Int. J. Nonlinear Anal. Appl. 14 (2023) 8, 237–247
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.7212

Existence and multiplicity of solutions for Neumann boundary value problems involving nonlocal $p(x)$-Laplacian equations

Maryam Mirzapour

Department of Mathematics, Faculty of Mathematical Sciences, Farhangian University, Tehran, Iran

(*Communicated by Javad Vahidi*)

Abstract

In this article, we study the nonlocal $p(x)$-Laplacian problem of the following form

$$\begin{cases} M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx\Big)\Big(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+|u|^{p(x)-2}u\Big)=\lambda f(x,u) & \text{in } \Omega,\\ M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx\Big)|\nabla u|^{p(x)-2}\nabla\frac{\partial u}{\partial \nu}=\mu g(x,u) & \text{on } \partial\Omega,\end{cases}$$

By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence and multiplicity of solutions for the problem.

Keywords: Generalized Lebesgue-Sobolev spaces, Nonlocal condition, Mountain pass theorem, Fountain theorem, Dual fountain theorem
2020 MSC: 35J60, 35B30, 35B40

1 Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx\Big)\Big(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+|u|^{p(x)-2}u\Big)=\lambda f(x,u) & \text{in } \Omega,\\ M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx\Big)|\nabla u|^{p(x)-2}\nabla\frac{\partial u}{\partial \nu}=\mu g(x,u) & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\Omega}$ is the outer unit normal derivative, $p(x)\in C(\overline{\Omega})$, $p(x)>1$, $\forall x\in\overline{\Omega}$ and $\lambda,\ \mu\in\mathbb{R}$. Throughout the paper, we assume that $\lambda^2+\mu^2\neq 0$. The operator $-\Delta_{p(x)}u=\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x)\equiv p$ (a constant). An essential difference between them is that the p-Laplacian operator is $(p-1)$-homogeneous, that is, $\triangle_p(\lambda u)=\lambda^{p-1}\triangle_p u$ for every $\lambda>0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [28], electrorheological fluids [5] or image restoration [6].

Problem (1.1) is called nonlocal because of the presence of the term M, which implies that the equation in (1.1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem

*Corresponding author
Email address: m.mirzapour@cfu.ac.ir (Maryam Mirzapour)

Received: March 2022 *Accepted:* September 2022

particulary interesting. For the physical and biological meaning of the nonlocal coefficients we refer the reader to [1, 2, 3, 7, 19, 20, 22] and the references therein.

2 Notations and preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [9, 10, 13, 14].

Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\};$$
$$p^+ = \max\{p(x);\ x \in \overline{\Omega}\}, \quad p^- = \min\{p(x);\ x \in \overline{\Omega}\};$$
$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\lambda > 0;\ \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\Big\},$$

with

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the natural norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u(x)|_{L^{p(x)}(\Omega)} + |\nabla u(x)|_{L^{p(x)}(\Omega)}.$$

We remember that $(W^{1,p(x)}(\Omega), \|\cdot\|)$ is a reflexive and separable Banach space. In this paper we will use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\| = \inf\Big\{\lambda > 0;\ \int_\Omega \frac{|\nabla u(x)|^{p(x)} + |u|^{p(x)}}{\lambda^{p(x)}}\,dx \le 1\Big\}.$$

Proposition 2.1 (See [9, 14]). (**i**)The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\int_\Omega |uv|\,dx \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}$$

(**ii**)If $p_1(x),\ p_2(x) \in C + \overline{\Omega},\ p_1(x) \le p_2(x),\ \forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2 (See [12, 14]). If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies

$$|f(x,s) \le a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \qquad \forall x \in \overline{\Omega},\ s \in \mathbb{R},$$

where $p_1(x),\ p_2(x) \in C_+(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \ge 0$ and $b \ge 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3 (See [11]). Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u|^{p(x)}\,dx$, then for $u,\ u_k \in W^{1,p(x)}(\Omega)$; we have

(**1**) $\|u\| < 1$ (respectively$= 1; > 1$) $\Longleftrightarrow \rho(u) < 1$ (respectively$= 1; > 1$);
(**2**) for $u \ne o, \|u\| = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$;
(**3**) if $\|u\| > 1$, then $\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;
(**4**) if $\|u\| < 1$, then $\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;
(**5**) $\|u\| \to 0$ (respectively $\to \infty$) $\Longleftrightarrow \rho(u) \to 0$(respectively $\to \infty$).

Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

Proposition 2.4 (See [14]). If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ ($q(x) < p^*(x)$) for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.5 (See [25]). If we denote

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N, \end{cases}$$

then the embedding from $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where $q(x) \in C_+(\partial\Omega)$ and $q(x) < p_*(x)$ for $x \in \partial\Omega$.

3 Existence of solutions

In this paper, we denote by $X = W^{1,p(x)}(\Omega)$; $X^* = (W^{1,p(x)}(\Omega))^*$, the dual space and $\langle \cdot, \cdot \rangle$, the dual pair.

Lemma 3.1 (See [15]). Denote

$$I(u) = \int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx, \quad \forall u \in X,$$

then $I(u) \in C^1(X, R)$ and the derivative operator I' of I is

$$\langle I'(u), v \rangle = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv)\, dx, \quad \forall u,\ v \in X,$$

and we have

(**1**) I is a convex functional.
(**2**) $I' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator,
(**3**) I' is a mapping of type (S_+), namely: $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} I'(u_n)(u_n - u) \leq 0$, implies $u_n \to u$.

The Euler-Lagrange functional associated to (1.1) is given by

$$J(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \Big) - \lambda \int_\Omega F(x,u)\, dx - \mu \int_{\partial\Omega} G(x,u)\, d\sigma,$$

where $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. Under proper assumptions on f and g, then

$$\langle J'(u), v \rangle = M\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \Big) \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv)\, dx$$
$$- \lambda \int_\Omega f(x,u) v\, dx - \mu \int_{\partial\Omega} g(x,u) v\, d\sigma,$$

for all $u,\ v \in X$, then we know that the weak solution of (1.1) corresponds to the critical point of the functional J, where F and G are denoted by

$$F(x,t) = \int_0^t f(x,s)\, ds, \qquad G(x,t) = \int_0^t g(x,s)\, ds.$$

Hereafter, $f(x,t)$, $g(x,t)$ and $M(t)$ are always supposed to verify the following assumption:

($\mathbf{f_0}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exist two constants $C_1 \geq 0$, $C_2 \geq 0$ such that

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha(x)-1}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}$$

where $\alpha(x) \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$,

($\mathbf{f_1}$) There exist $M_1 > 0$, $\theta_1 > \frac{p^+}{1-\mu}$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M_1$,

$$0 < \theta_1 F(x,t) \leq t f(x,t),$$

where μ comes from $(\mathbf{m_1})$ below.

($\mathbf{f_2}$) $f(x,t) = o(|t|^{p^+-1})$ as $t \to 0$ uniformly with respect to $x \in \Omega$.

($\mathbf{f_3}$) $f(x,-t) = -f(x,t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$.

($\mathbf{g_0}$) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exist two constants $C_1' \geq 0$, $C_2' \geq 0$ such that

$$|g(x,t)| \leq C_1' + C_2'|t|^{\beta(x)-1}, \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}$$

where $\beta(x) \in C_+(\overline{\Omega})$ and $\beta(x) < p_*(x)$ for all $x \in \partial\Omega$,

($\mathbf{g_1}$) There exist $M_2 > 0$, $\theta_2 > \frac{p^+}{1-\mu}$ such that for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M_2$,

$$0 < \theta_2 G(x,t) \leq t g(x,t),$$

where μ comes from $(\mathbf{m_1})$ below.

($\mathbf{g_2}$) $g(x,t) = o(|t|^{p^+-1})$ as $t \to 0$ uniformly with respect to $x \in \partial\Omega$.

($\mathbf{g_3}$) $g(x,-t) = -g(x,t)$, for all $x \in \partial\Omega$ and $t \in \mathbb{R}$.

($\mathbf{m_0}$) There exists $m_0 > 0$, such that $M(t) \geq m_0$.

($\mathbf{m_1}$) There exists $0 < \mu < 1$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$.

Remark 3.2. Under the conditions f_0 and g_0, the functional J is of class $C^1(X, \mathbb{R})$.

Remark 3.3. For simplicity, we use $C,\ M,\ K,\ K_i$, to denote the general nonnegative or positive constant (the exact value may change from line to line).

Theorem 3.4. If M satisfies $(\mathbf{m_0})$ and $(\mathbf{f_0})$, $(\mathbf{g_0})$ hold and $\alpha^+, \beta^+ < p^-$, then (1.1) has a weak solution.

Proof . From $(\mathbf{m_0})$ we have $\widehat{M}(t) \geq m_0 t$. For $(u_n) \in X$ such that $\|u_n\| \to +\infty$, we have

$$\begin{aligned}
J(u_n) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) - \lambda \int_\Omega F(x,u)\,dx - \mu \int_{\partial\Omega} G(x,u)\,d\sigma \\
&\geq m_0 \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx - |\lambda| \int_\Omega C(1 + |u_n|^{\alpha(x)})\,dx \\
&\quad - |\mu| \int_{\partial\Omega} C(1 + |u_n|^{\beta(x)})\,d\sigma \\
&\geq \frac{m_0}{p^+}\|u_n\|^{p^-} - |\lambda| C \|u_n\|^{\alpha^+} - |\mu| C \|u_n\|^{\beta^+} - M \to \infty \quad \text{as} \quad \|u_n\| \to \infty,
\end{aligned}$$

so J is coercive since $\alpha^+, \beta^+ < p^-$.

By Propositions 2.4 and 2.5, it is easy to verify that J is weakly lower semicontinuous. So J has a minimum point u in X and u is a weak solution of (1.1). $\square$

Corollary 3.5. Under the assumptions in Theorem 3.4 and $(\mathbf{m_1})$, if one of the following conditions hold, (1.1) has a nontrivial weak solution.

(1) If $\lambda,\ \mu \neq 0$, there exist two positive constants $d_1,\ d_2 < \frac{p^-}{1-\mu}$ such that
$\liminf_{t\to 0} \frac{sgn(\lambda)F(x,t)}{|t|^{d_1}} > 0$, for $x \in \Omega$ uniformly,
$\liminf_{t\to 0} \frac{sgn(\mu)G(x,t)}{|t|^{d_2}} > 0$, for $x \in \Omega$ uniformly,

(2) If $\lambda = 0,\ \mu \neq 0$, there exists a positive constant $d_2 < \frac{p^-}{1-\mu}$ such that $\liminf_{t\to 0} \frac{sgn(\mu)G(x,t)}{|t|^{d_2}} > 0$, for $x \in \Omega$ uniformly,

(3) If $\lambda \neq 0,\ \mu = 0$, there exists a positive constant $d_1 < \frac{p^-}{1-\mu}$ such that $\liminf_{t\to 0} \frac{sgn(\lambda)F(x,t)}{|t|^{d_1}} > 0$, for $x \in \Omega$uniformly.

Proof . From Theorem 3.4, we know J has a global minimum point u. We just need to show u is nontrivial. We only consider the case $\lambda, \mu \neq 0$ here. From (1), we know that for $0 < t < 1$ small enough, there exists a positive constant C such that

$$sgn(\lambda)F(x,t) \geq C|t|^{d_1}, \quad sgn(\mu)G(x,t) \geq C|t|^{d_2}.$$

When $t > t_0$ from $(\mathbf{m_1})$ we can easily obtain that

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{(1-\mu)}}} := Ct^{\frac{1}{(1-\mu)}},$$

where t_0 is an arbitrary positive constant. Choose $u_0 > 0$. For $0 < t < 1$ small enough, we have

$$\begin{aligned}
J(tu_0) &\leq C\Big(\int_\Omega \frac{1}{p(x)}(|t\nabla u_0|^{p(x)} + |tu_0|^{p(x)})dx\Big)^{\frac{1}{(1-\mu)}} - |\lambda|\int_\Omega sgn(\lambda)F(x,tu_0)\,dx \\
&\quad - |\mu|\int_{\partial\Omega} sgn(\mu)G(x,tu_0)\,d\sigma \\
&\leq C\Big(\frac{t^{p^-}}{p^-}\int_\Omega |u_0|^{p(x)}dx\Big)^{\frac{1}{(1-\mu)}} - |\lambda|\int_\Omega C|tu_0|^{d_1}\,dx \\
&\quad - |\mu|\int_{\partial\Omega} C|tu_0|^{d_2}\,d\sigma \\
&= K_1 t^{\frac{p^-}{1-\mu}} - |\lambda|K_2 t^{d_1} - |\mu|K_3 t^{d_2}.
\end{aligned}$$

Since $d_1, d_2 < \frac{p^-}{1-\mu}$, there exists $0 < t_0 < 1$ small enough such that $J(t_0u_0) < 0$. So the global minimum point u of J is nontrivial. □

Definition 3.6. We say that J satisfies (PS) condition in X, if any sequence (u_n) such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence, where (PS) means Palais-Smale.

Remark 3.7. We know that if we denote

$$\phi(u) = -\lambda\int_\Omega F(x,u)\,dx, \qquad \psi(u) = -\mu\int_{\partial\Omega} G(x,u)\,d\sigma,$$

then by Propositions 2.2, 2.4 and 2.5, they are both weakly continuous and their derivative operators are compact. By Lemma 3.1, we deduce that $J' = I' + \phi' + \psi'$ is also of type (S_+). By [15], to verify that J satisfies the (PS) condition on X, it is enough to verify that any (PS) sequence is bounded.

Lemma 3.8. If $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{m_0})$, $(\mathbf{m_1})$ hold and $\lambda,\ \mu \geq 0$, then J satisfies the (PS) condition.

Proof . Suppose that $(u_n) \subset X,\ |J(u_n)| \leq C$ and $J'(u_n) \to o$. Then

$$\begin{aligned}
C + 1 &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle + \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx\Big) - \lambda\int_\Omega F(x,u_n)dx - \mu\int_{\partial\Omega} G(x,u_n)d\sigma \\
&\quad - \frac{1}{\theta}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx\Big)\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx \\
&\quad - \lambda\int_\Omega f(x,u_n)u_n dx - \mu\int_{\partial\Omega} g(x,u_n)u_n d\sigma\Big] + \frac{1}{\theta}\langle J'(u_n), u_n\rangle
\end{aligned}$$

$$\begin{aligned}
&\geq (1-\mu)M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx\Big)\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx\\
&\quad -\lambda\int_\Omega F(x,u_n)dx-\mu\int_{\partial\Omega}G(x,u_n)d\sigma-\frac{1}{\theta}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx\Big)\\
&\quad -\lambda\int_\Omega f(x,u_n)u_n\,dx-\mu\int_{\partial\Omega}g(x,u_n)u_n\,d\sigma\Big]+\frac{1}{\theta}\langle J'(u_n),u_n\rangle\\
&\geq m_0\Big(\frac{1-\mu}{p^+}-\frac{1}{\theta}\Big)\|u_n\|^{p^-}-\frac{1}{\theta}\|J'(u_n)\|_{X^*}\|u_n\|-C\\
&\geq m_0\Big(\frac{1-\mu}{p^+}-\frac{1}{\theta}\Big)\|u_n\|^{p^-}-\frac{1}{\theta}\|u_n\|-C,
\end{aligned}$$

where $\theta=\min\{\theta_1,\theta_2\}$ and we have supposed that $\|u_n\|>1$ for convenience. Since $\theta>\frac{p^+}{1-\mu}$, we know that (u_n) is bounded in X. □

Theorem 3.9. If M satisfies $(\mathbf{m_0})$, $(\mathbf{m_1})$ and $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_2})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{g_2})$ hold and $\alpha^-,\beta^->p^+$; $\lambda,\mu\geq 0$, then (1.1) has a nontrivial weak solution.

Proof . Let us show that J satisfies the conditions of Mountain Pass Theorem (see Theorem 2.10 of [23]). By Lemma 3.8, J satisfies (PS) condition in X. Since

$$p^+<\alpha^-\leq\alpha(x)<p^*(x),\quad \forall x\in\overline{\Omega};\qquad p^+<\beta^-\leq\beta(x)<p_*(x),\quad \forall x\in\partial\Omega,$$

we have $X\hookrightarrow L^{p^+}(\Omega)$, $X\hookrightarrow L^{p^+}(\partial\Omega)$. Then there exists a constant $C>0$ such that

$$|u|_{L^{p^+}(\Omega)}\leq C\|u\|,\qquad |u|_{L^{p^+}(\partial\Omega)}\leq C\|u\|,\qquad \forall u\in X.$$

From $(\mathbf{f_0})$, $(\mathbf{f_2})$ and $(\mathbf{g_0})$, $(\mathbf{g_2})$, we have there exist an arbitrary constant $0<t<1$ and two positive constants (both denoted by $C(\epsilon)$) such that

$$\begin{aligned}
&|F(x,t)|\leq\epsilon|t|^{p^+}+C(\epsilon)|t|^{\alpha(x)}, &&\text{for all}\quad (x,t)\in\Omega\times\mathbb{R},\\
&|G(x,t)|\leq\epsilon|t|^{p^+}+C(\epsilon)|t|^{\beta(x)}, &&\text{for all}\quad (x,t)\in\partial\Omega\times\mathbb{R}.
\end{aligned}$$

In view of $(\mathbf{m_0})$ and above inequalities, for $\|u\|$ sufficiently small, noting Proposition 2.3, we have

$$\begin{aligned}
J(u)&\geq\frac{m_0}{p^+}\|u\|^{p^+}-\lambda\int_\Omega F(x,u)\,dx-\mu\int_{\partial\Omega}G(x,u)\,d\sigma\\
&\geq\frac{m_0}{p^+}\|u\|^{p^+}-\lambda\int_\Omega(\epsilon|u|^{p^+}+C(\epsilon)|u|^{\alpha(x)})\,dx-\mu\int_{\partial\Omega}(\epsilon|u|^{p^+}+C(\epsilon)|u|^{\beta(x)})\,d\sigma\\
&\geq\frac{m_0}{p^+}\|u\|^{p^+}-(\lambda\epsilon C+\mu\epsilon C)\|u\|^{p^+}-\lambda C(\epsilon)\|u\|^{\alpha^-}-\mu C(\epsilon)\|u\|^{\beta^-}.
\end{aligned}$$

Choose $\epsilon>0$ so small that $0<\lambda\epsilon C+\mu\epsilon C<\frac{m_0}{2p^+}$, we obtain

$$J(u)\geq\frac{m_0}{2p^+}\|u\|^{p^+}-C(\lambda,\mu,\epsilon)C(\|u\|^{\alpha^-}+\|u\|^{\beta^-}).$$

Since $\alpha^-,\beta^->p^+$, there exist $r>0$ small enough and $\delta>0$ such that $J(u)\geq\delta>0$ as $\|u\|=r$.

On the other hand, we have known that the assumption $(\mathbf{f_1})$, $(\mathbf{g_1})$ implies the following assertion:

$$\begin{aligned}
&F(x,t)\geq C|t|^{\theta_1}-M, &&\forall(x,t)\in\Omega\times\mathbb{R},\\
&G(x,t)\geq C|t|^{\theta_2}-M, &&\forall(x,t)\in\partial\Omega\times\mathbb{R}.
\end{aligned}$$

For $t>1$ large enough, we have

$$\begin{aligned}
J(t\widetilde{u})&=\widehat{M}\Big(\int_\Omega\frac{t^{p(x)}}{p(x)}(|\nabla\widetilde{u}|^{p(x)}+|\widetilde{u}|^{p(x)})\,dx\Big)-\lambda\int_\Omega F(x,\widetilde{u})\,dx-\mu\int_{\partial\Omega}G(x,\widetilde{u})\,d\sigma,\\
&\leq\Big(\frac{t^{p^+}}{p^-}\Big)^{\frac{1}{1-\mu}}\Big(\int_\Omega(|\nabla\widetilde{u}|^{p(x)}+|\widetilde{u}|^{p(x)})\,dx\Big)^{\frac{1}{1-\mu}}-\lambda Ct^\theta\int_\Omega|\widetilde{u}|^\theta\,dx-\mu Ct^\theta\int_{\partial\Omega}|\widehat{u}|^\theta\,d\sigma+C\\
&\quad\to-\infty\quad\text{as}\quad t\to+\infty,
\end{aligned}$$

due to $\theta = \min\{\theta_1, \theta_2\} > \frac{p^+}{1-\mu}$. $\square$

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [27]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span } \{e_j \ : j = 1, 2, ...\}}, \qquad X^* = \overline{\text{span } \{e_j^* \ : j = 1, 2, ...\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denote the duality product between X and X^*. We define

$$X_j = \text{ span } \{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 3.10. (Fountain Theorem, see [23]). Let $J \in C^1(X, \mathbb{R})$ be an even functional, where $(X, \|\cdot\|)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(**A1**) $\inf\{J(u) \ : \ u \in Z_k, \ \|u\| = r_k\} \to +\infty$ as $k \to +\infty$.
(**A2**) $\max\{J(u) \ : \ u \in Y_k, \ \|u\| = \rho_k\} \leq 0$.
(**A3**) J satisfies the (PS) condition for every $c > 0$.

Then J has an unbounded sequence of critical points.

Lemma 3.11 (See [16]). If $\alpha(x) \in C_+(\overline{\Omega}), \ \alpha(x) < p^*(x), \ \forall x \in \overline{\Omega}$ and $\beta(x) \in C_+(\partial\Omega), \ \beta(x) < p_*(x), \ \forall x \in \partial\Omega$, denote

$$\alpha_k = \sup\{|u|_{L^{\alpha(x)}(\Omega)}; \ \|u\| = 1, \ u \in Z_k\} \qquad \beta_k = \sup\{|u|_{L^{\beta(x)}(\partial\Omega)}; \ \|u\| = 1, \ u \in Z_k\},$$

then $\lim_{k\to\infty} \alpha_k = 0, \ \lim_{k\to\infty} \beta_k = 0$.

Theorem 3.12. If $(\mathbf{m_0})$, $(\mathbf{m_1})$, $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_3})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{g_3})$ hold and $\alpha^-, \beta^- > p^+, \ \lambda, \mu > 0$, then (1.1) has a sequence of solutions $(\pm u_k, \pm v_k)$ such that $J(\pm u_k, \pm v_k) \to +\infty$ as $k \to +\infty$.

Proof . According to the assumptions on f and g, Remark 3.7, Lemma 3.8, J is an even functional and satisfies Palais-Smale condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that $(A1)$ and $(A2)$ holding. Thus, the conclusion can be obtained from Fountain theorem.

(**A1**) For any $(u) \in Z_k, \ \|u\| > 1$, we have

$$\begin{aligned}
J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \Big) - \lambda \int_\Omega F(x,u)\, dx - \mu \int_{\partial\Omega} G(x,u)\, d\sigma \\
&\geq \frac{m_0}{p^+} \|u\|^{p^-} - \lambda \int_\Omega C(1 + |u|^{\alpha(x)})\, dx - \mu \int_{\partial\Omega} C(1 + |u|^{\beta(x)}) d\sigma \\
&\geq \frac{m_0}{p^+} \|u\|^{p^-} - \lambda C \max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} - \mu C \max\{|u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\Omega)}\} - C \\
&\geq \frac{m_0}{p^+} \|u\|^{p^-} - C(\lambda, \mu) \max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta}(\Omega)}\} - C.
\end{aligned}$$

If $\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta}(\Omega)}\} = |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$, we have

$$J(u) \geq \frac{m_0}{p^+} \|u\| \|u\|^{p^-} - C(\lambda, \mu) \alpha_k^{\alpha^+} \|u\|^{\alpha^+} - C.$$

At this stage, we fix r_k as follows:

$$r_k = \Big(\frac{\alpha^+ C(\lambda, \mu) \alpha_k^{\alpha^+}}{m_0} \Big)^{\frac{1}{p^- - \alpha^+}} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Consequently, if $\|u\| = r_k$ then

$$J(u) \geq m_0\Big(\frac{1}{p^+} - \frac{1}{\alpha^+}\Big) r_k^{p^-} - C \to +\infty \quad \text{as} \quad k \to +\infty,$$

due to $\alpha^+ > \alpha^- > p^+$.

(**A2**) From ($\mathbf{m_1}$), ($\mathbf{f_1}$) and ($\mathbf{g_1}$), we have

$$\begin{aligned}
&\widehat{M}(t) \leq Ct^{\frac{1}{1-\mu}},\\
&F(x,t) \geq C|t|^{\theta_1} - M, \quad \forall (x,t) \in \Omega \times \mathbb{R},\\
&G(x,t) \geq C|t|^{\theta_2} - M, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}.
\end{aligned}$$

Therefore, for any $u \in Y_k$ we have

$$\begin{aligned}
J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx\Big) - \lambda \int_\Omega F(x,u)dx - \mu \int_{\partial\Omega} G(x,u)\, d\sigma\\
&\leq \Big(\frac{1}{p^-}\Big)^{\frac{1}{1-\mu}} \|u\|^{\frac{p^+}{1-\mu}} - \lambda \int_\Omega (C|u|^{\theta_1} - M) - \mu \int_{\partial\Omega} (C|u|^{\theta_2} - M)\, d\sigma\\
&\leq \Big(\frac{1}{p^-}\Big)^{\frac{1}{1-\mu}} \|u\|^{\frac{p^+}{1-\mu}} - \lambda C \int_\Omega |u|^{\theta_1}\, dx - \mu C \int_{\partial\Omega} |u|^{\theta_2} d\sigma + K \to -\infty \quad \text{as} \quad \|u\| \to \infty,
\end{aligned}$$

since $\theta_1, \theta_2 > \frac{p^+}{1-\mu}$ and $\dim Y_k < \infty$. So $(A2)$ holds. From the proofs of $(A1)$ and $(A2)$, we can choose $\rho_k > r_k > 0$. The proof is completed. □

4 The case of concave-convex nonlinearity

In this section, we will obtain much better results with f and g in a special form. We have the following theorem:

Theorem 4.1. Assume the conditions ($\mathbf{m_0}$) and ($\mathbf{m_1}$) hold. And let $\alpha(x) \in C_+(\overline{\Omega})$, $\beta(x) \in C_+(\partial\Omega)$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, $\beta(x) < p_*(x)$ for any $x \in \partial\Omega$ with $\alpha^- > \frac{p^+}{1-\mu}$, $\beta^+ < p^-$ and $f(x,t) = |t|^{\alpha(x)-2}t$, $g(x,t) = |t|^{\beta(x)-2}t$, then we have

- (**i**) For every $\lambda > 0$, $\mu \in \mathbb{R}$, (1.1) has a sequence of weak solutions $(\pm u_k)$ such that $J(\pm u_k) \to +\infty$ as $k \to +\infty$.
- (**ii**) For every $\mu > 0$, $\lambda \in \mathbb{R}$, (1.1) has a sequence of weak solutions $(\pm v_k)$ such that $J(\pm v_k) \to 0$ as $k \to +\infty$.

We will use Lemma 3.10 to prove Theorem 4.1 (**i**) and the following "Dual fountain theorem" to prove Theorem 4.1 (**ii**), respectively.

Lemma 4.2. (Dual Fountain Theorem, see [23]). Assume (**A1**) is satisfied and there is $k_0 > 0$ so that, for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that

- (**B1**) $a_k = \inf\{J(u) \ : \ u \in Z_k,\ \|u\| = \rho_k\} \geq 0$.
- (**B2**) $b_k = \max\{J(u) \ : \ u \in Y_k,\ \|u\| = r_k\} < 0$.
- (**B3**) $d_k = \inf\{J(u) \ : \ u \in Z_k,\ \|u\| \leq \rho_k\} \to 0$ as $k \to +\infty$.
- (**B4**) J satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.

Then J has a sequence of negative critical values converging to 0.

Definition 4.3. We say that J satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$, contain a subsequence converging to a critical point of J.

Lemma 4.4. Assume that the conditions in Theorem 4.1 hold, then J satisfies the $(PS)_c^*$ condition.

Proof . Suppose $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$. Assume $\|u_{n_j}\| > 1$ for convenience. If $\lambda \geq 0$, for n large enough, we have

$$\begin{aligned}
C+1 &\geq J(u_{n_j}) - \frac{1}{\alpha^-}\langle J'(u_{n_j}), (u_{n_j})\rangle + \frac{1}{\alpha^-}\langle J'(u_{n_j}), (u_{n_j})\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})\,dx\Big) - \lambda \int_\Omega F(x, u_{n_j})dx - \mu \int_{\partial\Omega} G(x, u_{n_j})\,d\sigma \\
&\quad - \frac{1}{\alpha^-}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})\,dx\Big) - \lambda\int_\Omega f(x, u_{n_j})u_{n_j}\,dx \\
&\quad - \mu \int_{\partial\Omega} g(x,_{n_j})u_{n_j} d\sigma\Big] + \frac{1}{\alpha^-}\langle J'(u_{n_j}), u_{n_j}\rangle \\
&\geq \Big(\frac{1-\mu}{p^+} - \frac{1}{\alpha^-}\Big) M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})\,dx\Big)\int_\Omega (|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})\,dx \\
&\quad + \mu \int_{\partial\Omega}\Big(\frac{1}{\alpha^-} - \frac{1}{\beta(x)}\Big)|u_{n_j}|^{\beta(x)}\,d\sigma \\
&\geq m_0\Big(\frac{1-\mu}{p^+} - \frac{1}{\alpha^-}\Big)\|u_{n_j}\|^{p^-} - K\|u_{n_j}\|^{\beta^+}.
\end{aligned}$$

Since $p^- > \beta^+$ and $\alpha^- > \frac{p^+}{1-\mu}$, we deduce that (u_{n_j}) is bounded in X.
If $\lambda < 0$, for n large enough, we can consider the inequality below to get the boundedness of (u_{n_j}).

$$C+1 \geq J(u_{n_j}) - \frac{1}{\alpha^+}\langle J'(u_{n_j}), u_{n_j}\rangle + \frac{1}{\alpha^+}\langle J'(u_{n_j}), u_{n_j}\rangle.$$

Going if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
\lim_{n_j\to+\infty}\langle J'(u_{n_j}), u_{n_j} - u\rangle &= \lim_{n_j\to+\infty}\langle J'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \lim_{n_j\to+\infty}\langle J'(u_{n_j}), v_{n_j} - u\rangle \\
&= \lim_{n_j\to+\infty}\Big\langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\Big\rangle \\
&= 0.
\end{aligned}$$

As J' is of type (S_+), we can conclude $u_{n_j} \to u$, furthermore we have $J'(u_{n_j}) \to J'(u)$.
Let us prove $J'(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}
\langle J'(u), \omega_k\rangle &= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \langle J'(u_{n_j}), \omega_k\rangle \\
&= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \Big\langle (J|_{Y_{n_j}})'(u_{n_j}), \omega_k\Big\rangle.
\end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle J'(u), \omega_k\rangle = 0, \qquad \forall \omega_k \in Y_k,$$

so $J'(u) = 0$, this show that J satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$. $\square$

Proof of Theorem 4.1

(**i**) The proof is similar to that of Theorem 3.12 if we use the Fountain theorem, and the proof of the boundedness of (PS) sequence is same as in Lemma 4.4, we know that J satisfies (**A1**) and (**B4**), the assertion of conclusion can be obtained from Dual fountain theorem. Now, it remains to prove that there exist $\rho_k > r_k > 0$ such that if k is large enough (**B1**), (**B2**) and (**B3**) are satisfied.
(**B1**) Let $u \in Z_k$, then

$$\begin{aligned}
J(u) &\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx - \frac{\mu}{\beta^-}\int_{\partial\Omega}|u|^{\beta(x)}\,d\sigma \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{C|\lambda|}{\alpha^-}\|u\|^{\alpha^-} - \frac{\mu}{\beta^-}\max\Big\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\}.
\end{aligned}$$

There exists $0<\rho_1<1$ small enough such that $\frac{C|\lambda|}{\alpha^-}\|u\|^{\alpha^-}\le \frac{m_0}{p^+}\|u\|^{p^+}$ as $0<\rho=\|u\|\le\rho_1$. Then we have

$$J(u)\ge \frac{m_0}{p^+}\|u\|^{p^+}-\frac{\mu}{\beta^-}\max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)},|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\}.$$

If $\max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)},|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\}=|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}$, then

$$J(u)\ge \frac{m_0}{p^+}\|u\|^{p^+}-\frac{\mu}{\beta^-}\beta_k^{\beta^+}\|u\|^{\beta^+}.$$

Choose $\rho_k=\left(\frac{2p^+\mu\beta_k^{\beta^+}}{m_0\beta^-}\right)^{\frac{1}{p^+-\beta^+}}$, then

$$J(u)\ge \frac{m_0}{2p^+}(\rho_k)^{p^+}-\frac{m_0}{2p^+}(\rho_k)^{p^+}=0.$$

Since $p^->\beta^+$, $\beta_k\to 0$, we know $\rho_k\to 0$ as $k\to+\infty$. If $\max\left\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)},|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\right\}=|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}$, we can do the same work as the case above. So (**B1**) is satisfied.
(**B2**) For $u\in Y_k$ with $\|u\|\le 1$, we have

$$\begin{aligned}
J(u)&=\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx\Big)-\lambda\int_\Omega F(x,u)\,dx-\mu\int_{\partial\Omega}G(x,u)\,d\sigma\\
&\le M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\Big)\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})\,dx-\lambda\int_\Omega\frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx\\
&\quad-\mu\int_{\partial\Omega}\frac{1}{\beta(x)}|u|^{\beta(x)}\,d\sigma\\
&\le C\|u\|^{p^-}+\frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\,dx-\frac{\mu}{\beta^+}\int_{\partial\Omega}|u|^{\beta(x)}\,d\sigma.
\end{aligned}$$

Since $\dim Y_k=k$, conditions $\beta^+<p^-$ and $p^+<\frac{p^+}{1-\mu}<\alpha^-$ imply that there exists a $r_k\in(0,\rho_k)$ such that $J(u)<0$ when $\|u\|=r_k$. Hence $b_k=\max\{J(u)\ :\ u\in Y_k,\ \|u\|=r_k\}<0$, so (**B2**) is satisfied.
(**B3**) Because $Y_k\cap Z_k\ne\emptyset$ and $r_k<\rho_k$, we have

$$d_k=\inf\{J(u)\ :\ u\in Z_k,\ \|u\|\le\rho_k\}\ \le b_k=\max\{J(u)\ :\ u\in Y_k,\ \|u\|=r_k\}<0.$$

In view of the proof of (**B1**), we have $J(u)\ge -\frac{\mu}{\beta^-}\beta_k^{\beta^+}\|u\|^{\beta^+}$ or $-\frac{\mu}{\beta^-}\beta_k^{\beta^-}\|u\|^{\beta^-}$. Since $\beta_k\to 0$ and $\rho_k\to 0$ as $k\to+\infty$, (**B3**) is satisfied. The conclusion of Theorem 4.1 (**ii**) is reached by the Dual fountain theorem.

References

[1] G.A. Afrouzi, M. Mirzapour and N.T. Chung, *Existence and multiplicity of solutions for Kirchhoff type problems involving $p(\cdot)$-biharmonic operators*, Z. Anal. Anwend., **33** (2014), 289–303.

[2] G.A. Afrouzi, M. Mirzapour and N.T. Chung, *Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type equation*, Rend. Sem. Mat. Univ. Padova **136** (2016), 95–109.

[3] G.A. Afrouzi and M. Mirzapour, *Eigenvalue problems for $p(x)$-Kirchhoff type equations*, Electron. J. Differ. Equ. **253** (2013), 1–10.

[4] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[5] E. Acerbi and G. Mingione, *Gradient estimate for the $p(x)$-Laplacian system*, J. Reine Angew. Math. **584** (2005), 117–148.

[6] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.

[7] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. **30** (1997), 4619-4627.

[8] J.M.B. do O, *Nontrivial solutions for resonant cooperative elliptic systems via computations of the critical groups*, J. Diff. Equ. **11** (1997), 1–15.

[9] D.E. Edmunds, J. Rákosník, *Density of smooth functions in* $W^{k,p(x)}(\Omega)$, Proc. R. Soc. A **437** (1992), 229-236.

[10] D.E. Edmunds and J. Rákosník, *Sobolev embedding with variable exponent*, Studia Math. **143** (2000), 267–293.

[11] X.L. Fan and X.Y. Han, *Existence and multiplicity of solutions for* $p(x)$*-Laplacian equations in* $\mathbb{R}^N$, Nonlinear Anal. **59** (2004), 173–188.

[12] X.L. Fan and D. Zhao, *On the generalized Orlicz-Sobolev spaces* $W^{k,p(x)}(\Omega)$, J. Gansu Educ. College **12** (1998), no. 1, 1–6.

[13] X.L. Fan, J.S. Shen and D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. **262** (2001), 749–760.

[14] X.L. Fan and D. Zhao, *On the spaces* $L^{p(x)}$ *and* $W^{m,p(x)}$, J. Math. Anal. Appl., **263** (2001), 424–446.

[15] X.L. Fan and Q.H. Zhang, *Existence of solutions for* $p(x)$*-Laplacian Dirichlet problems*, Nonlinear Anal. **52** (2003), 1843–1852.

[16] X.L. Fan, *Solutions for* $p(x)$*-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl. **312** (2005), 464–477.

[17] D.D. Hai and R. Shivaji, *An existence result on positive solutions of* p*-Laplacian systems*, Nonlinear Anal. **56** (2004), 1007–1010.

[18] D.W. Huang and Y.Q. Li, *Multiplicity of solutions for a noncooperative* p*-Laplacian elliptic system in* $\mathbb{R}^N$, J. Differ. Equ. **215** (2005), no. 1, 206–223.

[19] J.L. Lions, *On some questions in boundary value problems of mathematical physics*, Proc. Int. Symp. Continuum Mech. Partial Differ. Equ., Rio de Janeiro 1977, in: de la Penha, Medeiros (Eds.), Math. Stud., North-Holland, **30** (1978), pp. 284–346.

[20] T.F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. **63** (2005), 1967–1977.

[21] S. Ma, *Nontrivial solutions for resonant cooperative elliptic systems via computations of the critical groups*, Nonlinear Anal. **73** (2010), 3856–3872.

[22] K. Perera and Z.T. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differ. Equ. **221** (2006), 246–255.

[23] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[24] T.F. Wu, *The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions*, Nonlinear Anal. **68** (2008), 1733–1745.

[25] J. Yao, *Solutions for Neumann boundary value problems involving* $p(x)$*-Laplace operators*, Nonlinear Anal. **68** (2008), 1271–1283.

[26] G. Zhang and Y. Wang, *Some existence results for a class of degenerate semilinear elliptic systems*, J. Math. Anal. Appl. **333** (2007), 904–918.

[27] J.F. Zhao, *Structure Theory of Banach Spaces*, Wuhan University Press, Wuhan, 1991.

[28] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv. **9** (1987), 33–66.

[29] N.B. Zographopoulos, *On a class of degenarate potential elliptic system*, Nonlinear Diff. Equ. Appl. **11** (2004), 191–199.

[30] N.B. Zographopoulos, p*-Laplacian systems on resonance*, Appl. Anal. **83** (2004), 509–519.

[31] N.B. Zographopoulos, *On the principal eigenvalue of degenerate quasilinear elliptic systems*, Math. Nachr. **281** (2008), no. 9, 1351–1365.