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Probabilistic Airy's type equation with third order minimum norm

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Abstract

In the present paper, we establish some generalized cyclic contraction results through p-number of subsets by using two different types of t-norm, viz. Hadzic type t-norm and minimum t-norm in the setting of 2- probabilistic metric spaces. Our results generalize some existing fixed point theorems in 2-Menger spaces. Some illustrative examples and an application to the existence of a solution to Airy's type differential equation are also provided.

Keywords: 2-probabilistic metric space, Cauchy sequence, Convergence sequence, cyclic contraction, p-cyclic contraction
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1 Introduction

Numerous extensions have been made to the idea of metric spaces. Gahler [13, 14] created one such extension, called a 2-metric space, in which every third member of the space is given a positive real integer. In these spaces, fixed point theory has developed quickly and various metric fixed point theory conclusions have been provided in these spaces (see [12, 8]).

In 1942, Menger introduced the concept of Menger space which is the particular type of probabilistic metric space in which the triangle inequality is postulated with the help of t-norm. Sehgal and Bharucha-Reid [21] generalized the famous Banach contraction mapping principle to probabilistic metric space. Schweizer and Sklar [19] have described several aspects of such spaces in their book. Some recent fixed point results on probabilistic metric spaces may be noted in [2, 3, 4, 5, 7].

The probabilistic extension of 2-metric spaces are 2-probabilistic metric spaces. A special case of the 2-probabilistic metric spaces are 2-Menger spaces.

The concerns with cyclic contractions and proximity point problems have been closely related. Other findings relating to proximity point issues and cyclic contractions in probabilistic metric and 2-probabilistic metric spaces may be found in [4, 5, 6, 7, 18].

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Main features of this paper are following:
1. Some new probabilistic fixed point results using p-cyclic contraction mappings have been discussed using different types of t-norm.
2. Some illustrative examples validate our results.
3. An integral application is also illustrated here.

2 Preliminaries

Now, we give some important definitions and mathematical preliminaries which are used in the main results.

Definition 2.1. A mapping $\Gamma : R \to R^+$ is called a distribution function (see **[16, 19]**) if it is non-decreasing and left continuous with $\inf_{\eta\in R} \Gamma(\eta) = 0$ and $\sup_{\eta\in R} \Gamma(\eta) = 1$, where R is the set of real numbers and R^+ is the set of non-negative real numbers.

Example 2.2. The Heaviside function is an example of a distribution function given by

$$H(\eta) = \begin{cases} 1, \eta > 0 \\ 0, \eta \leq 0. \end{cases}$$

Definition 2.3. Probabilistic metric space (briefly, PM-space) (see [16, 19]) is an ordered pair (S, Γ), where S is a non empty set and Γ is a mapping from $S \times S$ into the set of all distribution functions. The function $\Gamma_{\kappa,\mu}$ is assumed to satisfy the following conditions for all $\kappa, \mu, \nu \in S$,

(i) $\Gamma_{\kappa,\mu}(0) = 0$,
(ii) $\Gamma_{\kappa,\mu}(\eta) = 1$ for all $\eta > 0$ if and only if $\kappa = \mu$,
(iii) $\Gamma_{\kappa,\mu}(\eta) = \Gamma_{\mu,\kappa}(\eta)$ for all $\eta > 0$,
(iv) if $\Gamma_{\kappa,\mu}(\eta_1) = 1$ and $\Gamma_{\mu,\nu}(\eta_2) = 1$ then $\Gamma_{\kappa,\nu}(\eta_1 + \eta_2) = 1$ for all $\eta_1, \eta_2 > 0$.

Example 2.4. Suppose that $S = [0, 1]$ and $\Gamma_{\kappa,\mu}(\eta) = \frac{\eta}{\eta+|\kappa-\mu|}$. Then it is easy to see that (S, Γ) is a PM space.

Shi et al. [20] introduced the following definition of n-th order t-norm.

Definition 2.5. A mapping $T : \Pi_{i=1}^{n}[0, 1] \to [0, 1]$ is called a n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, \ldots, 0) = 0$, $T(a, 1, 1, \ldots, 1) = a$ for all $a \in [0, 1]$,
(ii) $T(a_1, a_2, a_3, \ldots, a_n) = T(a_2, a_1, a_3, \ldots, a_n) = T(a_2, a_3, a_1, \ldots, a_n)$
$= \ldots = T(a_2, a_3, a_4, \ldots, a_n, a_1)$,
(iii) $a_i \geq b_i$, i=1,2,3,...,n implies $T(a_1, a_2, a_3, \ldots, a_n) \geq T(b_1, b_2, b_3, \ldots, b_n)$,
(iv) $T(T(a_1, a_2, a_3, \ldots, a_n), b_2, b_3, \ldots, b_n)$
$= T(a_1, T(a_2, a_3, \ldots, a_n, b_2), b_3, \ldots, b_n)$
$= T(a_1, a_2, T(a_3, a_4, \ldots, a_n, b_2, b_3), b_4, \ldots, b_n)$
$\vdots$
$= T(a_1, a_2, \ldots, a_{n-1}, T(a_n, b_2, b_3, \ldots, b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

In this paper we use different types of 3rd order t-norm.
The following are some examples of different types of t-norm.
(i) The minimum t-norm, $\Delta = T_m$, defined by $T_m(a, b, c) = \min\{a, b, c\}$.
(ii) The product t-norm, $\Delta = T_p$, defined by $T_p(a, b, c) = a.b.c$.
(iii) The Lukasiewicz t-norm, $\Delta = T_L$, defined by $T_L(a, b, c) = \max\{a + b + c - 1, 0\}$.

Hadzic and Pap [16] introduced a new t-norm which is commonly known as Hadzic type t-norm.

Definition 2.6. t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as

$$\Delta^0(s) = 1,\ \Delta^{p+1}(s) = \Delta(\Delta^p(s), s) \text{ for all integer } p \geq 0.$$

t-norm is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0,1)$ such that

$$1 \geq s > \eta(\lambda) \text{ implies } \Delta^p(s) \geq 1 - \lambda \text{ for all integer } p \geq 0.$$

Definition 2.7. Menger space (see [16, 19]) is a triplet (S, Γ, Δ), where S is a non empty set, Γ is a function defined on $S \times S$ to the set of all distribution functions and Δ is a t-norm such that the following are satisfied:

(i) $\Gamma_{\kappa,\mu}(0) = 0$ for all $\kappa, \mu \in S$,
(ii) $\Gamma_{\kappa,\mu}(s) = 1$ for all $s > 0$ if and only if $\kappa = \mu$,
(iii) $\Gamma_{\kappa,\mu}(s) = \Gamma_{\mu,\kappa}(s)$ for all $\kappa, \mu \in S,\ s > 0$,
(iv) $\Gamma_{\kappa,\mu}(u + v) \geq \Delta\left(\Gamma_{\kappa,\nu}(u), \Gamma_{\nu,\mu}(v)\right)$ for all $u, v \geq 0$ and $\kappa, \mu, \nu \in S$.

A metric space becomes a Menger space if we write $\Gamma_{\kappa,\mu}(\eta) = H(\eta - d(\kappa, \mu))$ where H is the Heavyside function which is discussed earlier.

Definition 2.8. Let S be a non empty set. A real valued function d on $S \times S \times S$ is said to be a 2-metric on S if
(i) given distinct elements $\kappa, \mu \in S$, there exists an element ν of S such that $d(\kappa, \mu, \nu) \neq 0$,
(ii) $d(\kappa, \mu, \nu) = 0$ when at least two of κ, μ, ν are equal,
(iii) $d(\kappa, \mu, \nu) = d(\kappa, \nu, \mu) = d(\mu, \nu, \kappa)$ for all $\kappa, \mu, \nu \in S$ and
(iv) $d(\kappa, \mu, \nu) \leq d(\kappa, \mu, w) + d(\kappa, w, \nu) + d(w, \mu, \nu)$ for all $\kappa, \mu, \nu, w \in S$.
When d is a 2-metric on S, the ordered pair (S, d) is called a 2-metric space (see [13, 14]).

It is noticed that 2-metric is not a continuous function of its variables, whereas an ordinary metric is continuous. This leds Dhage [11] to introduce the notion of a D-metric space. 2-metric space is not topologically equivalent to a metric. So, the fixed point theorems in 2-metric spaces and metric spaces may not have any relationship. Dung et al [12] worked in this direction and some important remarks were noted regarding 2-metric spaces:
1. 2-metric is non-negative,
2. every 2-metric space contains at least three distinct points.

Example 2.9. Let $S = \{1, 2, 3\}$ and $d(\kappa, \mu, \nu) = \min\{|\kappa - \mu|, |\mu - \nu|, |\nu - \kappa|\}$, for all $\kappa, \mu, \nu \in S$. Then (S, d) is a 2-metric space.

Example 2.10. If we consider three vertices κ, μ, ν of a triangle, then area of triangle may be taken as $d(\kappa, \mu, \nu)$. Then the metric function d satisfies all the conditions of 2-metric.

Probabilistic 2-metric space is a probabilistic generalization of 2-metric space. Wen-Zhi Zeng [22] introduced the concept of probabilistic 2-metric space.

Definition 2.11. Probabilistic 2-metric space is an order pair (S, Γ) where S is an arbitrary set and Γ is a mapping from $S \times S \times S$ into the set of all distribution functions such that the following conditions are satisfied, for all $\kappa, \mu, \nu, w \in S$ and $\eta_1, \eta_2, \eta_3 > 0$,

(i) $\Gamma_{\kappa,\mu,\nu}(\eta) = 0$ for $\eta \leq 0$,
(ii) $\Gamma_{\kappa,\mu,\nu}(\eta_1) = 1$ if and only if at least two of κ, μ, ν are equal,
(iii) for distinct points $\kappa, \mu \in S$ there exists a point $\nu \in S$ such that $\Gamma_{\kappa,\mu,\nu}(\eta_1) \neq 1$,
(iv) $\Gamma_{\kappa,\mu,\nu}(\eta_1) = \Gamma_{\kappa,\nu,\mu}(\eta_1) = \Gamma_{\nu,\mu,\kappa}(\eta_1)$,
(v) $\Gamma_{\kappa,\mu,w}(\eta_1) = 1$, $\Gamma_{\kappa,w,\nu}(\eta_2) = 1$ and $\Gamma_{w,\mu,\nu}(\eta_3) = 1$ then $\Gamma_{\kappa,\mu,\nu}(\eta_1 + \eta_2 + \eta_3) = 1$.

Many researchers found many interesting results in this space see [1].

Example 2.12. Define the distribution function on S by

$$\Gamma_{\kappa,\mu,\nu}(\eta) = \begin{cases} \frac{\eta}{\eta + \min\{|\kappa-\mu|, |\kappa-\nu|, |\mu-\nu|\}} & \text{if } \eta > 0 \\ 0, & \text{if } \eta \leq 0 \end{cases}$$

for all $(\kappa, \mu, \nu) \in S^3$. Then (S, Γ) is a probabilistic 2-metric space.

The following is a special case of the above definition.

Definition 2.13. Let S be a nonempty set. A triplet (S, Γ, Δ) is said to be a 2-Menger space [8] if Γ is a mapping from $S \times S \times S$ into the set of all distribution functions satisfying the following conditions:

(i) $\Gamma_{\kappa,\mu,\nu}(0) = 0$,
(ii) $\Gamma_{\kappa,\mu,\nu}(\eta) = 1$ for all $\eta > 0$ if and only if at least two of $\kappa, \mu, \nu \in S$ are equal,
(iii) for distinct points $\kappa, \mu \in S$ there exists a point $\nu \in S$ such that $\Gamma_{\kappa,\mu,\nu}(\eta) \neq 1$ for $\eta > 0$,
(iv) $\Gamma_{\kappa,\mu,\nu}(\eta) = \Gamma_{\kappa,\nu,\mu}(\eta) = \Gamma_{\nu,\mu,\kappa}(\eta)$, for all $\kappa, \mu, \nu \in S$ and $\eta > 0$,
(v) $\Gamma_{\kappa,\mu,\nu}(\eta) \geq \Delta(\Gamma_{\kappa,\mu,w}(\eta_1), \Gamma_{\kappa,w,\nu}(\eta_2), \Gamma_{w,\mu,\nu}(\eta_3))$

where $\eta_1, \eta_2, \eta_3 > 0$, $\eta_1 + \eta_2 + \eta_3 = \eta$, $\kappa, \mu, \nu, w \in S$ and Δ is the 3rd order t norm.

In Menger space, we use a function Γ which is defined on $S \times S$ to the set of all distribution functions but in case of 2-Menger space (see [8]) we use the function Γ which is defined on $S \times S \times S$ to the set of all distribution functions.

Definition 2.14. A sequence $\{\kappa_n\}$ in 2-Menger space (S, Γ, Δ) is said to be converge [15] to a limit κ if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that
$$\Gamma_{\kappa_n,\kappa,a}(\epsilon) \geq 1 - \lambda \tag{1.1}$$
for all $n > N_{\epsilon,\lambda}$ and for every $a \in S$.

Definition 2.15. A sequence $\{\kappa_n\}$ in 2-Menger space (S, Γ, Δ) is said to be a Cauchy sequence [15] in S if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that
$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq 1 - \lambda \tag{1.2}$$
for all $m, n > N_{\epsilon,\lambda}$ and for every $a \in S$.

The equivalent of Definitions 2.14 and 2.15 is to replace $\geq$ with $>$ in (1.1) and (1.2) respectively. They are not written in this conventional way. We have presently given them the evidence from our theorems for our convenience.

In our main theorem we have used a complete 2-Menger spaces. Completeness property of spaces have an important role in our results.

Definition 2.16. A 2-Menger space (S, Γ, Δ) is said to be complete [15] if every Cauchy sequence is convergent in S.

We use the following control function Φ in our second theorem which was presented by Choudhury et al. [9].

Definition 2.17. A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

Example 2.18. $\phi(\eta) = \eta^2$, $\phi(\eta) = \sqrt{\eta}$, $\phi(\eta) = \eta$ are some examples of Φ-function.

In 2003, Kirk, Srinivasan and Veeramani [17] introduced the concept of cyclic contraction and cyclic contractive type mappings in the context of metric spaces. After that many authors establish various many results on this concept. In our main results, we use the concept of p-cyclic mapping in the setting of probabilistic 2-metric spaces.

Definition 2.19. Let $\{A_i\}_{i=1}^p$ be non-empty subsets of S. A p-cyclic mapping (see [17]) is a mapping $h : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ which satisfies the following conditions :
$$hA_i \subseteq A_{i+1} \text{ for } 1 \leq i < p,\ hA_p \subseteq A_1. \tag{1.3}$$

In the case $p = 2$, p-cyclic mapping reduces to cyclic mappings. Some recent works on cyclic and p-cyclic contractions may be seen in [4, 5, 7, 10, 18].

3 Hadzic t-norm and fixed points

In this section, we have established one fixed point theorem using Hadzic type t-norm.

Theorem 3.1. Let (S,Γ,Δ) be a complete 2-Menger space, where Δ is a Hadzic type t-norm. Also let $\{A_i\}$ be non empty closed subsets of S, and the mapping $h:\bigcup_{i=1}^{p}A_i\to\bigcup_{i=1}^{p}A_i$ is a p-cyclic mapping satisfying the following conditions:

(i) $hA_i\subseteq A_{i+1}$ for $1\le i<p$, $hA_p\subseteq A_1$,
(ii) for all $\kappa\in A_i$, $\mu\in A_j, i\neq j$, a $\in S$, we have

$$\frac{1}{\Gamma_{h\kappa,h\mu,a}(\eta)}-1\le \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{\eta}{c})}-1,\frac{1}{{}_{\kappa,h\kappa,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\mu,h\mu,a}(\frac{\eta}{c})}-1),$$

where $\eta>0$, $0<c<1$.

Then h has a unique fixed point in $\bigcap_{i=1}^{p}A_i$.

Proof . Let κ_0 be any arbitrary point in A_1. Now we define the sequence $\{\kappa_n\}_{n=0}^{\infty}$ in S by $\kappa_n=h\kappa_{n-1}$, $n\in N$ where N is the set of natural numbers.

By (i), we have $\kappa_o\in A_1$, $\kappa_1\in A_2$, $\kappa_2\in A_3$,. . . ,$\kappa_{p-1}\in A_p$ and in general

$$\kappa_{np}\in A_1,\kappa_{np+1}\in A_2,\dots,\kappa_{np+(p-1)}\in A_p, \tag{3.1}$$

for all $n\ge 0$. Now, we have from (ii) for $\eta>0$ and $c\in(0,1)$, $a\in S$, $\kappa_0\in A_1$ and $\kappa_1\in A_2$, we have

$$\frac{1}{\Gamma_{h\kappa_0,h\kappa_1,a}(\eta)}-1\le \min(\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\kappa_0,h\kappa_0,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\kappa_1,h\kappa_1,a}(\frac{\eta}{c})}-1),$$

that is,

$$\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\eta)}-1\le \min(\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})}-1). \tag{3.2}$$

Let us consider $\min(\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})}-1)=\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})}-1$. Then $\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\eta)}-1\le\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})}-1$, that is, $\Gamma_{\kappa_1,\kappa_2,a}(\eta)\ge\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})$. But by the monotone property of Γ, for all $c\in(0,1)$, $\frac{\eta}{c}>\eta$, $\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})\ge\Gamma_{\kappa_1,\kappa_2,a}(\eta)$. Hence we get a contradiction. So $\min(\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1,\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})}-1)=\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1$. Therefore,

$$\frac{1}{\Gamma_{\kappa_1,\kappa_2,a}(\eta)}-1\le\frac{1}{\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})}-1,$$

that is, $\Gamma_{\kappa_1,\kappa_2,a}(\eta)\ge\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c})$. Proceeding in a similar way, for all $\kappa_2\in A_3,\kappa_3\in A_4$, we have

$$\Gamma_{\kappa_2,\kappa_3,a}(\eta)\ge\Gamma_{\kappa_1,\kappa_2,a}(\frac{\eta}{c})\ge\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c^2}).$$

Now, repeating this process n times, we have

$$\Gamma_{\kappa_n,\kappa_{n+1},a}(\eta)\ge\Gamma_{\kappa_0,\kappa_1,a}(\frac{\eta}{c^n}). \tag{3.3}$$

Now, taking the limit $n\to\infty$ on both sides of (3.3), for all $\eta>0$, we obtain

$$\lim_{n\to\infty}\Gamma_{\kappa_{n+1},\kappa_n,a}(\eta)=1, \tag{3.4}$$

for all $a\in S$. Again, by repeated applications of (3.3), it follows that for all $\eta>0$, $n\ge 0$ and each $i\ge 1$,

$$\Gamma_{\kappa_{n+i},\kappa_{n+i+1},a}(\eta)\ge\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta}{c^i}). \tag{3.5}$$

We next prove that $\{\kappa_n\}$ is a Cauchy sequence, that is, we prove that for arbitrary $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N(\epsilon, \lambda)$ such that

$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq 1 - \lambda$$

for all $m, n \geq N(\epsilon, \lambda)$. Without loss of generality we can assume that $m > n$. Now,

$$\epsilon = \epsilon\frac{1-c}{1-c} > \epsilon(1-c)(1+c+c^2+\ldots+c^{m-n-1}).$$

Then, by the monotone increasing property of Γ, we have

$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq \Gamma_{\kappa_n,\kappa_m,a}(\epsilon(1-c)(1+c+c^2+\ldots+c^{m-n-1})),$$

that is,

$$\begin{aligned}\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq &\Delta(\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)), \Delta(\Gamma_{\kappa_{n+1},\kappa_{n+2},a}(\epsilon c(1-c)), \Delta(\ldots, \Delta(\Gamma_{\kappa_{m-2},\kappa_{m-1},a}\\ &(\epsilon c^{m-n-2}(1-c)), \Gamma_{\kappa_{m-1},\kappa_m,a}(\epsilon c^{m-n-1}(1-c)))\ldots))).\end{aligned} \tag{3.6}$$

Putting $\eta = (1-c)\epsilon c^i$ in (3.5), we get

$$\Gamma_{\kappa_{n+i},\kappa_{n+i+1},a}((1-c)\epsilon c^i) \geq \Gamma_{\kappa_n,\kappa_{n+1},a}((1-c)\epsilon).$$

Therefore, by (3.6), we have

$$\begin{aligned}\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq &\Delta(\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)), \Delta(\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)), \Delta(\ldots,\\ &\Delta(\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)), \Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)))\ldots))).\end{aligned}$$

Hence we get

$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq \Delta^{(m-n)}\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)). \tag{3.7}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equi-continuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geq 1 - \lambda \text{ whenever } \eta(\lambda) < s \leq 1. \tag{3.8}$$

Since, $\Gamma_{\kappa_0,\kappa_1,a}(\eta) \to 1$ as $\eta \to \infty$ and $0 < c < 1$, there exists an positive integer $N(\epsilon, \lambda)$ such that

$$\Gamma_{\kappa_0,\kappa_1,a}(\frac{(1-c)\epsilon}{c^n}) > \eta(\lambda), \tag{3.9}$$

for all $n \geq N(\epsilon, \lambda)$. From (3.5) and (3.9), with $n = 0$, $i = n$ and $\eta = (1-c)\epsilon$, we get

$$\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c)) > \Gamma_{\kappa_0,\kappa_1,a}(\frac{(1-c)\epsilon}{c^n}) > \eta(\lambda),$$

for all $n \geq N(\epsilon, \lambda)$. Then, from (3.8) with $s = \Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c))$, we have

$$\Delta^{(m-n)}(\Gamma_{\kappa_n,\kappa_{n+1},a}(\epsilon(1-c))) \geq 1 - \lambda.$$

It then follows from (3.7), that

$$\Gamma_{\kappa_n,\kappa_m,a}(\epsilon) \geq 1 - \lambda,$$

for all $m, n \geq N(\epsilon, \lambda)$. Thus $\{\kappa_n\}$ is a Cauchy sequence. Since S is complete, we have

$$\lim_{n\to\infty} \kappa_n = \nu. \tag{3.10}$$

By the construction of the sequence $\{\kappa_n\}$, we have $\kappa_p \in A_1$, $\kappa_{2p} \in A_1, \ldots, \kappa_{np} \in A_1$. Therefore the subsequences $\{\kappa_{np}\}$ of $\{\kappa_n\}$ which belongs to A_1, also converges to ν in A_1, since A_1 is closed. Similarly subsequences κ_{np+1} belongs to A_2 also converges to ν in A_2. Since A_3, $A_4, \ldots, A_p$ are closed sets , similarly we get $\nu \in A_3, A_4, \ldots, A_p$. Therefore $\nu \in \bigcap_{i=1}^{p} A_i$.

Now, we have to prove that $h\nu = \nu$. Since (S, Γ, Δ) is a complete 2- Menger space, $\kappa_n \to u$ as $n \to \infty$, for some $u \in S$. Moreover, for all $a \in S$, we get

$$\Gamma_{hu,u,a}(\epsilon) \geq \Delta(\Gamma_{hu,u,\kappa_{n+1}}(\frac{\epsilon}{3}), \Gamma_{hu,\kappa_{n+1},a}(\frac{\epsilon}{3}), \Gamma_{\kappa_{n+1},u,a}(\frac{\epsilon}{3})). \tag{3.11}$$

Again $\kappa_n \to u$ as $n \to \infty$ and hence there exists $n_0 \in N$ such that, for all $n > n_0$ (sufficiently large), $0 < \eta_2 < \frac{\epsilon}{3}$, we have

$$\begin{aligned}\frac{1}{\Gamma_{\kappa_{n+1},hu,a}(\frac{\epsilon}{3})} - 1 \leq &\frac{1}{\Gamma_{h\kappa_n,hu,a}(\eta_2)} - 1\\ \leq &\min(\frac{1}{\Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c})} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})} - 1, \frac{1}{\Gamma_{u,hu,a}(\frac{\eta_2}{c})} - 1).\end{aligned}$$

Taking $\kappa_n \in A_{n+1}$ and $u \in A_j$ where $n+1 \neq j$, we have $\frac{1}{\Gamma_{h\kappa_n,hu,a}(\eta_2)} - 1 \leq \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})} - 1$. Using (3.4), for sufficiently large n, we get $\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c}) \to 1$ and $\frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})} - 1 = 0$. So, $\min(\frac{1}{\Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c})} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})} - 1, \frac{1}{\Gamma_{u,hu,a}(\frac{\eta_2}{c})} - 1) = \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})} - 1$.

Hence $\Gamma_{\kappa_{n+1},hu,a}(\eta_2) \geq \Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c})$. Since $\eta_2 < \frac{\epsilon}{3}$, we have

$$\Gamma_{\kappa_{n+1},hu,a}(\frac{\epsilon}{3}) \geq \Gamma_{\kappa_{n+1},hu,a}(\eta_2) \geq \Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c}). \tag{3.12}$$

Similarly, $\kappa_n \to u$, $\Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c}) \to 1$. Therefore, $\Gamma_{\kappa_{n+1},hu,a}(\eta_2) \geq \Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c})$. Now for $\eta_2 < \frac{\varepsilon}{3}$, we have

$$\Gamma_{\kappa_{n+1},hu,a}(\frac{\epsilon}{3}) \geq \Gamma_{\kappa_{n+1},hu,a}(\eta_2) \geq \Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c}). \tag{3.13}$$

Now, from (3.11), we have

$$\Gamma_{hu,u,a}(\epsilon) \geq \Delta(\Gamma_{hu,u,\kappa_{n+1}}(\frac{\epsilon}{3}), \Gamma_{hu,\kappa_{n+1},a}(\frac{\epsilon}{3}), \Gamma_{\kappa_{n+1},u,a}(\frac{\epsilon}{3})).$$

Now using (3.12) and (3.13), for all $a \in S$, we have

$$\Gamma_{hu,u,a}(\epsilon) \geq \Delta(\Gamma_{\kappa_n,\kappa_{n+1},a}(\frac{\eta_2}{c}), \Gamma_{\kappa_n,u,a}(\frac{\eta_2}{c}), \Gamma_{\kappa_{n+1},u,a}(\frac{\epsilon}{3})). \tag{3.14}$$

As $n \to \infty$, $\eta_n \to u$ from (3.14), we have

$$\Gamma_{hu,u,a}(\epsilon) \geq \Delta(1,1,1) = 1,$$

for all $a \in S$, for every $\epsilon > 0$. Thus $hu = u$.

Next we establish the uniqueness of a fixed point. Let κ and μ be two distinct fixed points of h, that is, $h\kappa = \kappa$ and $h\mu = \mu$. We can take $s > 0$, such that $\Gamma_{\kappa,\mu,a}(s) > 0$ for all $a \in S$. Then, by an application of (i) and (ii), for $\kappa \in A_i$, $\mu \in A_j$ where $i \neq j$, we have

$$\begin{aligned}\frac{1}{\Gamma_{h\kappa,h\mu,a}(s)} - 1 \leq &\min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{s}{c})} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\frac{s}{c})} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\frac{s}{c})} - 1)\\ = &\min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{s}{c})} - 1, \frac{1}{\Gamma_{\kappa,\kappa,a}(\frac{s}{c})} - 1, \frac{1}{\Gamma_{\mu,\mu,a}(\frac{s}{c})} - 1)\\ = &\min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{s}{c})} - 1, 0, 0) = 0.\end{aligned}$$

Hence $\Gamma_{\kappa,\mu,a}(s) \geq 1$, $\kappa = \mu$. $\square$

4 Minimum t-norm and fixed points

In this section, we give another fixed point theorem using a control function Φ and α-admissible mapping.

Definition 4.1. Let (S,Γ,Δ) be a 2-Menger space, $h:S\to S$ be a given mapping and $\alpha : S\times S\times(0,\infty)\to R^+$ be a function, we say that h is α-admissible if for all $\kappa,\mu,a\in S$, and $\eta>0$, we have $\alpha(\kappa,\mu,\eta)\geq 1$ implies $\alpha(h\kappa,h\mu,\eta)\geq 1$.

Theorem 4.2. Let (S,Γ,Δ) be a complete 2-Menger space, where Δ is a 3-order min t-norm, that is, $\Delta(a,b,c)=\min\{a,b,c\}$. Also let $\{A_i\}$ be non empty closed subsets of X, and the mapping $h:\bigcup_{i=1}^{p}A_i\to\bigcup_{i=1}^{p}A_i$ is a p-cyclic mapping satisfying the following conditions:

(i) $hA_i\subseteq A_{i+1}$ for $1\leq i<p$, $hA_p\subseteq A_1$,

(ii) h is α-admissible,

(iii) for $\eta>0$, $0<c<1$, $\kappa\in A_i$, $\mu\in A_j, i\neq j$, a $\in S$, $\phi\in\Phi$, we have

$$\alpha(\kappa,\mu,\eta)(\frac{1}{\Gamma_{h\kappa,h\mu,a}(\phi(\eta))}-1)\leq\min(\frac{1}{\Gamma_{\kappa,\mu,a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa,h\kappa,a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\mu,h\mu,a}(\phi(\frac{\eta}{c}))}-1).$$

Then h has a unique fixed point in $\bigcap_{i=1}^{p}A_i$.

Proof . Let $\kappa_0\in S$ be such that $\alpha(\kappa_0,h\kappa_0,\eta)\geq 1$ for all $\eta>0$. We consider a sequence $\{\kappa_n\}$ in S so that $\kappa_{n+1}=h\kappa_n$, for all $n\in N$, where N is the set of natural numbers. Clearly, $\kappa_{n+1}\neq\kappa_n$ for all $n\in N$, otherwise h has trivially a fixed point. As h is α-admissible, we get

$$\alpha(\kappa_0,h\kappa_0,\eta)=\alpha(\kappa_0,\kappa_1,\eta)\geq 1 \text{ implies } \alpha(h\kappa_0,h\kappa_1,\eta)=\alpha(\kappa_1,\kappa_2,\eta)\geq 1.$$

By induction, we get

$$\alpha(\kappa_n,\kappa_{n+1},\eta)\geq 1,$$

for all $n\in N$ and for all $\eta>0$. From the properties of the function ϕ, we can find $\eta>0$ such that $\Gamma_{\kappa_0,\kappa_1,a}(\phi(\eta))>0$, for all $a\in S$.

Now, taking $\kappa_n\in A_{n+1}$, $\kappa_{n-1}\in A_n$ and using (i)-(iii) for all $a\in S$, $\eta>0$ and $c\in(0,1)$, we get

$$\begin{aligned}\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta))}-1=&\frac{1}{\Gamma_{h\kappa_n,h\kappa_{n-1},a}(\phi(\eta))}-1\\ \leq&\alpha(\kappa_n,\kappa_{n-1},\eta)(\frac{1}{\Gamma_{h\kappa_n,h\kappa_{n-1},a}(\phi(\eta))}-1)\\ \leq&\min(\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_n,h\kappa_n,a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_{n-1},h\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1)\\ =&\min(\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_{n-1},\kappa_n,a}(\phi(\frac{\eta}{c}))}-1)\\ =&\min(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1).\end{aligned}\tag{4.1}$$

The above inequality holds since $\alpha(\kappa_n,\kappa_{n-1},\eta)\geq 1$.

We now claim that for all $a\in S$, $\eta>0$, $n\geq 1$ and $c\in(0,1)$,

$$\min(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{\eta}{c}))}-1,\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1)=\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1,$$

holds.

If possible, let for some $s>0$,

$$\min(\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))}-1,\frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{s}{c}))}-1)=\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))}-1.$$

Using (4.1), we get

$$\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s))}-1 \le \frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c}))}-1,$$

that is,

$$\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s)) \ge \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c})),$$

which is impossible for all $c \in (0,1)$. Using strictly monotone increasing function $\phi(\frac{s}{c}) > \phi(s)$, that is, $\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\frac{s}{c})) \ge \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(s))$, by the monotone property of Γ.

Then, for all $\eta > 0$ and $a \in S$, we get

$$\frac{1}{\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta))}-1 \le \frac{1}{\Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))}-1.$$

Hence

$$\begin{aligned}\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) &\ge \Gamma_{\kappa_n,\kappa_{n-1},a}(\phi(\frac{\eta}{c}))\\ &\ge \Gamma_{\kappa_{n-1},\kappa_{n-2},a}(\phi(\frac{\eta}{c^2}))\\ &\vdots\\ &\ge \Gamma_{\kappa_1,\kappa_0,a}(\phi(\frac{\eta}{c^n})).\end{aligned}$$

Therefore,

$$\Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) \ge \Gamma_{\kappa_1,\kappa_0,a}(\phi(\frac{\eta}{c^n})). \tag{4.2}$$

Now, taking limit $n \to \infty$, for all $\eta > 0$ and $a \in S$, we obtain

$$\lim_{n\to\infty} \Gamma_{\kappa_{n+1},\kappa_n,a}(\phi(\eta)) = 1. \tag{4.3}$$

Now, we have to prove that $\{\kappa_n\}$ is a Cauchy sequence. On the contrary, there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{\kappa_{m(\ell)}\}$ and $\{\kappa_{n(\ell)}\}$ of $\{\kappa_n\}$ with $m(\ell) > n(\ell) > \ell$ such that

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon) < 1-\lambda. \tag{4.4}$$

We take $m(\ell)$ corresponding to $n(\ell)$ to be the smallest integer satisfying (4.4), so that

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\epsilon) \ge 1-\lambda. \tag{4.5}$$

If $\epsilon_1 < \epsilon$ then we have

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon_1) \le \Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\epsilon).$$

So, it is feasible to construct $\{\kappa_{m(\ell)}\}$ and $\{\kappa_{n(\ell)}\}$ with $m(\ell) > n(\ell) > \ell$ and satisfying (4.4), (4.5) whenever ϵ is replaced by a smaller positive value. By the continuity of ϕ at 0 and strictly monotone increasing property with $\phi(0) = 0$, it is possible to find $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$. Then, by the above condition, it is possible to get an increasing sequence of integers $\{m(\ell)\}$ and $\{n(\ell)\}$ with $m(\ell) > n(\ell) > \ell$ such that

$$\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2)) < 1-\lambda, \tag{4.6}$$

and

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2)) \ge 1-\lambda. \tag{4.7}$$

Now, from (4.6), we get

$$\begin{aligned}\frac{\lambda}{1-\lambda} <&\frac{1}{\Gamma_{\kappa_{m(\ell)},\kappa_{n(\ell)},a}(\phi(\epsilon_2))}-1\\ \leq&\alpha(\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\eta)(\frac{1}{\Gamma_{h\kappa_{m(\ell)-1},h\kappa_{n(\ell)-1},a}(\phi(\epsilon_2))}-1).\end{aligned}$$

Since $\alpha(\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\eta)\geq 1$ and applying the inequality (iii), for $\kappa_{m(l)-1}\in A_{m(l)}$ where $m(l)\neq n(l)$, we get

$$\begin{aligned}\frac{\lambda}{1-\lambda} \leq&\alpha(\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\eta)(\frac{1}{\Gamma_{h\kappa_{m(\ell)-1},h\kappa_{n(\ell)-1},a}(\phi(\epsilon_2))}-1)\\ \leq&\min(\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))}-1,\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1,\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1).\end{aligned} \tag{4.8}$$

Now, we can choose β_1, $\beta_2>0$ with $\phi(\frac{\epsilon_2}{c})=\beta_1+\beta_2+\phi(\epsilon_2)$ such that

$$\begin{aligned}&\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))\geq\\ &\Delta(\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\kappa_{n(\ell)}}(\beta_1),\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2)),\Gamma_{\kappa_{n(\ell)},\kappa_{n(\ell)-1},a}(\beta_2)),\end{aligned} \tag{4.9}$$

holds. Now, using (4.3) and (4.7), we have

$$\begin{cases}\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},\kappa_{n(\ell)}}(\beta_1)\geq 1-\lambda,\\ \Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)},a}(\phi(\epsilon_2))\geq 1-\lambda,\\ \Gamma_{\kappa_{n(\ell)},\kappa_{n(\ell)-1},a}(\beta_2)\geq 1-\lambda.\end{cases} \tag{4.10}$$

Since Δ is a min t-norm, using (4.10) in (4.9), we have

$$\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))\geq\Delta(1-\lambda,1-\lambda,1-\lambda)=1-\lambda.$$

Hence

$$\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))}-1\leq\frac{1}{1-\lambda}-1=\frac{\lambda}{1-\lambda}. \tag{4.11}$$

Again, using (4.3), we have

$$\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1\leq\frac{1}{1-\lambda}-1=\frac{\lambda}{1-\lambda}. \tag{4.12}$$

Similarly, we have

$$\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1\leq\frac{1}{1-\lambda}-1=\frac{\lambda}{1-\lambda}. \tag{4.13}$$

Now, using (4.11), (4.12) and (4.13) in (4.8), we have

$$\begin{aligned}\frac{\lambda}{1-\lambda} <&\min(\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{n(\ell)-1},a}(\phi(\frac{\epsilon_2}{c}))}-1,\frac{1}{\Gamma_{\kappa_{m(\ell)-1},\kappa_{m(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1,\frac{1}{\Gamma_{\kappa_{n(\ell)-1},\kappa_{n(\ell)},a}(\phi(\frac{\epsilon_2}{c}))}-1)\\ \leq&\min(\frac{\lambda}{1-\lambda},\frac{\lambda}{1-\lambda},\frac{\lambda}{1-\lambda})\\ =&\frac{\lambda}{1-\lambda},\end{aligned}$$

which is a contradiction. Hence $\{\kappa_n\}$ is a Cauchy sequence. Since (S,Γ,Δ) be a complete 2-Menger space, therefore $\kappa_n\to u$ as $n\to\infty$, for some $u\in S$. Moreover, we get

$$\Gamma_{hu,u,a}(\epsilon)\geq\Delta(\Gamma_{hu,u,\kappa_{n+1}}(\frac{\epsilon}{3}),\Gamma_{hu,\kappa_{n+1},a}(\frac{\epsilon}{3}),\Gamma_{\kappa_{n+1},u,a}(\frac{\epsilon}{3})).$$

Next, using the properties of function ϕ, we can find $\eta_2 > 0$ such that $\phi(\eta_2) < \frac{\epsilon}{3}$. Since $\kappa_n \to u$ as $n \to \infty$, there exists $n_0 \in N$ such that, for all $n > n_0$ (sufficiently large), $\kappa_n \in A_{n+1}$, $u \in A_j$ where $n+1 \neq j$ we have

$$\frac{1}{\Gamma_{\kappa_{n+1},hu,a}(\frac{\epsilon}{3})} - 1 \leq \frac{1}{\Gamma_{h\kappa_n,hu,a}(\phi(\eta_2))} - 1 \leq \alpha(\kappa_n, u, \eta_2)(\frac{1}{\Gamma_{h\kappa_n,hu,a}(\phi(\eta_2))} - 1).$$

Now, using the inequality (iii), we have

$$\begin{aligned}
&\alpha(\kappa_n, u, \eta_2)(\frac{1}{\Gamma_{h\kappa_n,hu,a}(\phi(\eta_2))} - 1)\\
&\leq \min(\frac{1}{\Gamma_{\kappa_n,u,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,h\kappa_n,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1)\\
&= \min(\frac{1}{\Gamma_{\kappa_n,u,a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{\kappa_n,\kappa_{n+1},a}(\phi(\frac{\eta_2}{c}))} - 1, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1).
\end{aligned}$$

Taking limit $n \to \infty$ on both sides, we have

$$\frac{1}{\Gamma_{u,hu,a}(\phi(\eta_2))} - 1 \leq \min(0, 0, \frac{1}{\Gamma_{u,hu,a}(\phi(\frac{\eta_2}{c}))} - 1) = 0.$$

Hence $\frac{1}{\Gamma_{u,hu,a}(\phi(\eta_2))} \leq 1$, $\Gamma_{u,hu,a}(\phi(\eta_2)) \geq 1$. Hence $hu = u$. So, it is proved that h has a fixed point. It is easy to see the uniqueness of the fixed point.

□

Now, we give some consequences of our obtained results.
Taking $p = 2$ and $\phi(\eta) = \eta$, $\alpha(\kappa, \mu, \eta) = 1$ we get the following result.

Corollary 4.3. Let (S, Γ, Δ) be a complete 2-Menger space, where Δ is a 3-order min t-norm, that is, $\Delta(a, b, c) = min\{a, b, c\}$. Also let A_1, A_2 are non-empty closed subsets of S, and the mapping $h : A_1 \bigcup A_2 \to A_1 \bigcup A_2$ is a cyclic mapping satisfying the following conditions:

(i) $hA_1 \subseteq A_2$ and $hA_2 \subseteq A_1$,
(ii) for all $\kappa \in A_1, \mu \in A_2, a \in S$, where $\eta > 0, 0 < c < 1$, we have

$$(\frac{1}{\Gamma_{h\kappa,h\mu,a}(\eta)} - 1) \leq \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\frac{\eta}{c})} - 1).$$

Then h has a unique fixed point in $A_1 \cap A_2$.

Taking $p = 2$ we get the following example.

Example 4.4. Let $S = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \beta, \gamma\}$, $B = \{\gamma, \delta\}$, the t-norm Δ is a 3rd order Lukasiewicz t-norm and Γ be defined as

$$\Gamma_{\alpha,\beta,\gamma}(\eta) = \Gamma_{\alpha,\beta,\delta}(\eta) = \begin{cases} 0, & \text{if } \eta \leq 0, \\ 0.40, & \text{if } 0 < \eta < 4, \\ 1, & \text{if } \eta \geq 4, \end{cases}$$

$$\Gamma_{\alpha,\gamma,\delta}(\eta) = \Gamma_{\beta,\gamma,\delta}(\eta) = \begin{cases} 0, & \text{if } \eta \leq 0, \\ 1, & \text{if } \eta > 0. \end{cases}$$

Then (S, Γ, Δ) is a complete 2-Menger space. If we define $\Gamma : S \to S$ as follows: $\Gamma\alpha = \delta, \Gamma\beta = \gamma, \Gamma\gamma = \gamma, \Gamma\delta = \gamma$ then the mapping Γ satisfies all the conditions of the Theorem 3.1 (see Figure 1) and γ is the unique fixed point of Γ in $A \bigcap B$.

Taking $p = 3$ and $\phi(\eta) = \eta, \alpha(\kappa, \mu, \eta) = 1$ we get the following consequence.

Corollary 4.5. Let (S, Γ, Δ) be a complete 2-Menger space, where Δ is a 3-order min t-norm, that is, $\Delta(a, b, c) = \min\{a, b, c\}$. Also let A_1, A_2, A_3 are non-empty closed subsets of S, and the mapping $h : A_1 \bigcup A_2 \bigcup A_3 \to A_1 \bigcup A_2 \bigcup A_3$ is a 3-cyclic mapping satisfying the following conditions:

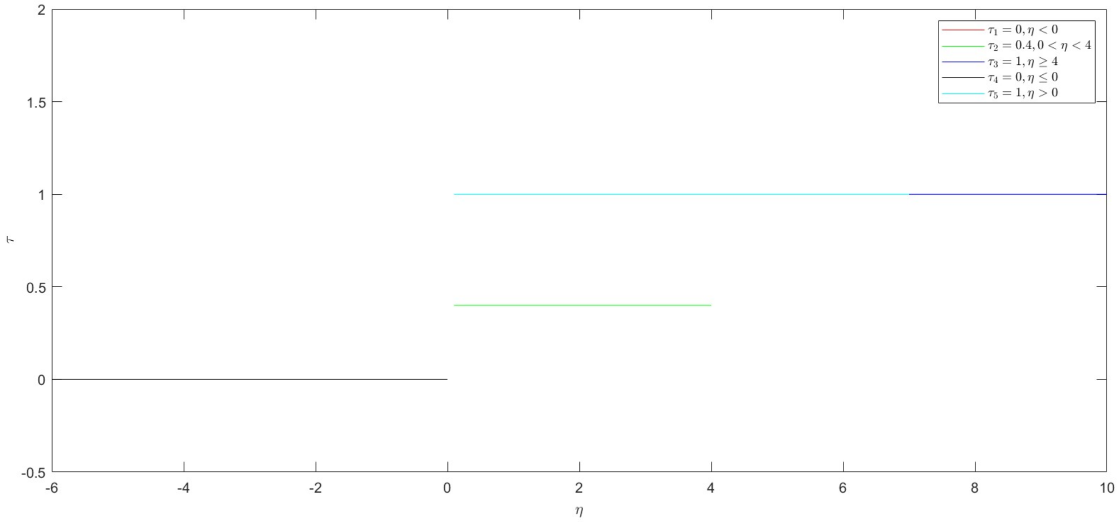

Figure 1:

(i) $hA_1 \subseteq A_2$, $hA_2 \subseteq A_3$ and $hA_3 \subseteq A_1$,
(ii) for all $\kappa \in A_i, \mu \in A_j, i \neq j, a \in S$, where $\eta > 0, 0 < c < 1$, we have

$$(\frac{1}{\Gamma_{h\kappa,h\mu,a}(\eta)} - 1) \leq \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\frac{\eta}{c})} - 1)$$

Then h has a unique fixed point in $A_1 \cap A_2 \cap A_3$.

Taking $p = 3$, we get the following example to validate the theorem 2.1.

Example 4.6. Let $S = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \gamma, \delta\}$, $B = \{\alpha, \beta\}$, $C = \{\alpha, \gamma\}$ the t-norm Δ is a 3rd order minimum t-norm and Γ be defined as

$$\Gamma_{\alpha,\beta,\gamma}(\eta) = \Gamma_{\alpha,\beta,\delta}(\eta) = \begin{cases} 0, & \text{if} \quad \eta \leq 0, \\ 0.40, & \text{if} \quad 0 < \eta < 7, \\ 1, & \text{if} \quad \eta \geq 7, \end{cases}$$
$$\Gamma_{\alpha,\gamma,\delta}(\eta) = \Gamma_{\beta,\gamma,\delta}(\eta) = \begin{cases} 0, & \text{if} \quad \eta \leq 0, \\ 0.95, & \text{if} \quad 0 < \eta < 1, \\ 1, & \text{if} \quad \eta \geq 1, \end{cases}$$

then (S, Γ, Δ) is a complete 2-Menger space. If we define $\Gamma : S \to S$ as follows: $\Gamma\alpha = \alpha, \Gamma\beta = \alpha, \Gamma\gamma = \alpha, \Gamma\delta = \alpha$ then the mapping Γ satisfies all the conditions of the Theorem 3.1 (see Figure 2) and α is the unique fixed point of Γ in $A \cap B \cap C$.

Here we also get another corollary. The corollary is based on fixed point result on 2-probabilistic metric space.

Corollary 4.7. Let (S, Γ, Δ) be a complete 2-menger space, where Δ is the third order min t-norm and $\Gamma : S \to S$ be a self mapping satisfying the following inequality

$$\frac{1}{\Gamma_{h\kappa,h\mu,a}(\eta)} - 1 \geq \min(\frac{1}{\Gamma_{\kappa,\mu,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\kappa,h\kappa,a}(\frac{\eta}{c})} - 1, \frac{1}{\Gamma_{\mu,h\mu,a}(\frac{\eta}{c})} - 1), \tag{4.14}$$

for all $\kappa, \mu, a \in S, \eta > 0$. Then Γ has a unique fixed point in S.

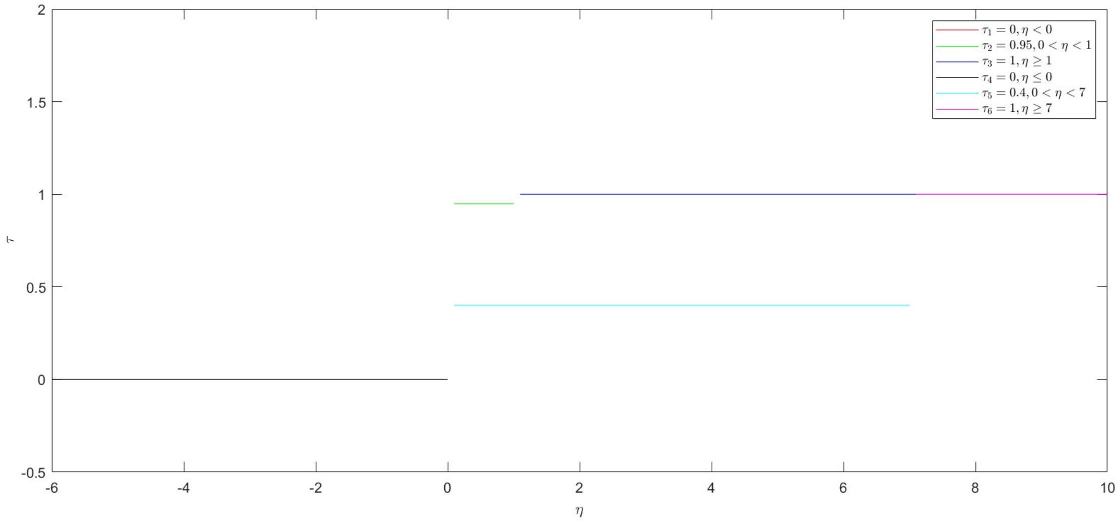

Figure 2:

5 Application to Airy's type differential equation

In this section, we obtain the solution of the following second order boundary value problem:

$$-\frac{d^2x}{dt^2} = g(t,\ x(t)), \quad t \in [0,1]$$

$$x(0) = x(1) = 0,$$

where $g : [0,1] \times R \longrightarrow R$ is a continuous function.

Let $X = C([0,1], R)$, where $C([0,1], R)$ be the collection of continuous functions from $I := [0,1] \to R$ and Δ be the 3rd order minimum t-norm, that is, $\Delta(a,b,c) = \min(a,b,c)$. Choose the distribution function defined as $F_{x,y,z}(t) = \frac{t}{t+d(x,y,z)}$ where $x, y, z \in X$, $t > 0$, d is a 2-metric given by

$$\begin{aligned} d(x,y,z) &= \min\{\|x-y\|_\infty, \|y-z\|_\infty, \|z-x\|_\infty\} \\ &= \min\{\max_{t\in I}|x(t)-y(t)|, \max_{t\in I}|y(t)-z(t)|, \max_{t\in I}|z(t)-x(t)|\}. \end{aligned}$$

Here, it is easy to see that $(X = C([0,1], R), F, \Delta)$ is a complete 2-Menger space.

Now, Green's function $G(t,s)$ exists for the associated boundary-values problem is given by

$$G(t,s) = \begin{cases} a_1t + a_2, & 0 \leq t < s \\ b_1t + b_2, & s < t \leq 1 \end{cases}$$

The Green's function must satisfy the following three properties:

i) $G(t,s)$ is continuous at $x = s$, that is, $b_1s + b_2 = a_1s + a_2$ implies $s(b_1 - a_1) + b_2 - a_2 = 0$.

ii) The determination of G has a discontinuity of magnitude $-\frac{1}{p_0(s)}$ at the point $x = s$ where $p_0(t) =$ co-efficient of the highest order derivative, that is,

$$\left(\frac{\partial G}{\partial t}\right)_{t=s+0} - \left(\frac{\partial G}{\partial t}\right)_{t=s-0} = -1 \text{ implies } b_1 - a_1 = -1.$$

iii) $G(t,s)$ must satisfy the boundary conditions $G(0,s) = 0$ implies $a_2 = 0$ and $G(1,s) = 0$ implies $b_1 + b_2 = 0$.

Therefore, $G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ -st + s, & 0 \leq s \leq t \leq 1. \end{cases}$

We have to show that the above mentioned differential equation satisfies the following inequality

$$\frac{1}{F_{fx,fy,a}(t)} - 1 \geq \min\left(\frac{1}{F_{x,y,a}(\frac{t}{c})} - 1, \frac{1}{F_{x,fx,a}(\frac{t}{c})} - 1, \frac{1}{F_{y,fy,a}(\frac{t}{c})} - 1\right), \tag{5.1}$$

for all $x, y, a \in X, t > 0$.
Taking $F_{x,y,z}(t) = \frac{t}{t+d(x,y,z)}$, we get

$$\frac{t + d(fx, fy, a)}{t} - 1 \geq \min\left(\frac{\frac{t}{c} + d(x, y, a)}{\frac{t}{c}} - 1, \frac{\frac{t}{c} + d(x, fx, a)}{\frac{t}{c}} - 1, \frac{\frac{t}{c} + d(y, fy, a)}{\frac{t}{c}} - 1\right). \tag{5.2}$$

It implies that

$$d(fx, fy, a) \leq c. \min\{d(x, y, a), d(x, fx, a), d(y, fy, a)\}.$$

Now, It is well known that $x \in C^2(I)$ is a solution of given differential equation is equivalent to that $x \in C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s, a)g(s, x(s), a)ds, \text{for all } \ t \in I. \tag{5.3}$$

Define the operator $f : C(I) \to C(I)$ by

$$f(x(t)) = \int_0^1 G(t, s, a)g(s, x(s), a)ds, \text{for all } \ t \in I,$$

with

$$|g(s, x(s), a) - g(s, y(s), a)| \leq c \cdot \min\{d(x, y, a), d(x, fx, a), d(y, fy, a)\}.$$

Consider

$$\begin{aligned}
|f(x(t)) - f(y(t))| &= |\int_0^1 G(t, s, a)[g(s, x(s), a) - g(s, y(s), a)]ds| \\
&\leq \int_0^1 G(t, s, a)|g(s, x(s), a) - g(s, y(s), a)|ds \\
&\leq \int_0^1 G(t, s, a)\, c \cdot \min\{d(x, y, a), d(x, fx, a), d(y, fy, a)\}ds \\
&= c \cdot \min\{d(x, y, a), d(x, fx, a), \ d(y, fy, a)\} \int_0^1 G(t, s, a)ds \\
&\leq c \cdot \min\{d(x, y, a), d(x, fx, a), d(y, fy, a)\} \times \frac{1}{8} \\
&= 0.
\end{aligned}$$

Note that for all $t \in I$,

$$\int_0^1 G(t, s, a)ds = -\frac{t^2}{2} + \frac{t}{2},$$

which implies that,

$$\sup_{t \in I} \int_0^1 G(t, s, a)ds = \frac{1}{8}.$$

Also, $\min\{d(x, y, a), d(x, fx, a), d(y, fy, a)\} = \min\{d(x, y, a), 0, 0\} = 0$. Therefore by Corollary 4.7, $t \geq 0$ for all $x, y, a \in C([0, 1], R)$ and $t > 0$, we conclude the uniqueness of a operator f, $fx^* = x^* \in C([0, 1], R)$, which is also a solution of the proposed integral equation.

6 Conclusion

In our present discussion we have introduced two p-cyclic contraction results on 2-Menger spaces. Two different types t-norms have been used here. In our first theorem we have used Hadzic type t-norm while minimum t-norm have been used in our second theorem.

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