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The distance-based critical node detection in the symmetric travelling salesman problem and its application to improve the approximate solutions

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Abstract

The traveling salesman problem is one of the well-known NP-hard problems, and there are various versions of the problem with respect to its different specifications of the constraints and assumptions. Especially, the symmetric traveling salesman problem has been considered in numerous routing models. The critical node detection problem has received increasing attention throughout the routing models. The critical node has the most important role in the routing problems, and if it is out of service then the optimal solution will be hit by a large undesirable cost. The critical node is defined as the node whose deletion from the network results in the largest decrease in the optimal cost. It is proved the critical node of the network is the critical node for the optimal tour, too. Thus, the critical node is considered to obtain a good approximate solution in a reasonable iteration. The 2-opt heuristic is applied by the critical node in the symmetric traveling salesman problem and the iterations are reduced significantly. Then, the pseudo-critical node is defined and detected in the approximate solution, whose removal results in the largest decrease of the approximate cost. So, the 2-opt heuristic is applied by the pseudo-critical node and the optimal or a nearby optimal solution is obtained.

Keywords: Critical node, Travelling salesman problem, Approximation algorithm, 2-opt algorithm, Approximate solution
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1 Introduction

The critical node is an important issue for decision makers in the optimization models because it could impose the largest difference in the optimal cost as well as the connectivity of the network as soon as it is out of service. There are various definitions of the critical node according to its applications; however, the most addressed critical (or important) node is related to the connectivity of the network with respect to a connectivity criteria [13, 8, 15, 2]. We consider the critical node as the node its removal results in the maximum decrease of the optimal cost, which is as the distance based critical node. So, let $C^*(n)$ be the optimal cost of network $G = (N, A)$ and $C^*(n-1)$ be the optimal cost of the reduced network $\bar{G} = (\bar{N}, \bar{A})$ by removing the critical node $\bar{v}$ from G, then $\bar{G} = G \setminus \bar{v}$.

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In the general case the critical node detection problem (CNDP) is NP-complete [3] and it was shown for trees the problem is NP-complete, too [7]. Since the present definition of the critical node is based on the difference in the cost of the optimal solution, so it remains NP-complete, too. However, there are some polynomial time approximation algorithms for the traveling salesman problem (TSP), where the costs are satisfying the triangle inequality [5, 10].

Shirdel and Abdolhosseinzadeh [16] formulated the the critical node problem as an established discrete time Markov chain in the stochastic networks. Jiang et al. [8] applied a nonconvex quadratically constrained quadratic programming model instead of integer linear programming model to formulate CNDP; they determined approximate solutions by semidefinite programming technique. Santos et al. [15] studied p critical nodes problem according to the connectivity of the network and the solution computationally were improved. Veremyev et al. [18] considered two deterministic and probabilistic versions of CNDP, and they studied a mixed integer linear programming for the deterministic one, and based on Markov chain process a scenario based formulation was presented for the probabilistic one. Aringhieri et al. [2] studied some classes of CNDP where the objective function is impacted by the distances of the node pairs; they showed in the general cases the problem is NP-complete. Li et al. [12] studied a bi-objective CNDP based on the psychology of the decision makers and the pairwise connectivity of the network, and they proved the problem is NP-Hard in the general case.

Chen et al. [4] considered the negative CNDP where the larger edge weights demonstrate the weaker relationship between nodes; their objective was simultaneously the minimization of the pairwise connectivity and the maximization of the weights between the network's nodes. Alozie et al. [1] developed an algorithm for distance based CNDP for separating the problem by breadth first search tree generation. Zhou et al. [19] considered node weighted critical node problem, and they applied a local search procedure and a late acceptance strategy to find a local optimal solution. Shukla [17] developed an algorithm to solve a three dimensional CNDP.

2 Distance critical node detection problem formulation

In this paper, the critical node is defined according to the most decreasing in the optimal cost of TSP, which is known as the distance critical node detection problem (D-CNDP). The network $G = (N, A)$ is a complete network with symmetric arc costs, so $c_{ij} = c_{ji}$ for all $(i, j) \in A$. Thus, it is assumed the arc costs c_{ij} are satisfying the triangle inequality; by the assumption of the triangle inequality there are some approximate solution in polynomial time [5, 9, 10].

Consider the following TSP formulation to obtain minimum Hamiltonian path in the network

$$min \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij} \tag{2.1}$$

$$\sum_{i=1, i \neq j}^{n} x_{ij} = 1, j = 1, 2, ..., n \tag{2.2}$$

$$\sum_{j=1, j \neq i}^{n} x_{ij} = 1, i = 1, 2, ..., n \tag{2.3}$$

$$\sum_{i \in V} \sum_{j \neq i, j \in V} x_{ij} \leq |V| - 1, \forall V \subsetneq \{1, 2, ..., n\}, |V| \geq 2 \tag{2.4}$$

The objective function 2.1 determines a minimum length Hamiltonian cycle; the constraints 2.2 and 2.3 imply any node to traverse exactly once, and the last constraint 2.7 implies any pure subset of the nodes constructs a path. To determine the critical node in the network the following formulation is presented the above problem should be solved for any node $i \in N$, and in the reduced network $G \setminus \{i\}$, in the general form.

Theorem 2.1. The critical node for the original network $G = (N, A)$ is the critical node for the optimal tour P^* in G.

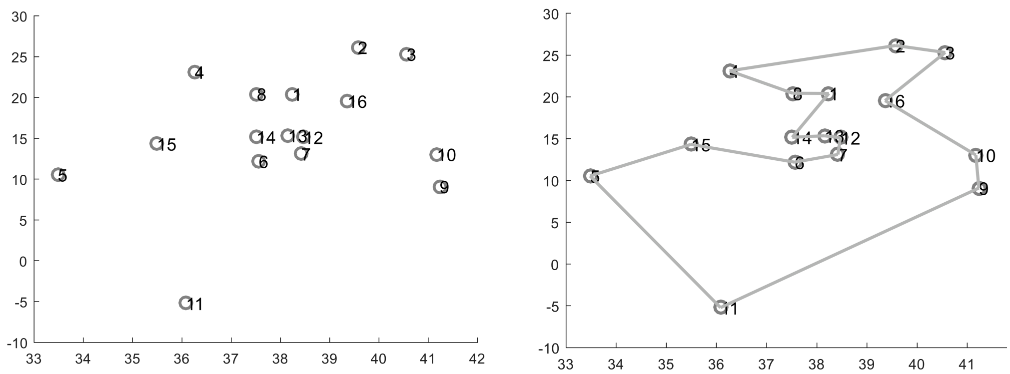
Proof . Let node $\bar{v}$ be the critical node in the network G. if node $\bar{v}$ is out of service in the network, then let $\bar{P}^*$ be the optimal tour in the reduced network $\bar{G} = G \setminus \{\bar{v}\}$. By triangle inequality, for $\tilde{P} = P \setminus \{\bar{v}\}$ the following inequality is satisfied by the costs of the reduced tour $\tilde{P}$ and the optimal cost of P^*: $C(\tilde{P}) \leq C(P^*)$.

On the other hand, $\bar{v}$ is the critical node and $\tilde{P}$ is a feasible solution for the reduced network $\bar{G}$, by the definition we have

$$C(P^*) - C(\bar{P}^*) = max_{\forall v_i \in N}(C(P^*) - C(\bar{P}_i^*)) \Rightarrow C(\bar{P}^*) = min_{\forall v_i \in N}(C(\bar{P}_i^*)) \tag{2.5}$$

and obviously, $C(\bar{P}^*) \leq C(\tilde{P})$; where, $\bar{P}_i^*$ is the optimal solution in the reduced network $\bar{G}_i = G \backslash \{v_i\}$. Let u be the critical node of the optimal solution P^* and $\bar{P}_u = P^* \setminus \{u\}$, then $C(\bar{P}_u) < C(\tilde{P})$ and $C(P^*) - C(\bar{P}_u) > C(P^*) - C(\tilde{P})$ is a contradiction with 2.5 and that node $\bar{v}$ is the critical node of network G; So, node v is the critical node of the optimal solution P^*, too. □

So, by theorem 2.1 the removal of the critical node in the optimal solution P^* in network G results the optimal solution $\bar{P}$ in the reduced network $\bar{G}$; it should be noticed, the optimal solution P^* contains n nodes and the D-CNDP could be solved in $O(n)$, by the worst-case analysis. For example, consider the instance network ulysses16 (Figure 1(a)); node 11 is detected as the critical node both in the network and in the optimal solution (Figure 1(b)).



(a) (b)

Figure 1: The instance network ulysses16 (a), the optimal solution (b)

For the instance network ulysses16 (see Figure 1 (a)), there are 16 nodes in the network with the symmetric and 2D Euclidean lengths for the arc costs; the critical node is detected as node 11 and the length of the optimal tour is 6859 and the optimal cost in the case the critical node 11 being out of service is obtained 5216.

Thus, in the symmetric traveling salesman problem (S-TSP), the forward direction costs of the arcs are equal to the backward directions; network $G = (N, A)$ is a complete network and the triangle inequality is satisfied by the arc costs parameters.

Theorem 2.2. The optimal solution P^* of S-TSP in the original network $G = (N, A)$ is reduced to the optimal solution $\bar{P}^*$ in the reduced network $\bar{G} = (\bar{N}, \bar{A})$, where $\bar{v}$ is the critical node in G and $\bar{G} = G \setminus \{\bar{v}\}$.

Proof . Let $P^* = \{v_1, v_2, ..., v_{i-1}, v_i = \bar{v}, v_{i+1}, ..., v_n\}$, by removal of the critical node $v_i = \bar{v}$ there is $\bar{P}^* = \{v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n\}$, by theorem 2.1 the critical node $\bar{v}$ in the network G is the critical node in the optimal solution P^*, too. Thus, by 2.5 $C(\bar{P}^*) = min_{\forall v_i \in N}(C(\bar{P}_i^*))$ and $C(\bar{P}^*)$ is the optimal cost in the reduced network $\bar{G}$. □

Therefore, by theorems 2.1 and 2.2 the distance based critical node is determined as the equation 2.6; let $\bar{v}$ be the critical node in G, then for any failure node $v_i \in N$ the critical node is

$$\bar{v} = argmax_{\forall v_i \in N}\{C(P^*) - (C(v_{i-1}, v_i) + C(v_i, v_{i+1})) + C(v_{i-1}, v_{i+1})\} \tag{2.6}$$

where $C(v_{i-1}, v_i)$, $C(v_i, v_{i+1})$ and $C(v_{i-1}, v_{i+1})$ are the costs of the arcs (v_{i-1}, v_i), (v_i, v_{i+1}) and (v_{i-1}, v_{i+1}), respectively. Figure 2 shows the D-CNDP algorithm that implemented to find the critical node fo S-TSP in the complete network with triangle inequality of the arc costs.

By Theorem 2.2, the optimal solution for network G is transformed into the optimal solution of the reduced network $\bar{G}$ by removal the critical node from the optimal solution P^* (see Figure 2). So, in the instance network ulysses16, the optimal solution 1,14,13,12,7,6,15,5,11,9,10,16,3,2,4,8 is transformed into the optimal solution 1,14,13,12,7,6,15,5,9,10,16,3,2,4,8, for the reduced network

Figure 2: The D-CNDP algorithm

Input the optimal tour P^*
Let $C*$ be the optimal cost of P^*
Let $\bar{C}^* = \infty$
for $v = 1$ to $n = |P^*|$ **do**
  $\Delta_v = C(v_{i-1}, v_i) + C(v_i, v_{i+1}) - C(v_{i-1}, v_{i+1})$
  Delete node v from P^*
  $\bar{C}(v) = C^* - \Delta_v$
  if $\bar{C}(v) < \bar{C}^*$ **then**
    $\bar{C}^* = \bar{C}(v)$
    $\bar{v} = v$
  end if
end for

2.1 Pseudo-critical node in the approximate solution

The TSP optimization models are in NP-hard classes, so it is not easy to obtain the exact optimal solutions. However, Sahni and Gonzales [14] showed that the TSP without the triangle inequality could not be solved by a constant ratio ρ-approximation algorithm. However, there is a 2-approximation solution by the triangle inequality assumption [6]. So, Christofides [5] provided a 3/2-approximation algorithm, and in some special cases the obtained approximation ratio will be 4/3 by Held-Krap relaxation [9]. Then, our assumption is $1 \leq \rho \leq 2$, and we applied Christofides' algorithm, regularly.

According to the complexity of S-TSP, the approximate solution for the problem is respected in the polynomial time. So, the pseudo-critical node is defined similarly to the critical node, but it is related to the approximate solution. Thus, $\tilde{v}$ is the pseudo-critical node with respect to the approximate solution $\tilde{P}$, if its removal results the largest decrease in the length of the approximate solution.

$$\tilde{v} = argmax_{\forall v_i \in N}\{C(\tilde{P}) - (C(v_{i-1}, v_i) + C(v_i, v_{i+1})) + C(v_{i-1}, v_{i+1})\} \tag{2.7}$$

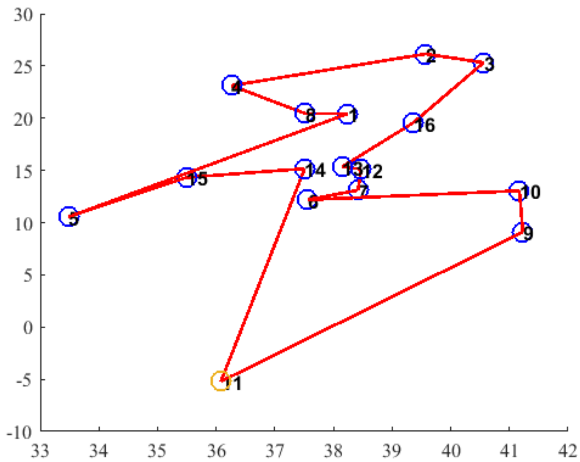


Figure 3: The approximate solution for the instance network ulysses16

The 2-opt algorithm applies two point exchanges in a feasible tour for TSP, and it attempts to improve the tour lengths alliteratively. We apply the 2-opt algorithm by the pseudo-critical node and it results in fewer iterations against the general algorithm. For the instance network ulysses16, the obtained approximate solution by Christofides' algorithm [5] is shown in Figure 3. The node 11 is detected as the pseudo-critical node. After 4 iterations of the 2-opt algorithm, the approximate solution 1,8,4,2,3,16,13,12,7,6,10,9,11,14,15,5 with length 7788 is transformed into the optimal solution (see Figure 4).

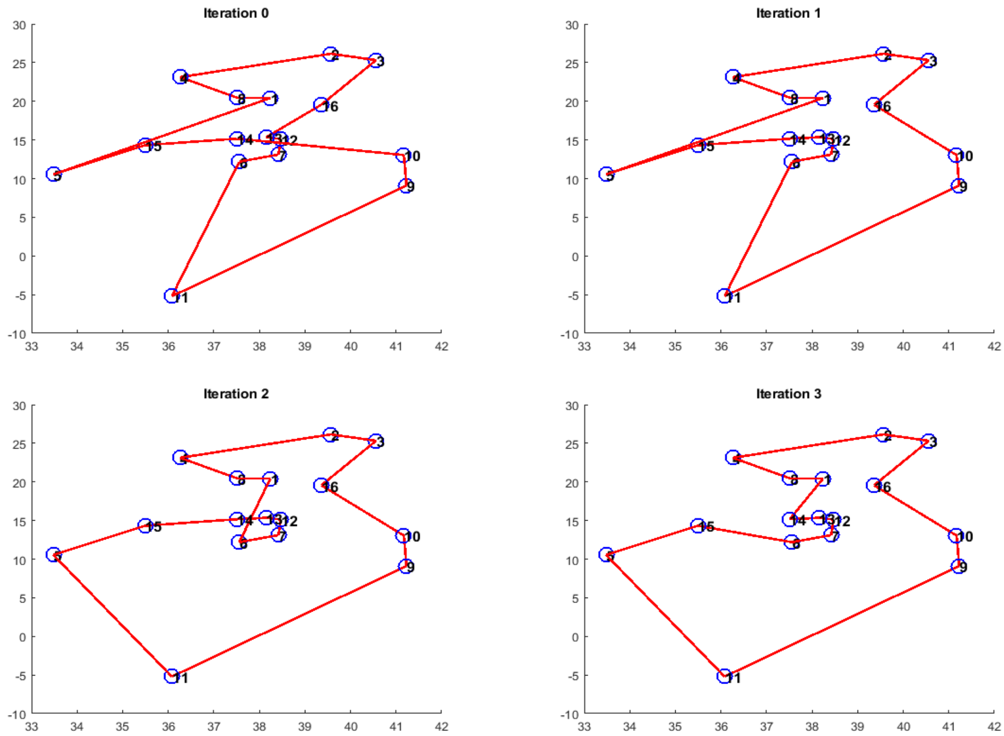


Figure 4: The instance network ulysses16 and the iterations of the 2-opt algorithm

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