamma, (b_\varepsilon^{-1}\varepsilon)^{\frac{q}{p}}$. Then for $||x - x_0||_{bv_p} \leq \gamma$ and $s > \tilde{n}$ we have $|x(s) - x(s-1)| \leq \delta_\varepsilon$, and hence according to (3),

$$|f(s, x_0(s)) - f(s-1, x_0(s-1))| \leq (a_\varepsilon(s) - a_\varepsilon(s-1)) + b_\varepsilon |x_0(s) - x_0(s-1)|^{\frac{p}{q}},$$

and

$$|f(s, x(s)) - f(s-1, x(s-1))| \leq (a_\varepsilon(s) - a_\varepsilon(s-1)) + b_\varepsilon |x(s) - x(s-1)|^{\frac{p}{q}}.$$

From the above, it follows that

$$\begin{aligned}
\|F_f(x) - F_f(x_0)\|_{bv_q} &= \Big(\sum_{s=1}^{\infty} |\big(F_f(x) - F_f(x_0)\big)(s) - \big(F_f(x) - F_f(x_0)\big)(s-1)|^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\infty} |\big(f(s,x(s)) - f(s,x_0(s))\big) - \big(f(s-1,x(s-1)) - f(s-1,x_0(s-1))\big)|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=1}^{\tilde{n}} |f(s,x(s)) - f(s,x_0(s))|^q\Big)^{\frac{1}{q}} + \Big(\sum_{s=1}^{\tilde{n}} |f(s-1,x(s-1)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \\
&+ \Big(\sum_{s=\tilde{n}+1}^{\infty} |f(s,x(s)) - f(s-1,x(s-1))|^q\Big)^{\frac{1}{q}} + \Big(\sum_{s=\tilde{n}+1}^{\infty} |f(s,x_0(s)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=1}^{\tilde{n}} |f(s,x(s)) - f(s,x_0(s))|^q\Big)^{\frac{1}{q}} + \Big(\sum_{s=1}^{\tilde{n}} |f(s-1,x(s-1)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \\
&+ \Big(\sum_{s=\tilde{n}+1}^{\infty} |a_\varepsilon(s) - a_\varepsilon(s-1)|^q\Big)^{\frac{1}{q}} + b_\varepsilon\Big(\sum_{s=\tilde{n}+1}^{\infty} |x(s) - x(s-1)|^q\Big)^{\frac{1}{q}} \\
&+ \Big(\sum_{s=\tilde{n}+1}^{\infty} |a_\varepsilon(s) - a_\varepsilon(s-1)|^q\Big)^{\frac{1}{q}} + b_\varepsilon\Big(\sum_{s=\tilde{n}+1}^{\infty} |x_0(s) - x_0(s-1)|^q\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\tilde{n}} |f(s,x(s)) - f(s,x_0(s))|^q\Big)^{\frac{1}{q}} + \Big(\sum_{s=1}^{\tilde{n}} |f(s-1,x(s-1)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \\
&+ 2||a_\varepsilon||_{bv_q} + b_\varepsilon ||P_{\tilde{n}} x||_{bv_p}^{\frac{p}{q}} + b_\varepsilon ||P_{\tilde{n}} x_0||_{bv_p}^{\frac{p}{q}} \\
&\leq \Big(\sum_{s=1}^{\tilde{n}} |f(s,x(s)) - f(s,x_0(s))|^q\Big)^{\frac{1}{q}} + \Big(\sum_{s=1}^{\tilde{n}} |f(s-1,x(s-1)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \\
&+ 2\varepsilon + \varepsilon + b_\varepsilon\big((b_\varepsilon^{-1}\varepsilon)^{\frac{q}{p}} + ||x - x_0||_{bv_p}\big)^{\frac{p}{q}}.
\end{aligned}$$

Since the functions $f(s,\cdot)$, for each $s \in \mathbb{N}$, are continuous, we can consider $\mu \in (0, \gamma)$ such that for $||x - x_0||_{bv_p} \leq \mu$, we have

$$\Big(\sum_{s=1}^{\tilde{n}} |f(s,x(s)) - f(s,x_0(s))|^q\Big)^{\frac{1}{q}} \leq \varepsilon$$

and

$$\Big(\sum_{s=1}^{\tilde{n}} |f(s-1,x(s-1)) - f(s-1,x_0(s-1))|^q\Big)^{\frac{1}{q}} \leq \varepsilon,$$

in addition, without loss of generality we can assume that $b_\varepsilon\big((b_\varepsilon^{-1}\varepsilon)^{\frac{q}{p}} + ||x - x_0||_{bv_p}\big)^{\frac{p}{q}} \leq 2\varepsilon$. Therefore, for $||x - x_0||_{bv_p} \leq \mu$, we have

$$\|F_f(x) - F_f(x_0)\|_{bv_q} \leq 7\varepsilon,$$

and hence the operator superposition F_f is continuous at the point x_0.

Conversely, Suppose that the superposition operator $F_f : bv_p \longrightarrow bv_q$ is continuous. Let $i_s : \mathbb{R} \longrightarrow bv_p$ be the embedding defined for each $t \in \mathbb{R}$ by $i_s(t) = t\chi_{\{s\}} \in bv_p$ and the surjective function $\pi_s : bv_q \longrightarrow \mathbb{R}$ for every $v \in bv_q$ defined by $\pi_s(v) = v(s)$. Then for each $s \in \mathbb{N}$ the function $f(s, \cdot)$ factors as follows

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f(s,\cdot)} & \mathbb{R} \\ i_s \big\downarrow & & \big\uparrow \pi_s \\ bv_p & \xrightarrow[F_f]{} & bv_q. \end{array}$$

Since the functions i_s and π_s are continuous, the continuity of the operator $F_f : bv_p \longrightarrow bv_q$ implies the continuity of the function $f(s, \cdot)$. The theorem is proved. $\square$

By Theorems 3.1, we can easily obtain locally boundedness for superposition operators.

Corollary 3.2. Let $1 \leq p, q < \infty$ and $f : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ be a real function. Then superposition operator $F_f : bv_p \longrightarrow bv_q$, generated by function f, is locally bounded if and only if for each $s \in \mathbb{N}$ all functions $f(s, \cdot)$ are bounded.

Example 3.3. Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ can be defined by $f(s, x(s)) = (x(s))^s$ and superposition operator $F_f : bv_p \longrightarrow bv_q$ generated by the function f. Then clearly the superposition operator F_f is continuous. However, it is not bounded on any sphere with radius greater than 1.

In the following theorem, we give necessary and sufficient conditions for the boundedness of a superposition operator.

Theorem 3.4. Let $1 \leq p, q < \infty$ and $f : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ be a real function. Then superposition operator $F_f : bv_p \longrightarrow bv_q$, generated by function f, is bounded if and only if for $u \in bv_p$ and each $r > 0$ there exists a function $a_r \in bv_q$ and $b_r \geq 0$ such that

$$|f(s, u(s)) - f(s-1, u(s-1))| \leq \big(a_r(s) - a_r(s-1)\big) + b_r|u(s) - u(s-1)|^{\frac{p}{q}}, \tag{6}$$

where $|u(s) - u(s-1)| \leq r$. Furthermore,

$$\phi_f(r) \leq \psi_f(r) \leq \|F_f(0)\|_{bv_q} + (2^{\frac{1}{q}} + 1)\phi_f(r),$$

where $\phi_f(r) = \sup_{||u||_{bv_p} \leq r} \|F_f(u)\|_{bv_q}$, and for $|u(s) - u(s-1)| \leq r$,

$$\begin{aligned}\psi_f(r) = \inf\{&||a||_{bv_q} + br^{\frac{p}{q}} : |f(s, u(s)) - f(s-1, u(s-1))| \\ &\leq (a(s) - a(s-1)) + b|u(s) - u(s-1)|^{\frac{p}{q}}\}.\end{aligned}$$

Proof . Suppose that for $r > 0$, (6) is satisfied and for $u \in bv_p$ we have $||u||_{bv_p} \leq r$, then

$$\begin{aligned}\|F_f(u)\|_{bv_q} &= \Big(\sum_{s=1}^{\infty} |F_f(u)(s) - F_f(u)(s-1)|^q\Big)^{\frac{1}{q}} \\ &= \Big(\sum_{s=1}^{\infty} |f(s, u(s)) - f(s-1, u(s-1))|^q\Big)^{\frac{1}{q}} \\ &\leq \Big(\sum_{s=1}^{\infty} |a_r(s) - a_r(s-1)|^q\Big)^{\frac{1}{q}} + b_r\Big(\sum_{s=1}^{\infty} |u(s) - u(s-1)|^q\Big)^{\frac{1}{q}} \\ &= ||a_r||_{bv_q} + b_r||u||^{\frac{p}{q}}_{bv_p} \\ &\leq ||a_r||_{bv_q} + b_r r^{\frac{p}{q}},\end{aligned}$$

and therefore $\|F_f(u)\|_{bv_q} \leq \psi_f(r)$, and hence $\phi_f(r) \leq \psi_f(r)$. Now suppose that the superposition operator F_f is bounded. Define the function

$$\begin{aligned}|f_r(s, u(s)) - f_r(s-1, u(s-1))| = \max\{0, &|f(s, u(s)) - f(s-1, u(s-1))| - |f(s, 0) - f(s-1, 0)| \\ &- 2^{\frac{1}{q}} r^{-\frac{p}{q}} \phi_f(r)|u(s) - u(s-1)|^{\frac{p}{q}}\}.\end{aligned} \tag{7}$$

Moreover, for every function $x \in bv_p$, in order to satisfy the condition $|x(s) - x(s-1)| \leq r$, put

$$D(x) := \{s \in \mathbb{N}; |f(s,x(s)) - f(s-1,x(s-1))| > |f(s,0) - f(s-1,0)| + 2^{\frac{1}{q}} r^{-\frac{p}{q}} \phi_f(r) |x(s) - x(s-1)|^{\frac{p}{q}}\},$$

and

$$y := P_{D(x)} x.$$

According to Lemma 2.2, the function y can be represented in the form of pairwise difference disjoint functions $y_1, ..., y_m$, such that $m \leq 2r^{-p} ||y||^p_{bv_p} + 1$ and $||y_j||_{bv_p} \leq r$, where $j = 1, \ldots, m$. Then

$$\begin{aligned}
\sum_{s=1}^{\infty} |f_r(s,x(s)) - f_r(s-1,x(s-1))|^q &= \sum_{s=1}^{\infty} |f_r(s,y(s)) - f_r(s-1,y(s-1))|^q \\
&= \sum_{s=1}^{\infty} |f_r(s, \sum_{j=1}^{m} y_j(s)) - f_r(s-1, \sum_{j=1}^{m} y_j(s-1))|^q \\
&\leq \sum_{j=1}^{m} \Big(\sum_{s=1}^{\infty} |f(s,y_j(s)) - f(s-1,y_j(s-1))|^q \\
&\quad - \sum_{s=1}^{\infty} |f(s,0) - f(s-1,0))|^q - 2r^{-p} \phi_f^q(r) \sum_{s=1}^{\infty} |y_j(s) - y_j(s-1)|^p \Big) \\
&\leq (2r^{-p} ||y||^p_{bv_p} + 1) \phi_f^q(r) - 2r^{-p} \phi_f^q(r) ||y||^p_{bv_p}.
\end{aligned}$$

Therefore, it follows from $x \in bv_p$ and $|x(s) - x(s-1)| \leq r$ that

$$\sum_{s=1}^{\infty} |f_r(s,x(s)) - f_r(s-1,x(s-1))|^q \leq \phi_f^q(r).$$

Hence, as in the proof of Theorem 2.1, for $|u(s) - u(s-1)| \leq r$, we have

$$|f_r(s,u(s)) - f_r(s-1,u(s-1))|^q \leq a_r(s) - a_r(s-1), \tag{8}$$

where $||a_r||_{bv_q} \leq \phi_f(r)$. Then it follows from (7) and (8) that

$$|f(s,u(s)) - f(s-1,u(s-1))| \leq |f(s,0) - f(s-1,0))| + \big(a_r(s) - a_r(s-1)\big) + 2^{\frac{1}{q}} r^{-\frac{p}{q}} \phi_f(r) |u(s) - u(s-1)|^{\frac{p}{q}},$$

if we put $b_r := 2^{\frac{1}{q}} r^{-\frac{p}{q}} \phi_f(r)$. Hence

$$|f(s,u(s)) - f(s-1,u(s-1))| \leq a_r(s) - a_r(s-1) + b_r |u(s) - u(s-1)|^{\frac{p}{q}}.$$

Moreover, for each $r > 0$, we have

$$\begin{aligned}
\psi_f(r) \leq ||a_r||_{bv_q} + b_r r^{\frac{p}{q}} &\leq \|F_f(0)\|_{bv_q} + ||a_r||_{bv_q} + b_r r^{\frac{p}{q}} \\
&= \|F_f(0)\|_{bv_q} + ||a_r||_{bv_q} + 2^{\frac{1}{q}} r^{-\frac{p}{q}} \phi_f(r) r^{\frac{p}{q}} \\
&\leq \|F_f(0)\|_{bv_q} + (2^{\frac{1}{q}} + 1) \phi_f(r).
\end{aligned}$$

The proof is complete. □

In the following example, it is shown that a continuous superposition operator is not necessarily uniformly continuous.

Example 3.5. Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ can be defined by $f(s,x(s)) = x(s) sin \pi s x(s)$, where x is a real function of the space bv_1 on $\mathbb{N}$. Then the operator $F_f : bv_1 \longrightarrow bv_1$, defined by $F_f(x)(s) = f(s,x(s))$, is a continuous superposition operator. But it is not uniformly continuous, since for the sequences $u_n := \frac{2n+1}{2n}\chi_{\{n\}}$ and $v_n := \frac{2n-1}{2n}\chi_{\{n\}}$ we have

$$\begin{aligned}||u_n||_{bv_1} &= \sum_{s=1}^{\infty}\Big|\frac{2n+1}{2n}\chi_{\{n\}}(s) - \frac{2n+1}{2n}\chi_{\{n\}}(s-1)\Big| \\ &= \frac{2n+1}{2n}\Big(\sum_{s=1}^{\infty}\big|\chi_{\{n\}}(s) - \chi_{\{n\}}(s-1)\big|\Big) \\ &= \frac{2n+1}{2n}||\chi_{\{n\}}||_{bv_1} \leq \frac{3}{2},\end{aligned}$$

and

$$\begin{aligned}||v_n||_{bv_1} &= \sum_{s=1}^{\infty}\Big|\frac{2n-1}{2n}\chi_{\{n\}}(s) - \frac{2n-1}{2n}\chi_{\{n\}}(s-1)\Big| \\ &= \frac{2n-1}{2n}\Big(\sum_{s=1}^{\infty}\big|\chi_{\{n\}}(s) - \chi_{\{n\}}(s-1)\big|\Big) \\ &= \frac{2n-1}{2n}||\chi_{\{n\}}||_{bv_1} < 1,\end{aligned}$$

then $u_n, v_n \in bv_1$, and thus

$$\begin{aligned}||u_n - v_n||_{bv_1} &= \sum_{s=1}^{\infty}\big|(u_n - v_n)(s) - (u_n - v_n)(s-1)\big| \\ &= \sum_{s=1}^{\infty}\Big|\Big(\frac{2n+1}{2n}\chi_{\{n\}}(s) - \frac{2n-1}{2n}\chi_{\{n\}}(s)\Big) - \Big(\frac{2n+1}{2n}\chi_{\{n\}}(s-1) - \frac{2n-1}{2n}\chi_{\{n\}}(s-1)\Big)\Big| \\ &= \frac{1}{n}\Big(\sum_{s=1}^{\infty}\big|\chi_{\{n\}}(s) - \chi_{\{n\}}(s-1)\big|\Big) = \frac{1}{n}.\end{aligned}$$

But we have

$$\begin{aligned}\|F_f(u_n) - F_f(v_n)\|_{bv_1} &= \sum_{s=1}^{\infty}\big|(F_f(u_n) - F_f(v_n))(s) - (F_f(u_n) - F_f(v_n))(s-1)\big| \\ &= \sum_{s=1}^{\infty}\big|(f(s,u_n(s)) - f(s,v_n(s))) - (f(s-1,u_n(s-1)) - f(s-1,v_n(s-1)))\big| \\ &= \sum_{s=1}^{\infty}\big|\big(u_n(s)sin\pi s u_n(s) - v_n(s)sin\pi s v_n(s)\big) \\ &- \big(u_n(s-1)sin\pi(s-1)u_n(s-1) - v_n(s-1)sin\pi(s-1)v_n(s-1)\big)\big| \\ &= \sum_{s=1}^{\infty}\Big|\Big(\frac{2n+1}{2n}\chi_{\{n\}}(s)sin\pi s\frac{2n+1}{2n}\chi_{\{n\}}(s) - \frac{2n-1}{2n}\chi_{\{n\}}(s)sin\pi s\frac{2n-1}{2n}\chi_{\{n\}}(s)\Big) \\ &- \Big(\frac{2n+1}{2n}\chi_{\{n\}}(s-1)sin\pi(s-1)\frac{2n+1}{2n}\chi_{\{n\}}(s-1) \\ &- \frac{2n-1}{2n}\chi_{\{n\}}(s-1)sin\pi(s-1)\frac{2n-1}{2n}\chi_{\{n\}}(s-1)\Big)\Big| \\ &= 2\Big(\sum_{s=1}^{\infty}\big|\chi_{\{n\}}(s) - \chi_{\{n\}}(s-1)\big|\Big) = 2.\end{aligned}$$

Let the operator $F_f : bv_p \longrightarrow bv_q$, generated by function f, is the superposition operator on bv_p for $1 \leq p, q < \infty$. Then for $r, \delta \geq 0$, $\omega_f(r,\delta)$ is the continuity modulus of the operator F_f and is defined by

$$\omega_f(r,\delta) = \sup_{||u||_{bv_p},||v||_{bv_p}\leq r,||u-v||_{bv_p\leq\delta}} \|F_f(u) - F_f(v)\|_{bv_q},$$

where $u, v \in bv_p$, also for the function $a \in bv_q$ and constants $b, c, d \geq 0$, $\nu_f(r,\delta)$ is the function defined by

$$\nu_f(r,\delta) = \inf \big\{ ||a||_{bv_q} + (b+c)r^{\frac{p}{q}} + d\delta^{\frac{p}{q}} : \big|\big(f(s,u(s)) - f(s-1,u(s-1))\big) - \big(f(s,v(s)) - f(s-1,v(s-1))\big)\big| \\ \leq \big(a(s) - a(s-1)\big) + b\big|u(s) - u(s-1)\big|^{\frac{p}{q}} + c\big|v(s) - v(s-1)\big|^{\frac{p}{q}} + d\big|(u-v)(s) - (u-v)(s-1)\big|^{\frac{p}{q}} \big\},$$

where $|u(s) - u(s-1)|, |v(s) - v(s-1)| \leq r$ and $\big|(u-v)(s) - (u-v)(s-1)\big| \leq \delta$.
In the following, we give necessary and sufficient conditions for the uniformly continuity of a superposition operator and we compare the functions $\omega_f(r,\delta)$ with $\nu_f(r,\delta)$.

Theorem 3.6. Let $1 \leq p, q < \infty$ and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a real function. Then superposition operator $F_f : bv_p \longrightarrow bv_q$, generated by function f, is uniformly continuous if and only if for $u, v \in bv_p$ and each $r, \delta \geq 0$ and $\varepsilon > 0$ there exists a function $a_{r,\delta} \in bv_q$ such that

$$\big|\big(f(s,u(s)) - f(s-1,u(s-1))\big) - \big(f(s,v(s)) - f(s-1,v(s-1))\big)\big|$$

$$\leq \big(a_{r,\delta}(s) - a_{r,\delta}(s-1)\big) + b_{r,\delta}\big|u(s) - u(s-1)\big|^{\frac{p}{q}} + c_{r,\delta}\big|v(s) - v(s-1)\big|^{\frac{p}{q}} + d_{r,\delta}\big|\big(u-v)(s) - (u-v)(s-1)\big|^{\frac{p}{q}}. \qquad (9)$$

where $||a_{r,\delta}||_{bv_q} + (b_{r,\delta} + c_{r,\delta})r^{\frac{p}{q}} \leq \varepsilon$, $d_{r,\delta} \geq 0$ and $|u(s) - u(s-1)|, |v(s) - v(s-1)| \leq r$ and $\big|(u-v)(s) - (u-v)(s-1)\big| \leq \delta$. Furthermore,

$$\omega_f(r,\delta) \leq \nu_f(r,\delta) \leq (2^{\frac{1+q}{q}} + 1)\ \omega_f(r,\delta).$$

Proof . Let for each $r, \delta \geq 0$, (9) is satisfied. Then for $u, v \in bv_p$ such that $||u||_{bv_p}, ||v||_{bv_p} \leq r$, and $||u - v||_{bv_p} \leq \delta$, we have

$$\begin{aligned}
\|F_f(u) - F_f(v)\|_{bv_q} &= \Big(\sum_{s=1}^{\infty} \big|\big(F_f(u) - F_f(v)\big)(s) - \big(F_f(u) - F_f(v)\big)(s-1)|^q \Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\infty} \big|\big(f(s,u(s)) - f(s,v(s))\big) - \big(f(s-1,u(s-1)) - f(s-1,v(s-1))\big)\big|^q \Big)^{\frac{1}{q}} \\
&= \Big(\sum_{s=1}^{\infty} \big|\big(f(s,u(s)) - f(s-1,u(s-1))\big) - \big(f(s,v(s)) - f(s-1,v(s-1))\big)\big|^q \Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=1}^{\infty} |a_{r,\delta}(s) - a_{r,\delta}(s-1)|^q \Big)^{\frac{1}{q}} + b_{r,\delta}\Big(\sum_{s=1}^{\infty} |u(s) - u(s-1)|^q \Big)^{\frac{1}{q}} \\
&\quad + c_{r,\delta}\Big(\sum_{s=1}^{\infty} |v(s) - v(s-1)|^q \Big)^{\frac{1}{q}} + d_{r,\delta}\Big(\sum_{s=1}^{\infty} |(u-v)(s) - (u-v)(s-1)|^q \Big)^{\frac{1}{q}} \\
&\leq ||a_{r,\delta}||_{bv_q} + (b_{r,\delta} + c_{r,\delta})r^{\frac{p}{q}} + d_{r,\delta}\delta^{\frac{p}{q}}.
\end{aligned}$$

Hence, according to the above inequality, we have

$$\|F_f(u) - F_f(v)\|_{bv_q} \leq \nu_f(r,\delta).$$

Therefore the operator superposition F_f is uniformly continuous and $\omega_f(r,\delta) \leq \nu_f(r,\delta)$.

Conversely, suppose that the operator superposition F_f, for $u \in bv_p$, is uniformly continuous on $||u||_{bv_p} \leq r$. Consider the function

$$g_{r,\delta}(s,u(s)) - g_{r,\delta}(s-1,u(s-1)) = \sup_{|t(s)-t(s-1)| \leq \delta, -r-u \leq t \leq r-u} \max \big\{0, \big|\big(f(s,(u+t)(s)) - f(s-1,(u+t)(s-1))\big) \\ -\big(f(s,u(s)) - f(s-1,u(s-1))\big)\big| - 2^{\frac{1}{q}}\delta^{-\frac{p}{q}}\omega_f(r,\delta)|t(s) - t(s-1)|^{\frac{p}{q}} \big\}. \qquad (10)$$

By repeating the arguments given in the proof of Theorem 3.4, it can be easily shown that the superposition operator is bounded on $||u||_{bv_p} \leq r$ by $\omega_f(r,\delta)$. Once again, using arguments similar in the proof of Theorem 3.4, for $|u(s) - u(s-1)| \leq r$, we have

$$|g_{r,\delta}(s,u(s)) - g_{r,\delta}(s-1,u(s-1))| \leq (a_{r,\delta}(s) - a_{r,\delta}(s-1)) + 2^{\frac{1}{q}}r^{-\frac{p}{q}}\omega_f(r,\delta)|u(s) - u(s-1)|^{\frac{p}{q}}, \qquad (11)$$

where $||a_{r,\delta}||_{bv_q} \leq \omega_f(r,\delta)$. Hence, it follows from (10) and (11) that
$\big|\big(f(s,(u+t)(s)) - f(s-1,(u+t)(s-1))\big) - \big(f(s,u(s)) - f(s-1,u(s-1))\big)\big|$

$$\leq \big(a_{r,\delta}(s) - a_{r,\delta}(s-1)\big) + 2^{\frac{1}{q}} r^{-\frac{p}{q}} \omega_f(r,\delta)|u(s)-u(s-1)|^{\frac{p}{q}}$$
$$+ 0|(u+t)(s) - (u+t)(s-1)|^{\frac{p}{q}} + 2^{\frac{1}{q}} \delta^{-\frac{p}{q}} \omega_f(r,\delta)|t(s)-t(s-1)|^{\frac{p}{q}}.$$

If we put $v := u + t$ and $b_{r,\delta} = 2^{\frac{1}{q}} r^{-\frac{p}{q}} \omega_f(r,\delta)$, $c_{r,\delta} = 0$, $d_{r,\delta} = 2^{\frac{1}{q}} \delta^{-\frac{p}{q}} \omega_f(r,\delta)$. Then inequality (9) is established. Moreover, for each $r, \delta > 0$, we have

$$\begin{aligned} \nu_f(r,\delta) &\leq ||a_{r,\delta}||_{bv_q} + (b_{r,\delta} + c_{r,\delta}) r^{\frac{p}{q}} + d_{r,\delta} \delta^{\frac{p}{q}} \\ &\leq \omega_f(r,\delta) + 2^{\frac{1}{q}} r^{-\frac{p}{q}} r^{\frac{p}{q}} \omega_f(r,\delta) + 2^{\frac{1}{q}} \delta^{-\frac{p}{q}} \delta^{\frac{p}{q}} \omega_f(r,\delta) \\ &= \Big(1 + 2^{\frac{q+1}{q}}\Big) \omega_f(r,\delta). \end{aligned}$$

This ends the proof. □

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