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Volatility in the Black-Scholes equation

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Abstract

Generally, the Black-Scholes model and its analyses can be presented in several different ways, ranging from the highly theoretical to the very applied approach. In this article, we will show in detail the applied methodology, the calculations and which effects the applied stress will have on the Black-Scholes option pricing model. One of the aims of nonlinear analysis is to investigate related topics to the analysis of partial differential equations and their applications. To provide for the further development of the Black-Scholes model and the Black-Scholes partial differential equation, we study some related problems. For example, we conclude that the number of call options and volatility increases at the same time.

Keywords: Black-Scholes equation, Call option, Volatility, Nonlinear analysis
2020 MSC: 62P05, 65N40, 65N12, 65N15

1 Introduction

The Black-Scholes equation is well-known in financial mathematics. In 1955, M.F.M. Osborne wrote an article entitled "Brownian Motion in the Stock market". In this year, Richard Kruizenga wrote a similar article entitled "Put and Call Option: A Theoretical and Market Analysis" [13]. In fact, the famous Black-Scholes equation is a modified heat equation. This model was first published by Fischer Black and Myron Scholes in their paper "The Pricing of Options Corporate Liabilities" [4]. In this article, they derived a partial differential equation, now called the Black-Scholes equation. The equation and model are named after economists Fisher Black and Myron Schulz. Robert C. Merton, who first wrote an academic paper on the subject, is sometimes credited. This equation estimates the price of an option over time. The Black-Scholes formula can be deduced, which provides a theoretical estimate of the price of European-style options and shows that the option has a unique price. The key idea behind this model is to hedge the options in an investment portfolio by buying and selling the underlying asset (such as stocks) in just the right way and as a consequence, eliminate risk.

Notice that a security giving the right to buy or sell an asset, in a period of time is called an option [26]. An option which grants its holder the right to buy the underlying assets at a specified price in the future is called a call option. On the other hand, an option which grants its holder the right to sell the underlying assets at a specified price in the future is called a put option. In addition, an amount of variation in the price of a financial instrument over time is called volatility [2].

In [18], a geometric Brownian motion is defined as following: A continuous time stochastic process when the logarithm of the randomly varying quantity follows a Brownian motion. It is an important example of stochastic

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processes satisfying a stochastic differential equation (SDE); in special case, a geometric Brownian motion is used in mathematical finance to model stock prices in the Black-Scholes model. For more specific information in this regard refer to the references [3], [15] and [23].

The key idea behind this model is to cover the option in a proper way by buying and selling the underlying asset and thus eliminate the risk. This type of coverage is called "continuous revised delta hedging" and is the basis of more sophisticated coverage strategies, such as those pursued by investment banks and hedge funds.

The Black-Scholes formula has only one parameter that cannot be directly observed in the market. Since the value of the option (whether inserting or calling) in this parameter is increasing, it can be reversed to produce an "escape level" which is then used to calibrate other models.

In mathematical finance, the Black-Scholes equation is a partial differential equation (PDE) that governs the price evolution of a European option or European call option under the Black-Scholes model. Hence, the term may refer to a similar PDE that can be derived for different options or more generally derivatives. Simulation of Brownian geometric movements with market data parameters for a European call or placement of a major stock that does not pay dividends. The conversion equation is expressed by the following equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.1}$$

When, V is the option price as a function of stock price S and time t. Also, r is the risk-free interest rate, and σ is the volatility of the stock.

In recent years, various studies have been performed on the Black-Scholes equation. This equation has been studied in several articles both in terms of pure mathematical calculations and in terms of applied mathematics. There are key works that will assist the interested reader in becoming more familiar with these topics. We refer readers to the references [1], [14], [16], [17],[19], [20], [21], [22], [24], [25], and [27]. In this paper, we tried to obtain some approximations of this equation.

2 Main Results

Before proving the following theorem, we must recall some basic facts. The dividend yield is a financial ratio that tells you the percentage of a company's stock price that it pays as dividends each year. Dividend yield or the ratio of dividends to the price of a share is the division of earnings per share by the price per share. Also, the total dividends paid annually by a company are divided by its market value, assuming the amount of shares is constant. It is often expressed as a percentage. Dividend payment for preferred stock is announced in the booklet. The name of a preferred stock usually includes its nominal return on the issue price: for example, a preferred stock. However, dividends may be approved or reduced under certain conditions. Current yield is the ratio of annual profit to current market price that changes over time.

Some preferred stock issues can be redeemed at a future date, a feature that allows the issuer to repurchase the stock, starting at a specified futures date with a fixed call price. These factors are stated in the publication manual. The recourse figure for such a stock is the current effective return calculated on the assumption that the issuer immediately applies the recall on the recall date and returns the recall price to the shareholder. Return on call option is implicitly a current measure of future value that calculates the difference between a future call price versus the current market price. Since the current market price may be higher or lower than the call price, the returns may be lower or higher than the current returns. For more information in this regard, see [11]. Also, long call options are simply standard call options where the buyer has the right, but not the obligation, to buy the stock at the price of a future strike. The advantage of a long call option is that it allows you to plan ahead to buy stocks at a lower price. Let D be a long call option and S be the asset price. If the price of an asset, S, underlies it, the value of D decreases and the buying position loses. To counteract the decline in the price of the underlying asset, we want to shorten the Δ units of the underlying asset. Let P is the value of the portfolio of a long one call option and short Δ units, then we obtain

$$P = D - \Delta S. \tag{2.1}$$

Proposition 2.1. Assume that

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\alpha)S\frac{\partial V}{\partial S}-rV=0$$

is a special case of the Black-Scholes equation. When, V is the option price as a function of stock price S and time t. Also, r is the risk-free interest rate, and σ is the volatility of the stock. In addition, α is the dividend yield. Let D be a long call option and S be the asset price. Let P is the value of the portfolio of a long one call option such that

$$P=D-\Delta S.$$

Then, we have:

$$dP\approx\frac{\partial V}{\partial t}dt+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}.$$

When, $0<S<\infty$.

Proof . We know that

$$P=D-\Delta S.$$

The function P should not be sensitive to minor changes in the S variable. Therefore,

$$\frac{\partial P}{\partial S}=0.$$

$$\frac{\partial D}{\partial S}-\Delta\frac{\partial S}{\partial S}=0.$$

We obtain that

$$\Delta=\frac{\partial D}{\partial S}.$$

Using Taylor series approximation, we have

$$V(S+dS,t+dt)=V(S,t)+dS\frac{\partial V}{\partial S}+dt\frac{\partial V}{\partial t}+\frac{1}{2}(dS)^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}(dt)^2\frac{\partial^2 V}{\partial t^2}+dSdt\frac{\partial^2 V}{\partial t\partial S}. \tag{2.2}$$

We know that $dV=V(S+dS,t+dt)-V(S,t)$. Approximately, set

$$dS^2\approx\sigma^2 S^2 dt.$$

Thus, we conclude that

$$dV\approx dS\frac{\partial V}{\partial S}+dt\frac{\partial V}{\partial t}+\frac{1}{2}dt\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2. \tag{2.3}$$

Therefore,

$$P=V-\Delta S.$$

So, if we remove the sensitivity of prices, we may obtain that

$$V\approx D.$$

Using this, we conclude that

$$dP=dV-\Delta dS.$$

Approximately,

$$dP\approx dS\frac{\partial V}{\partial S}+dt\frac{\partial V}{\partial t}+\frac{1}{2}dt\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2-\Delta dS. \tag{2.4}$$

Hence,

$$\Delta=\frac{\partial V}{\partial S}.$$

So, we conclude that

$$dP \approx \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, \tag{2.5}$$

as we claimed. □

We know that a call option is a contract. This contract is between a buyer and a seller to purchase a certain stock at a certain price up until a defined expiration date. The buyer of a call option has the right, not the obligation.

In fact, in such transactions, the delivery of goods is usually not intended. The profit that traders make in such transactions is due to price fluctuations. Therefore, it is very important to study the issue of price fluctuations in this type of trading. In other words, traders, by predicting the future of the market, practically bet on prices. The profit or loss that traders receive as a result of these fluctuations is much greater than the profit or loss that a trader will receive in a real trading market. Therefore, examining price changes in such transactions is one of the important issues.

The Delta of an option is mathematically equivalent to the rate of change of the option price with respect to the change in the asset price. Notice that in theorem 2.1, we obtain how this relates to the amount of stock that needs to be sold to prevent a change in position. The Delta of the option is $\Delta = \frac{\partial V}{\partial S}$. The Gamma operator is

$$\Gamma = \frac{\partial^2 V}{\partial S^2}. \tag{2.6}$$

This is an important concept because in practice infinite trades do not take place over a limited period of time. As you might expect, the larger the Γ in absolute value, the greater the discretization error of the Δ hedging strategy. Before proving the next result, we have the following definitions and preliminaries.

Definition 2.2. The Vega of the Black-Scholes equation is the sensitivity of the price to a change in volatility. The vega of Black-Scholes equation satisfies:

$$\nu = \frac{\partial V}{\partial \sigma}. \tag{2.7}$$

$$\nu = SN'(d_1)\sqrt{T-t}. \tag{2.8}$$

It is well known that the Gamma has the same value for call and put options. Therefore, the Vega has the same value for call and put options. We can see this directly from put-call parity. Notice that the difference of a put option and a call option is linear in S and independent from σ. In addition, N is the standard normal probability density function.

In practice, some sensitivities are usually mentioned in smaller expressions to match the scale of possible changes in parameters. For example, Vega is often divided by 100 (volume change 1).

Consider Black-Scholes differential operator L_{BS} as:

$$L_{BS} = \partial_t + rx\partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2 - r. \tag{2.9}$$

When r and σ are positive constant. In Proposition 8.11.1 of [8], it is proved that if $f(x,t)$ and $g(x,t)$ be two functions in $C^{2,1}$, then:

$$L_{BS}(fg) = fL_{BS}(g) + gL_{BS}(f) + (f,g), \tag{2.10}$$

where,

$$(f,g) = rfg + \sigma^2 x^2 \partial_x f \partial_x g. \tag{2.11}$$

It can be checked that for the Black Scholes differential operator L_{BS} :

$$L_{BS}\nu = \sigma S^2 \partial^2 V \partial S^2 = \sigma^2 S^2 \Gamma. \tag{2.12}$$

Hence, $\sigma^2 S^2 \Gamma$ acts as a source for the Vega.

Before proving the following theorem, we need to talk about a few symbols. We use these symbols in the proof text theorem. As we have discussed so far, our intention is to use the computational argument to obtain applied results on the Black-Scholes model. The Black-Scholes equation calculates the price of European put and call options. We denote by: $V(S,t)$, the option price as a function of the underlying asset S, at time t.
$C(S,t)$ is the European call option price.
$P(S,t)$ the European put option price.
T is the time of option expiration.
τ is the time until maturity, which is equal to $\tau = T - t$.
p is the strike price of the option.

In this regard, the following relations have been proved.

$C(0,T) = 0, \forall t > 0$.
$C(S,T) = S$, when $S \to \infty$.
$C(S,T) = max(S - p, 0)$.

In addition, when

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}[ln(\frac{S_t}{p}) + (r\frac{\sigma^2}{2}(T-t))], \tag{2.13}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{2.14}$$

then, we have

$$C(S_t, t) = N(d_1)S_t - N(d_2)pe^{-r(T-t)}. \tag{2.15}$$

Also,

$$P(S_t, t) = pe^{-r(T-t)} - S_t + C(S_t, t) = N(-d_2)pe^{-r(T-t)} - N(-d_1)S_t. \tag{2.16}$$

When, $N(x)$ is the standard normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dz.$$

Theorem 2.3. Assume that $S(t)$ is the price of the underlying asset at time t of a Black-Scholes model. We denoted it as S_t. Also, consider that p is the strike price of the option, T is the time of option expiration and σ be the volatility of the stock. Then, we have:

$$S_0 N(d_2 + \sigma\sqrt{T}) = E(S_T \mid S_T > p)(1 - F(S_T(p))).$$

When $F(S_T)$ is the distribution function of S_T. Therefore, we conclude from these calculations that the amount of the call option and volatility increases at the same time.

Proof . As we saw above, Vega of the Black-Scholes equation is a positive function, meaning that the value of the option is an incremental function of the oscillations, because Vega is a derivative of the value relative to the oscillation σ. Assume that σ is the volatility and r is the risk-free rate. Consider that the dividend yield is 0. We know that the underlying asset price has a geometric Brownian motion process in the Black-Scholes model. The geometric Brownian motion specifies that the instantaneous percentage change in the exchange rate has a constant drift, μ_B, and volatility, σ_B, so we have:

$$\frac{S_t}{S_t} = \mu_B dt + \sigma_B dB_t. \tag{2.17}$$

The error,dB_t, is a standard Brownian motion. We can write this in the form

$$S_t = S_0 e^{\mu_B t + \sigma_B B_t}. \tag{2.18}$$

If $\tau > 0$ is the interval between observations, then the $\tau-$period logarithmic return

$$ln(\frac{S_{t+\tau}}{S_t}) \equiv x(\tau) = \mu_B \tau + \sigma_B (B_{t+\tau} - B_t)$$

has a normal distribution. This means

$$x(\tau) \sim N(\mu_B \tau, \sigma_B^2 \tau). \tag{2.19}$$

After stating these preliminaries, we have

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})T} e^{\sigma \sqrt{T} \omega} \tag{2.20}$$

where, $\omega \sim N(0,1)$ is a standard normal random variable. Also, we have

$$d_2 = \frac{log(\frac{S_0 e^{rT}}{p})}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T}. \tag{2.21}$$

But, S_T has a lognormal distribution. Consequently,

$$P(S_T > p) = P(log(S_T) > log(p)) = P(\omega > -d_2).$$

According to the characteristics of distribution,

$$P(\omega > -d_2) = P(\omega < d_2) = N(d_2). \tag{2.22}$$

Therefore, d_2 decreases for an at-the-money-forward option respect to σ. In addition, it is decreasing for any strike price p. Hence, the probability of exercise for a call option is $N(d_2)$ and it is decreasing respect to volatility.

Notice that:

$$P(F, \tau) = e^{-r\tau}(N(d_+)F - N(d_-)p). \tag{2.23}$$

where,

$$d_+ = \frac{1}{\sigma \sqrt{\tau}}[ln(\frac{F}{p}) + \frac{1}{2}\sigma^2 \tau], \tag{2.24}$$

$$d_- = d_+ - \sigma \sqrt{\tau}. \tag{2.25}$$

When, $D = e^{-r\tau}$ is the discount factor and $F = e^{r\tau} S = \frac{S}{D} S$ is the forward price of the underlying asset. In addition, we know that Vega is always positive and, hence, the amount of an option (call or put) should increase, when the volatility is increased. This means that the expected value of the $E(\max(S_T - K, 0))$ increases as σ is increased for a lognormal distribution.

On the other hand, since the S_T has a lognormal distribution, it is always a nonnegative random variable. When the volatility increases, we conclude that the dispersion of the distribution increases and the mass is more likely to accumulate in the interval $[0, K]$ than in the interval $[K, \infty)$. Note that

$$e^{rT} V + pN(d_2) = S_0(d_2 + \sigma \sqrt{T}). \tag{2.26}$$

If this option expires in cash, the holder actually pays the strike price and receives the underlying asset. The strike price is fixed, but subject to enforcement, the delivered underlying asset has unlimited upside potential. The second statement is simply the expected value of the payment. The strike price is 0 when $S_T \leq p$ and also a constant $-p$ if $S_T > p$. Therefore, this takes the form $-pN(d_2)$ and this will move closer to 0 With increasing fluctuations, because it is unlikely to be realized. In addition, the first term is positive and we explain it as:

$$S_0 N(d_2 + \sigma\sqrt{T}) = E(S_T \mid S_T > p)(1 - F(S_T(p))). \tag{2.27}$$

When $F(S_T)$ is the distribution function of S_T. This is our claim in the theorem. The above term increases with σ. Therefore, we conclude from these calculations that the amount of the call option and volatility increases at the same time.

A numerical example with an applied approach. Black and Scholes attempted to apply the formula to the markets, but incurred financial losses, due to a lack of risk management in their trades. In the year 1970, they return to the academic environment. Scholes received the 1997 Nobel Memorial Prize in Economic Sciences, the committee citing their discovery of the risk neutral dynamic revision as a breakthrough that separates the option from the risk of the underlying security. Nowadays, we know that many classic finance, such as the Black-Scholes option pricing model, has the following equation:

$$\frac{1}{P} dP = \mu dt + \sigma dW, \tag{2.28}$$

for the change in the relative price $P^{-1}dP$ in terms of the expected return, μ, the standard deviation of the return, σ, and independent increments of Brownian motion, dW. The SDE can solve this equation analytically and the solution has the following form:

$$P(t) = P(0)exp([\mu - \frac{\sigma^2}{2}]t + \sigma W(t)),$$

when $(W(t))_{t\geq 0}$ is a Brownian motion. On the other hand, Black-Scholes is a pricing model used to determine the fair price or theoretical value of a buy or sell option based on six variables such as fluctuations, type of option, stock price, time, strike price and risk-free rate. Quantum is more speculation about stock market derivatives, and therefore proper pricing of options eliminates the possibility of any arbitrage. There are two important models for option pricing, the binomial model and the Black-Scholes model. This model is used to determine the price of a European purchase option, which simply means that this option is only valid on the expiration date. For more information in this regard, we refer dear readers to references [7], [5],[9], [10], [12] and [6].

In this article, we intend to examine the effects of the Black-Scholes model on the volume of trading on the stock exchange of Iran. Table (1) contains the volume of monthly transactions in Iran Stock Exchange. We get data through the Tse Clint software.

Table 1: Your table's caption

Month	Monthly volume of trading on the stock exchange of Iran(Trillion Tomans)	Returns
1	9600	
2	13292	0.384583333
3	16697	0.256169124
4	23380	0.400251542
5	14725	-0.370188195
6	10989	-0.253718166
7	6587	-0.400582401
8	10855	0.647942918
9	16618	0.530907416
10	8922	-0.463112288
11	8668	-0.266843563
12	6355	0.073550047

Hence, we have $\mu = 0.039721888$ and $\sigma = 4754.129416$. We know that

$$dP = \mu P dt + \sigma P dW.$$

Hence, in the interval $[a, b]$, we have

$$\int_a^b dP = \int_a^b \mu P dt + \int_a^b \sigma P dW.$$

Consequently, considering the numerical approximation $\int_a^b f(t) = f(a)(b-a)$, we have $P(b) = (1+\mu)P(a) + \sigma P(a)(W(b) - W(a))$. We know that $W(t)$ is a Brownian motion. Therefore,

$$W(b) - W(a) \sim N(0, b-a).$$

In this manner, we obtain that

$$P(b) = (1+\mu)P(a) + \sigma P(a)N(0, b-a)).$$

So, if we use Table 1, we can get the following relation beyond the volume of trading on the stock exchange of Iran.

$$P(1) = (1+\mu)P(0) + \sigma P(a)N(0,1)) = P(0)((1.0.039721888) + 4754.129416N(0,1)).$$

When $P(1)$ is the volume of trading on the stock exchange of Iran for one year later. This relation can be used as an approximation method using numerical methods and can practically be considered as one of the available approximate methods to predict the future trend of the volume of trading on the stock exchange of Iran.

3 Conclusion

In this paper, we tried to obtain some approximations of the Black-Scholes equation. Then, we prove that:

$$S_0 N(d_2 + \sigma\sqrt{T}) = E(S_T \mid S_T > p)(1 - F(S_T(p)).$$

When $F(S_T)$ is the distribution function of S_T. Therefore, we conclude from these calculations that the amount of the call option and volatility increases at the same time.

Also, assume that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\alpha)S\frac{\partial V}{\partial S} - rV = 0$$

is a special case of the Black-Scholes equation. When, α is the dividend yield and let D be a long call option and S be the asset price. Let P is the value of the portfolio of a long one call option such that

$$P = D - \Delta S.$$

Then, we have:

$$dP \approx \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

when, $0 < S < \infty$. □

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