Int. J. Nonlinear Anal. Appl. 14 (2023) 8, 95–105
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.7226

Solving two-dimensional nonlinear Volterra integral equations using rationalized Haar functions

Majid Erfanian[a,*], Hamed Zeidabadi[b]

[a]Department of Science, School of Mathematical Sciences, University of Zabol, Zabol, Iran

[b]Faculty of Engineering, Sabzevar University of New Technology, Sabzevar, Iran

(Communicated by Asadollah Aghajani)

Abstract

In this paper, we have introduced a computational method for a class of two-dimensional nonlinear Volterra integral equations, based on the expansion of the solution as a series of Haar functions. To achieve this aim it is necessary to define the integral operator. The Banach fixed point theorem guarantees that under certain assumptions this operator has a unique fixed point, we have introduced an orthogonal projection and by interpolation property, we have achieved an operational matrix of integration. Also, by using the Banach fixed point theorem, we get an upper bound for the error of our method. Since our examples in this article are selected from different references, so should be the numerical results obtained here can be compared with other numerical methods.

Keywords: Two-dimensional integral equations, Rationalized Haar wavelet, Operational matrix, Fixed point theorem, Error analysis
2020 MSC: 47A56, 45B05, 47H10, 42C40

1 Introduction and preliminaries

The base of our work is to apply the Rationalized Haar wavelet (RHW) for approximating the two-dimensional linear Volterra integral equations(VIE). In fact, the wavelet theory has been opened a new and powerful area in mathematical research, science, and engineering, such as Darboux equation, Painleve equations, telegraph equations, consolidation equation [29, 13, 14, 15, 27]. A Haar wavelet is a group of square waves with magnitude $-2^{j/2}$, $2^{j/2}$ and 0 for $j = 0, 1, ...$, and the simplest type of wavelet, that was proposed in 1909 by Alfred Haar[28]. One of the pioneering researchers in developing of the HW is Ülo Lepik. He introduced the integration techniques for the HWM [21, 22]. Lynch and Reis's [20] have Rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two, and are composed of only three amplitude $+1$, -1, and 0. Recently, several articles have been published to approximate some equations containing integral operators using RHW method[17, 16]. An Eextension of RHW to the multidimensional cases is also introduced in [1, 3, 2]. Brunner and Kauthen in [8], One of the pioneering researchers for introduced collocation and iterated collocation methods for the solution of VIE. For approximating the solution of two-dimensional integral equations several methods have been used such as: the meshless scheme [4], the Bernstein collocation method [23], the Haar wavelet in [6, 12], using Legendre polynomials[26], Extrapolation

*Corresponding author
Email addresses: erfaniyan@uoz.ac.ir (Majid Erfanian), h.zeidabadi@yahoo.com (Hamed Zeidabadi)

Received: March 2022 *Accepted:* September 2022

method[18].
In this work we approximating the solution of nonlinear second-kind VIE as follows:

$$u(t,s) = f(t,s) + \int_0^s \int_0^t W_1(t,s,x,y,u(x,y))dxdy + \alpha \int_0^s W_2(t,s,y,u(t,y))dy + \beta \int_0^t W_3(t,s,x,u(x,s))dx, \quad (1.1)$$

that $\alpha, \beta \in \mathbb{R}$ and unknown function to be determined. Also,

$$\begin{aligned} &f : [0,1]^2 \to \mathbb{R}^2, && t,s \in [0,1], \\ &W_1 : [0,1]^4 \times \mathbb{R}^2 \to \mathbb{R}^2, && x,y \in [0,1], \\ &W_2, W_3 : [0,1]^3 \times \mathbb{R}^2 \to \mathbb{R}^2, && \end{aligned} \quad (1.2)$$

are assumed to be known continuous functions satisfying the Lipschitz condition

$$|W_1(t,s,x,y,v_1(x,y)) - W_1(t,s,x,y,v_2(x,y))| \leqslant M_1|v_1 - v_2|,$$

$$|W_2(t,s,y,v_1(t,y)) - W_2(t,s,y,v_2(x,y))| \leqslant M_2|v_1 - v_2|,$$
$$|W_3(t,s,x,v_1(x,s)) - W_3(t,s,x,v_2(x,y))| \leqslant M_3|v_1 - v_2|,$$

where $v_1, v_2 \in \mathbb{R}^2$, and $M_1, M_2, M_3 \geqslant 0$.
At first, we give two real examples of the applications of this equation.

Darboux problem

the Darboux problem has been considered for $a, b, a_0, b_0 \in [0, \infty)$ as [19]

$$\begin{aligned} D_{xy}z(x,y) &= f(x,y,z(x,y)), \quad (x,y) \in \ E = (0,a] \times (0,b], \\ z(x,y) &= \phi(x,y) \quad for \quad (x,y) \in ([-a_0,a] \times [-b_0,b]) \setminus E, \\ D_{xy}z &= \frac{\partial^2 z}{\partial x \partial y}, \quad B = [-a_0,0] \times [-b_0,b]. \end{aligned}$$

and

$$f : \overline{E} \times C(B,R) \to R, \quad z : E^0 \cup E \to R,$$

If $E = (0,1] \times (0,1]$ this problem is equivalent to[11]

$$u(t,s) = u(t,0) + u(0,s) + \int_0^s \int_0^t W(x,y,u(x,y))dxdy - u(0,0). \quad (1.3)$$

Telegraph equation
Another example can be cited as a telegraph equation

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx},$$

where $c^2 = \frac{1}{LC}$, $\alpha = \frac{G}{C}$, $\beta = \frac{R}{L}$, which consists of a resistor of resistance Rdx, a coil of inductance, Ldx, a resistor of conductance Gdx or a capacitor of capacitance Cdx.
In [25] it has been shown that the telegraph equation can be reduced to an equation of the form (1), if $\alpha, \beta \neq 0$, and

$$W_1(t,s,x,y,u(x,y)) = g_1(x,y,u(x,y)),$$

$$W_2(t,s,y,u(t,y)) = g_2(t,y,u(t,y)),$$
$$W_3(t,s,x,u(x,s)) = g_3(x,s,u(x,s)).$$

2 Properties of the Rationalized Haar functions

In this section, we introduce briefly the RHWs and then construct the operational matrices of iteration for the RHWs set.

Definition 2.1. The RH wavelet is the function defined on the real line $\mathbb{R}$ as follows:

$$H(t)=\begin{cases} 1, & 0<t\leq \frac{1}{2}, \\ -1, & \frac{1}{2}<t<1, \\ 0, & otherwise. \end{cases} \tag{2.1}$$

Definition 2.2. The RH functions $h_n(t)$ for any $n=1,2,\dots$, where $n=2^i+j$, for $i=0,1,\dots$ and $j=0,1,\dots,2^i-1$, are defined by

$$h_n(t)=H(2^it-j)=\begin{cases} 1 & j.2^{-i}\leq t<(j+\frac{1}{2})2^{-i} \\ -1 & (j+\frac{1}{2})2^{-i}\leq t<(j+1)2^{-i} \\ 0 & \text{otherwise.} \end{cases}$$

Also, we can expand for any function such as $f(t)\in C[0,1]$ with RH function as

$$f(t)=\sum_{n=0}^{\infty} a_n h_n(t), \tag{2.2}$$

where

$$a_n=2^j\int_0^1 f(t)h_n(t)dx.$$

Thus the series $\sum_n 2^j\langle f,h_n\rangle h_n$, will converge to f, see e.g. [29], if (2.2) truncated up to its first m terms that $m=2^{\lambda+1}$, $\lambda=0,1,\dots$ where $j=0,1,\dots,\lambda$ then:

$$f(t)=\sum_{n=0}^{m-1} a_n h_n(t)=\mathbf{A}^T\mathbf{h}(t), \tag{2.3}$$

where $\mathbf{A}=[a_0,a_1,...,a_{m-1}]^T$, and $\mathbf{h}(t)=[h_0(t),h_1(t),\dots,h_{m-1}(t)]^T$. Also, for any function $f(t,s)$ we can approximated by RH functions as

$$f(t,s)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} f_{ij}h_{ij}(t,s)=\mathbf{F}^T\mathbf{h}(t,s), \tag{2.4}$$

that

$$h_{ij}(t,s)=h_i(t)h_j(s),$$

and

$$\mathbf{F}(t,s)=[f_{00},f_{01},...,f_{m-1,m-1}]^T, \qquad \mathbf{h}(t,s)=[h_{00},h_{01},...,h_{m-1,m-1}]^T. \tag{2.5}$$

In addition, f_{ij} are defined in (2.4) are given by

$$f_{ij}=\frac{\langle f(t,s),h_{ij}(t,s)\rangle}{\|h_{ij}(t,s)\|_2^2}.$$

3 Numerical approximation of the solution

In this section for achieve it is necessary to define the integral operator, $T:(X,\|.\|_\infty)\to(X,\|.\|_\infty)$. By applying this operator in Eq (1), we have

$$T(u(t,s))=f(t,s)+\int_0^s\int_0^t W_1(t,s,x,y,u(x,y))dxdy+\alpha\int_0^s W_2(t,s,y,u(t,y))dy+\beta\int_0^t W_3(t,s,x,u(x,s))dx. \tag{3.1}$$

In book of Atkinson by using of Banach fixed point theorem he is proved under certain assumptions T has an unique fixed point [5]. Since, in general it is not possible to calculate u explicitly from the sequence of functions $\{T^n(u)\}_{n\in\mathbb{N}}$, so we define a new sequence of functions, denoted by $\{u_i\}_{i\in\mathbb{N}}$, obtained recursively by using interpolation and RH basis.

$$u_i(t,s) = f(t,s) + \int_0^s \int_0^t W_1(t,s,x,y,u_{i-1}(x,y))dxdy + \alpha \int_0^s W_2(t,s,y,u_{i-1}(t,y))dy + \beta \int_0^t W_3(t,s,x,u_{i-1}(x,s))dx. \tag{3.2}$$

In this section we assume:

$$\psi_1^{(i-1)}(t,s,x,y) := W_1(t,s,x,y,u_{i-1}(x,y)), \tag{3.3}$$

$$\psi_2^{(i-1)}(t,s,y) := W_2(t,s,y,u_{i-1}(t,y)), \tag{3.4}$$

$$\psi_3^{(i-1)}(t,s,x) := W_3(t,s,x,u_{i-1}(x,s)), \tag{3.5}$$

we can expand $\psi_i^{(i-1)}$ for $i = 1,2,3$ in terms of RH functions as

$$\psi_1^{(i-1)}(t,s,x,y) = h^T(t,s)K_1h(x,y),$$
$$\psi_2^{(i-1)}(t,s,y) = h^T(t,s)K_2h(t,y),$$
$$\psi_1^{(i-1)}(t,s,x) = h^T(t,s)K_3h(x,s),$$

if Q_m is an orthogonal projection with following interpolation property we have

$$Q_m(\psi_1^{(i-1)}(t,s,x,y)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k^1_{ijrq}h_{ij}(t,s)h_{rq}(x,y), \tag{3.6}$$

$$Q_m(\psi_2^{(i-1)}(t,s,y)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k^2_{ijrq}h_{ij}(t,s)h_{rq}(t,y), \tag{3.7}$$

$$Q_m(\psi_3^{(i-1)}(t,s,x)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k^3_{ijrq}h_{ij}(t,s)h_{rq}(x,s). \tag{3.8}$$

Thus K_1, K_2, K_3 are block matrices of the form

$$K_s = [K_s^{(i,j)}]_{i,j=0}^{m-1}, \quad s = 1,2,3, \tag{3.9}$$

where

$$K_s^{(i,j)} = [K^s_{ijrq}]_{i,j,r,q=0}^{m-1}, \quad s = 1,2,3, \tag{3.10}$$

and for coefficients $K^s_{ijrq},\ s = 1,2,3$ we have:

$$k^1_{ijrq} = \frac{\langle W_1(t,s,x,y,u(x,y)), h_{rq}(x,y)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(x,y), h_{rq}(x,y)}, \tag{3.11}$$

$$k^2_{ijrq} = \frac{\langle W_2(t,s,y,u(t,y)), h_{rq}(t,y)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(t,y), h_{rq}(t,y)}, \tag{3.12}$$

$$k^3_{ijrq} = \frac{\langle W_3(t,s,x,u(x,s)), h_{rq}(x,s)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(x,s), h_{rq}(x,s)}. \tag{3.13}$$

The row vector below has been introduced by Chen and Hsiao [10]

$$\mathbf{h}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T. \tag{3.14}$$

Now we have

$$\int_0^t \mathbf{h}(s)ds = \mathbf{Ph}(t), \tag{3.15}$$

Chen and Hsiao in [10] shown that the following recursive formula for operational matrix of integration $\mathbf{P}_{m\times m}$ are holds

$$\mathbf{P}_{m\times m} = \frac{1}{2m}\begin{pmatrix} 2k\ \mathbf{P}_{(\frac{m}{2})\times(\frac{m}{2})} & -\hat{\mathbf{\Phi}}_{(\frac{m}{2})\times(\frac{m}{2})} \\ \hat{\mathbf{\Phi}}^{-1}_{(\frac{m}{2})\times(\frac{m}{2})} & \mathbf{0} \end{pmatrix}, \tag{3.16}$$

and

$$\hat{\mathbf{\Phi}}_{m\times m} = [\mathbf{h}(\frac{1}{2m}), \mathbf{h}(\frac{3}{2m}), \ldots, \mathbf{h}(\frac{2m-1}{2m})]. \tag{3.17}$$

For example

$$\hat{\mathbf{\Phi}}_{8\times 8} = \begin{pmatrix} \mathbf{h}_0(t) \\ \mathbf{h}_1(t) \\ \vdots \\ \mathbf{h}_7(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{3.18}$$

Which $\hat{\mathbf{\Phi}}_{1\times 1} = [1]$, $\mathbf{P}_{1\times 1} = [\frac{1}{2}]$, and

$$\hat{\mathbf{\Phi}}^{-1}_{k\times k} = (\frac{1}{k}).\hat{\mathbf{\Phi}}^{\mathbf{T}}_{k\times k}.\mathbf{diag}\Big(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{\frac{k}{2}}\Big). \tag{3.19}$$

Thus for the nonlinear second-kind VIE we have:

$$u_i(t,s) = f(t,s) + \int_0^s \int_0^t Q_m(\psi_1^{(i-1)}(t,s,x,y))dxdy + \alpha \int_0^s Q_m(\psi_2^{(i-1)}(t,s,y))dy + \beta \int_0^t Q_m(\psi_3^{(i-1)}(t,s,x))dx. \tag{3.20}$$

4 Error analysis

Suppose that, two-dimensional function $u(t,s) : [0,1]^2 \to \mathbf{R}^2$ be an arbitrary continuous function, we define

$$||u||_\infty = sup\{|u(t,s)|; (t,s) \in [0,1] \times [0,1]\}. \tag{4.1}$$

So, in this section, by using the Banach fixed point theorem, we proved our method is convergence and the order of method is 2

Lemma 4.1. Let $W_1 \in C([0,1]^4 \times \mathbb{R}^2)$, and $W_2, W_3 \in C([0,1]^3 \times \mathbb{R}^2)$ are Lipschitz functions with respect to their fifth and fourth variables, with Lipchitz constants M_1, M_2 and M_3, then T has an unique fixed point and for all $u_0 \in C([0,1]^2)$

$$\|u - T^i(u_0)\|_\infty \leq \|T(u_0) - u_0\|_\infty \times \sum_{j=i}^{\infty} q^j, \tag{4.2}$$

where $q = max\{M_1, |\alpha|M_2, |\beta|M_3\} < \frac{1}{4}$, and u is the fixed point of T.

Proof: For $u, v \in C([0,1]^2)$, we have:

$$\begin{aligned} &|T(u(t,s)) - T(v(t,s))| = |\int_0^s \int_0^t (W_1(t,s,x,y,u(x,y)) - W_1(t,s,x,y,v(x,y)))dxdy \\ &+ \alpha \int_0^s (W_2(t,s,y,u(t,y)) - W_2(t,s,y,v(t,y)))dy + \beta \int_0^t (W_3(t,s,x,u(x,s)) - W_3(t,s,x,v(x,s)))dx| \\ &\leq \int_0^s \int_0^t |W_1(t,s,x,y,u(x,y)) - W_1(t,s,x,y,v(x,y))|dxdy + |\alpha| \int_0^s |W_2(t,s,y,u(t,y)) - W_2(t,s,y,v(t,y))|dy \\ &+|\beta| \int_0^t |W_3(t,s,x,u(x,s)) - W_3(t,s,x,v(x,s))|dx \\ &\leq M_1 \int_0^s \int_0^t |u(x,y) - v(x,y)|dxdy + M_2|\alpha| \int_0^s |u(t,y) - v(t,y)|dy + M_3|\beta| \int_0^t |u(x,s) - v(x,s)|dx \\ &\leq M_1||u-v||_\infty + M_2|\alpha|||u-v||_\infty + |\beta|M_3||u-v||_\infty \leq (M_1 + |\alpha|M_2 + |\beta|M_3)||u-v||_\infty. \end{aligned}$$

By induction, for all $n \in \mathbb{N}$ we have

$$\|T^n(u) - T^n(v)\|_\infty \leq q^n \|u - v\|_\infty,$$

since $q < 1$ we have:

$$\sum_{n=1}^{\infty} \|T^n(u) - T^n(v)\|_\infty < \infty.$$

Then T has a unique fixed point and (3.1) has a unique solution. □

Theorem 4.2. Assume that $\psi_1^{(i-1)} \in C([0,1]^4)$, and $\psi_2^{(i-1)}, \psi_3^{(i-1)} \in C([0,1]^3)$ and $\{u_i\}_{i\geq 1}$ is a subset of $C([0,1]^2)$, and $W_1 \in C([0,1]^4 \times \mathbb{R}^2)$, and $W_2, W_3 \in C([0,1]^3 \times \mathbb{R}^2)$ are Lipschitz functions with respect to their fifth and fourth variables, then we have

$$\|u - u_i\|_\infty \leq \|T(u_0) - u_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j, \tag{4.3}$$

Proof: If

$$L_{i-1} = \max\{\|\frac{\partial \psi_k^{i-1}}{\partial t}\|_\infty, \|\frac{\partial \psi_k^{i-1}}{\partial s}\|_\infty \|\frac{\partial \psi_k^{i-1}}{\partial x}\|_\infty, \|\frac{\partial \psi_k^{i-1}}{\partial y}\|_\infty\}, \tag{4.4}$$

for $k = 1, 2, 3$ and $m = 2^{i+1}$ for $i = 1, \ldots,$ then

$$\begin{aligned}
\|T(u_{i-1}) - u_i\|_\infty \leq & \| \int_0^s \int_0^t \psi_1^{(i-1)}(t,s,x,y) - Q_m(\psi_1^{(i-1)}(t,s,x,y))dxdy\|_\infty \\
& + |\alpha| \; \| \int_0^s \psi_2^{(i-1)}(t,s,y) - Q_m(\psi_2^{(i-1)}(t,s,y))dy\|_\infty \\
& + |\beta| \; \| \int_0^t \psi_3^{(i-1)}(t,s,x) - Q_m(\psi_3^{(i-1)}(t,s,x))dx\|_\infty \\
\leq & \|\psi_1^{i-1} - Q_m(\psi_1^{i-1})\|_\infty + |\alpha| \; \|\psi_2^{i-1} - Q_m(\psi_2^{i-1})\|_\infty + |\beta| \; \|\psi_3^{i-1} - Q_m(\psi_3^{i-1})\|_\infty.
\end{aligned}$$

If we define

$$g(t,s,x,y) := \psi^{i-1} - Q_m(\psi^{i-1}),$$

by using interpolating property and the mean-value theorem for four variables with $t_0, s_0, x_0, y_0 = 0$ and

$$\begin{aligned}
t_i &= \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \quad for \quad i = 2^{n_1} + v_1, \\
s_j &= \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \quad for \quad j = 2^{n_2} + v_2, \\
x_k &= \frac{1}{2^{n_3+1}} + \frac{v_3}{2^{n_3}}, \quad for \quad k = 2^{n_3} + v_3, \\
y_l &= \frac{1}{2^{n_4+1}} + \frac{v_4}{2^{n_4}}, \quad for \quad l = 2^{n_4} + v_4,
\end{aligned}$$

where $n_1, n_2, n_3, n_4 \geq 1$, and $i, j, k, l \leq m-1$, we have

$$\begin{aligned}
\|\psi^{i-1} - Q_m(\psi^{i-1})\|_\infty =& \|g(t_i, s_j, x_k, y_l) + \frac{\partial g}{\partial t}(\xi, \gamma, \tau, \upsilon)(\xi - t_i) + \frac{\partial g}{\partial s}(\xi, \gamma, \tau, \upsilon)(\gamma - s_j) + \\
& \frac{\partial g}{\partial x}(\xi, \gamma, \tau, \upsilon)(\tau - x_k) + \frac{\partial g}{\partial y}(\xi, \gamma, \tau, \upsilon)(\upsilon - y_l)\|_\infty \\
=& \|(I - Q_m)\frac{\partial \psi^{i-1}}{\partial t}(\xi, \gamma, \tau, \upsilon) + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial s}(\xi, \gamma, \tau, \upsilon) \\
& + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial x}(\xi, \gamma, \tau, \upsilon) + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial y}(\xi, \gamma, \tau, \upsilon)\|_\infty \\
& \max\{\|\xi - t_i\|_\infty, \|\gamma - s_j\|_\infty, \|\tau - x_k\|_\infty, \|\upsilon - y_l\|_\infty\} \\
\leq & \frac{2}{2^i}\|(I - Q_m)\|_\infty \|\|\frac{\partial \psi^{i-1}}{\partial t}(\xi, \gamma, \tau, \upsilon) + \frac{\partial \psi^{i-1}}{\partial s}(\xi, \gamma, \tau, \upsilon) \\
& + \frac{\partial \psi^{i-1}}{\partial x}(\xi, \gamma, \tau, \upsilon) + \frac{\partial \psi^{i-1}}{\partial y}(\xi, \gamma, \tau, \upsilon)\|_\infty \leq \frac{8L_{i-1}}{2^i}.
\end{aligned}$$

The same proof holds for ψ_k^{i-1} for $k = 2, 3$. We have

$$\|T(u_{i-1}) - u_i\|_\infty \leq (4 + 3|\alpha| + 3|\beta|)\frac{2L_{i-1}}{2^i}. \tag{4.5}$$

For certain constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i > 0$ that $i \geq 1$, if

$$(4 + 3|\alpha| + 3|\beta|)\frac{2L_{i-1}}{2^i} < \varepsilon_k \quad for \quad k = 1, 2, ..., i,$$

we have

$$\|T(u_{i-1}) - u_i\|_\infty < \varepsilon_i, \quad i \geq 1. \tag{4.6}$$

By applying the triangle inequality, we achieve

$$\|u - u_i\|_\infty \leq \|u - T^i(u_0)\|_\infty + \sum_{j=1}^{i} q^j \|T(u_{j-1}) - u_j\|_\infty. \tag{4.7}$$

From (4.2) and (4.6) we conclude

$$\|u - u_i\|_\infty \leq \|T(u_0) - u_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j. \tag{4.8}$$

Also, since $q < \frac{1}{4}$, the geometric series $\sum_{j=n}^{\infty} q^j = \frac{q^n}{1-q}$ is convergent. we set

$$q = \frac{1}{4} - \frac{1}{2^{l+2}} < \frac{1}{4}, \qquad l \in \mathbb{N}, \tag{4.9}$$

By using Eqs. (4.9), (4.8), and (4.5), we have

$$\begin{aligned}
\|u - u_i\|_\infty &\leq \|Tu_0 - u_0\|_\infty \frac{q^i}{1-q} + \sum_{j=1}^{i}(\frac{1}{4} - \frac{1}{2^{l+2}})^{i-j}(4 + 3|\alpha| + 3|\beta|)\frac{2L_{j-1}}{2^j} \\
&= \|Tu_0 - u_0\|_\infty \frac{q^i}{1-q} + \sum_{j=1}^{i}(\frac{1}{4} - \frac{1}{2^{l+2}})^{i}(\frac{1}{4} - \frac{1}{2^{l+2}})^{-j}(4 + 3|\alpha| + 3|\beta|)\frac{2L_{j-1}}{2^j} \\
&= \|Tu_0 - u_0\|_\infty \frac{q^i}{1-q} + 2(4 + 3|\alpha| + 3|\beta|)(\frac{1}{4} - \frac{1}{2^{l+2}})^i \sum_{j=1}^{i}(\frac{1}{4} - \frac{1}{2^{l+2}})^{-j}\frac{L_{j-1}}{2^j},
\end{aligned} \tag{4.10}$$

in which, from Eq. (4.4), the sequence L_{j-1} for any $j \in \mathbb{N}$ is uniformly bounded. Therefore, for every $1 \leq j \leq n$, there exists $N < \infty$ such that $|L_{j-1}| \leq N$. Thus from Eq. (4.10), we have

$$\begin{aligned}\|u-u_i\|_\infty &\leq \|Tu_0-u_0\|_\infty \frac{q^i}{1-q} + 2(4+3|\alpha|+3|\beta|)(q)^i \sum_{j=1}^{i}(\frac{1}{4}-\frac{1}{2^{l+2}})^{-j}\frac{N}{2^j} \\ &= \|Tu_0-u_0\|_\infty \frac{q^i}{1-q} + 2(4+3|\alpha|+3|\beta|)N(q)^i \sum_{j=1}^{i}(1+\frac{1}{2^l-1})^j 2^j. \end{aligned} \tag{4.11}$$

Since $(1+\frac{1}{2^l-1}) \leq 2$ for any $l \in \mathbb{N}$ and by using (4.11), we have

$$\begin{aligned}\|u-u_i\|_\infty &\leq \|Tu_0-\mathbf{u}_0\|_\infty \frac{q^i}{1-q} + 2(4+3|\alpha|+3|\beta|)N(q)^i \sum_{j=1}^{i} 2^j \\ &\leq \|Tu_0-\mathbf{u}_0\|_\infty \frac{q^i}{1-q} + 2(4+3|\alpha|+3|\beta|)N(q)^i i 4^i. \end{aligned} \tag{4.12}$$

If $n \to \infty$ and $q < \frac{1}{4}$, then the last summand of the right-hand side of (4.12) vanishes, and so we have

$$\|u-u_i\|_\infty \leq 2(4+3|\alpha|+3|\beta|)Ni(4q)^i. \tag{4.13}$$

Thus the rate of convergence of our method is $O(i(4q)^i)$.

5 Numerical examples

In this section we solve three examples with our method. for this purpose we lets $(x_i, t_i) = (\frac{1}{2^i}, \frac{1}{2^i})$ for $i = 1, 2, ...6$, and numerical results obtained here can be compare with another numerical methods.

Example 5.1. For the first example, consider the special case of Cauchy problem equation on $X = [0,1] \times [0,1]$ (see [25])

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_0^t\int_0^x & sin(u(y,z))cos(\frac{y-z}{2})sin(\frac{y+z}{2}) + cos(u(y,z))sin(\frac{y-z}{2})cos(\frac{y+z}{2})dydz \\ & -\frac{1}{4}\int_0^x sin(u(y,t)) + cos(u(y,t)) - sin(u(y,0)) - cos(u(y,0))dy \\ & +\frac{1}{4}\int_0^t cos(u(x,z)) + cos(u(x,z)) - cos(u(0,z)) + sin(u(0,z))dz \\ & +sin^2(\frac{x}{2}) - sin^2(\frac{t}{2}), \end{aligned}$$

with exact solution $u(x,t) = sin(\frac{x+t}{2})sin(\frac{x-t}{2})$. Numerical results for Example 1 are displayed in Table 1.

Table 1. Numerical results for Example 1.

(x,t)	Legendre polynomials method([26]) $M=6$	Presented method $m=16$	Presented method $m=32$
(0.0,0.2)	1.5×10^{-6}	0	0
(0.2,0.4)	6.4×10^{-6}	2.19×10^{-6}	5.89×10^{-7}
(0.3,0.6)	5.6×10^{-6}	2.93×10^{-6}	7.36×10^{-7}
(0.4,0.8)	3.5×10^{-6}	3.53×10^{-6}	9.29×10^{-7}
(0.8,1.0)	3.3×10^{-6}	2.80×10^{-6}	7.26×10^{-7}

Example 5.2. Let us consider the nonlinear two dimensional Volterra integral equation of the second kind (see [26],[7]).

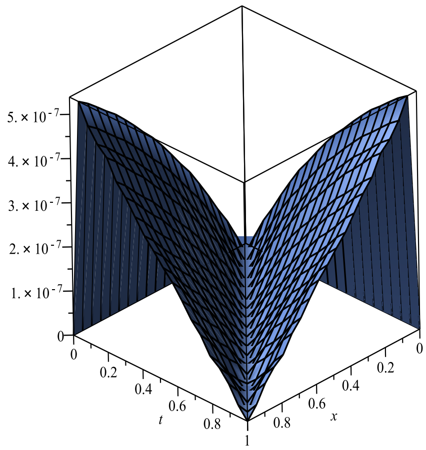

Figure 1: Absolute errors for Example 1 with grid=[20,20]

$$u(x,t) = x + t - \frac{1}{12}xt(x^3 + 4x^2t + 4xt^2 + t^3) + \int_0^t \int_0^x (x + t - y - z)u^2(y,z)dydz, \tag{5.1}$$

with exact solution $u(x,t) = x + t$, and $0 \leq x, t < 1$. Numerical results for Example 2 are displayed in Table 2.

Table 2. Numerical results for Example 2.

$(x,t) = (\frac{1}{2^i}, \frac{1}{2^i})$	Legendre polynomials method ([26]) $M = 2$	Haar wavelet method ([7]) $m = 32$	Presented method $m = 16$
i=1	3.5×10^{-3}	3.1×10^{-2}	2.15×10^{-4}
i=2	4.5×10^{-4}	3.1×10^{-2}	6.71×10^{-4}
i=3	6.1×10^{-4}	3.1×10^{-2}	2.08×10^{-5}
i=4	5.7×10^{-4}	3.1×10^{-2}	6.38×10^{-7}
i=5	3.6×10^{-4}	3.1×10^{-2}	1.73×10^{-8}
i=6	2.0×10^{-4}	2.2×10^{-9}	7.76×10^{-10}

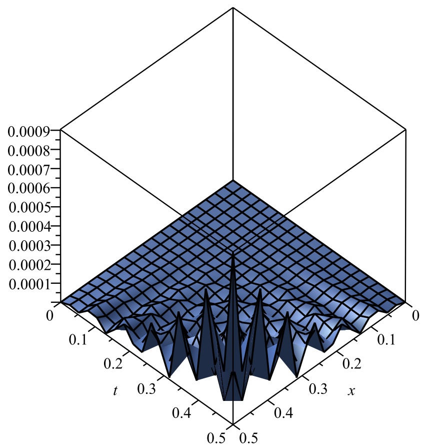

Figure 2: Absolute errors for Example 2 with grid=[20,20]

Example 5.3. Finally let us, consider the following nonlinear two-dimensional Volterra integral equation (see [24],[18],[7])

$$u(x,t) = x^2 + t^2 - \frac{1}{45}xt(9x^4 + 10x^2t^2 + 9t^4) + \int_0^t \int_0^x u^2(y,z)dydz, \tag{5.2}$$

with exact solution $u(x,t) = x^2 + t^2$, and $0 \leq x, t < 1$. Numerical results for Example 3 are displayed in Table 3.

Table 3. Numerical results for Example 3.

$(x,t)=(\frac{1}{2^i},\frac{1}{2^i})$	2D-BPFs method([24]) $m=32$	Haar wavelet method ([7]) $m=32$	Presented method $m=32$
i=2	1.0×10^{-1}	1.6×10^{-2}	5.90×10^{-5}
i=3	7.0×10^{-2}	8.5×10^{-3}	9.06×10^{-7}
i=4	5.8×10^{-2}	4.6×10^{-3}	1.29×10^{-8}
i=5	5.3×10^{-2}	2.6×10^{-3}	1.43×10^{-10}
i=6	2.1×10^{-4}	1.6×10^{-4}	2.00×10^{-12}

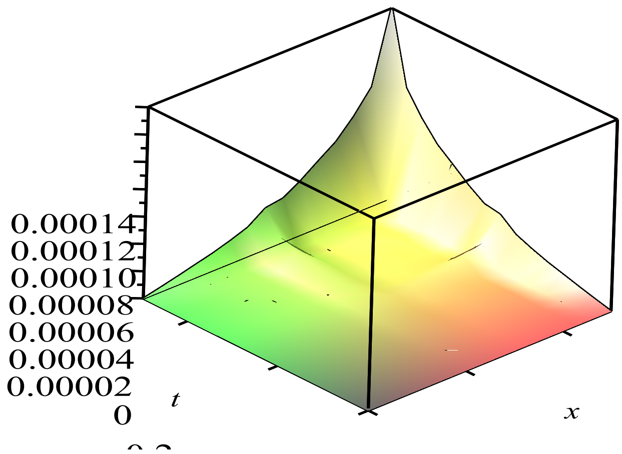

Figure 3: Absolute errors for Example 3 with grid=[20,20]

6 Conclusions

In this work we used RHW function for solving two-dimensional nonlinear Volterra integral equations. One of the advantages of the proposed method is the use of operational matrices for single and double integration of basic functions without the need for direct integration. In addition, the numerical estimation of the order of convergence of the results are provided. In section Numerical examples, we solve three examples of some references and we compare with another methods. Absolute error for all examples are shown.

References

[1] I. Aziz and R. Amin, *Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet*, Appl. Math. Model. **40** (2016), 10286–10299.

[2] I. Aziz, S. Islam and M. Asif, *Haar wavelet collocation method for three-dimensional elliptic partial differential equations*, Comput. Math. Appl. **73** (2017), 2023–2034.

[3] I. Aziz and I. Khan, *Numerical solution of diffusion and reaction–diffusion partial integro-differential equations.* Int. J. Comput. Meth. **15** (2018), 1850047.

[4] P. Assari, *On the numerical solution of two-dimensional integral equations using a meshless local discrete Galerkin scheme with error analysis*, Eng. Comput. **35** (2019), no. 3, 893–916.

[5] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 1997.

[6] A. Babaaghaie and K. Maleknejad, *Numerical solutions of nonlinear two-dimensional partial Volterra integro-differential equations by Haar wavelet*, J. Comput. Appl. Math. **317** (2017), 643–651.

[7] E. Babolian, S. Bazm and P. Lima, *Numerical solution of nonlinear two-dimensional integral equations using Rationalized Haar functions*, Commun. Nonl. Sci. Numer. Simul. **16** (2011), no. 3, 1164–1175.

[8] H. Brunner and J.-P. Kauthen, *The numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation*, IMA J. Numer. Anal. **9** (1989), no. 1, 47–59.

[9] H. Brunner, *On the numerical solution of nonlinear Volterra–Fredholm integral equations by collocation methods*, SIAM J. Numer. Anal. **27** (1990), no. 4, 987–1000

[10] C.F. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed parameter systems*, IEE Proc. Contr. Theor. Appl. **144** (1997), 87–94.

[11] H.J. Dobner, *Bounds for the solution of hyperbolic problems*, Comput. **38** (1987), 209–218

[12] M. Erfanian, M. Parsamanesh and A. Akrami, *Solving two-dimensional nonlinear Fredholm integral equations using rationalized Haar functions in the complex plane*, Int. J. Appl. Comput. Math. **5** (2019), 47.

[13] M. Erfanian and M. Gachpazan, *A new method for solving of telegraph equation with Haar wavelet*, Int. J. Comput. Sci. **3** (2016), 6–10 .

[14] M. Erfanian, M. Gachpazan and S. Kosari, *A new method for solving of Darboux problem with Haar Wavelet*, SeMA J. **74** (2017), 475–487.

[15] M. Erfanian and A. Mansoori, *Rationalized Haar wavelet bases to approximate the solution of the first Painleve equations*, J. Math. Model. **7** (2019), 107–116

[16] M. Erfanian, *The approximate solution of nonlinear integral equations with the RH wavelet bases in a complex plane*, Int. J. Appl. Comput. Math. **4** (2018), 31.

[17] M. Erfanian,*The approximate solution of nonlinear mixed Volterra-Fredholm Hammerstein integral equations with RH wavelet bases in a complex plane*, Math. Method Appl. Sci. **41** (2018), 8942–8952.

[18] G.Q. Han, K. Hayami, K. Sugihara and J. Wang, *Extrapolation method of iterated collocation solution for two-dimensional non-linear Volterra integral equation*, Appl. Math. Comput. **112** (2000), 49–61.

[19] Z. Kamont and H. Leszczynski, *Numerical solutions to the Darboux problem with functional dependence*, Georgian Math. **5** (1998), no. 1, 71–90

[20] R.T. Lynch and J.J. Reis, *Haar transform image conding*, Proc. Nat. Telecommun. Conf., Dallas, TX, 1976, pp. 441–443.

[21] Ü. Lepik, *Haar wavelet method for nonlinear integro-differential equations*, Appl. Math. Comput. **176** (2006), 324–333.

[22] Ü, Lepik, *Solving fractional integral equations by the Haar wavelet method*, Appl. Math. Comput. **214** (2009), 468–478.

[23] F. Mirzaee and N. Samadyar, *Numerical solution based on two-dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order*, Appl. Math. Comput. **344** (2019), 191–203.

[24] K. Maleknejad, S. Sohrabi and B. Baranji, *Two-dimensional PCBFs: application to nonlinear Volterra integral equations*, Proc. Worldcong. Engin. (WCE), vol II. July 1–3, London, UK. 2009.

[25] S. Mckee, T. Tang and T. Diogo, *An Euler-type method for two-dimensional Volterra integral equations of the first kind*, IMA J. Numer. Anal. **20** (2000), 423–440.

[26] S. Nemati, P.M. Lima and Y. Ordokhani, *Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials*, J. Comput. Appl. Math. **242** (2013), 53–69.

[27] M. Rastegar, A. Bazrafshan Moghaddam, M. Erfanian and B.B. Moghaddam, *Using matrix-based rationalized Haar wavelet method for solving consolidation equation*, Asian-Eur. J. Math. **12** (2019), 1950086.

[28] M. Razzaghi and J. Nazarzadeh, *Walsh functions*, Wiley Encycl. Electric. Electron. Engin. **23** (1999), 429–440.

[29] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge, 1997.