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On a wave equation containing nonlinear integral terms: Existence and asymptotic expansion of solutions

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Abstract

In this paper, we consider an initial-boundary problem for a wave equation containing nonlinear integral terms. By the linear approximate method associated with the Faedo-Galerkin method, the existence and uniqueness of solutions for the proposed problem are proved. Moreover, a high-order asymptotic expansion in a small parameter of the weak solution is also discussed.

Keywords: Nonlinear integral term, Wave equation, Faedo-Galerkin method, Reccurence sequence; Asymptotic expansion
2020 MSC: 35L20, 35L70

1 Introduction

In this paper, we consider the following problem for a nonlinear wave equation with nonlinear integral terms

$$\begin{aligned} u_{tt} - \frac{\partial}{\partial x}\left[\mu\left(x,t,\int_0^1 \sigma(x,y,t,u(y,t),u_x(y,t))dy\right)u_x\right] \\ = f\left(x,t,u,u_x,u_t,\int_0^1 g(x,y,t,u(y,t),u_x(y,t),u_t(y,t))dy\right), \end{aligned} \tag{1.1}$$

$$u_x(0,t) - h_0 u(0,t) = u(1,t) = 0,\ t>0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x),\ 0<x<1, \tag{1.3}$$

where μ, σ, f, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0 \geq 0$ is a given constant.

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The equation (1.1) can be considered as a generalized model of Kirchhoff-Carrier type equations that some specific cases have been studied in the literature. Indeed, as $\sigma(x,y,t,u,u_x) = u_x^2$, $\mu\left(x,t,\int_0^1 \sigma(x,y,t,u(y,t),u_x(y,t))dy\right) = \mu\left(\|u_x\|^2\right)$ and $f=0$, it becomes the Kirchhoff equation (see [5])

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.4}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t, L is the length of the string, h is the cross-section area, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. The equation (1.4) is an extension of the classical D'Alembert's wave equation which describes vibrations of a string under the effects that can make changes in length of the string. Another special case of (1.1) with $\sigma(x,y,t,u,u_x) = u^2, \mu\left(x,t,\int_0^1 \sigma(x,y,t,u(y,t),u_x(y,t))dy\right) = \mu\left(\|u\|^2\right)$ and $f=0$, is called the Carrier equation [2] describing vibrations of an elastic string when changes in tension are not small

$$v_{tt} - \left(P_0 + P_1\int_0^L v^2(y,t)dy\right) v_{xx} = 0, \tag{1.5}$$

where P_0, P_1 are constants. Afterward, the Kirchhoff-Carrier type equations have been extensively studied by many authors, for example, we refer the reader to some previous studies as in [3], [4], [6], [9], [11], [13], [19]-[21] and the references therein. In these works, numerous of interesting results about the local or global existence, the asymptotic expansion, the decayed behavior and the blow-up property of solutions were obtained.

In [3], Cavalcanti et.al. studied the existence of global solutions and exponential decay for the following nonlinear problem

$$\begin{cases} u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u - \Delta u_t = f, \text{ in } Q = \Omega \times (0,\infty), \\ u = 0, \text{ on } \Sigma_1 = \Gamma_1 \times (0,\infty), \\ M\left(\int_\Omega |\nabla u|^2 dx\right)\dfrac{\partial u}{\partial \upsilon} + \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial \upsilon}\right) = g, \text{ on } \Sigma_0 = \Gamma_0 \times (0,\infty), \\ u(0) = u_0,\ \dfrac{\partial u}{\partial t}(0) = u_1, \text{ in } \Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with C^2 boundary Γ and M is a C^1 function, $M(\lambda) \geq \lambda_0 > 0$, $\forall \lambda \geq 0$.

In [21], Triet et.al. used the linear approximate method associated with the Faedo-Galerkin method for proving the local existence and uniqueness of solutions for the following Kirchhoff-Carrier wave equation

$$u_{tt} - \frac{\partial}{\partial x}\left[\mu(x,t,u,\|u\|^2,\|u_x\|^2)u_x\right] = f(x,t,u,u_x,u_t),\ 0 < x < 1,\ t > 0, \tag{1.7}$$

where $\|u(t)\|^2 = \displaystyle\int_0^1 u^2(x,t)\,dx$. Furthermore, the $(N+1)^{th}$-order asymptotic expansion in small parameters of the weak solution of the equation (1.7) has been considered.

Recently, some authors have paid attention to the studies of the initial-boundary value problems with nonlinear integral terms, see [7], [8] and [17]. In [8], Hao proved the general decay of solutions for the time varying-delay viscoelastic equation with the nonlinear integral term $\displaystyle\int_\Omega \nabla u \nabla u_t dx$ named Balakrishnan-Taylor damping

$$\begin{cases} u_{tt} - \left(a + b\|\nabla u\|^2 + \sigma\displaystyle\int_\Omega \nabla u\nabla u_t dx\right)\Delta u \\ \quad +\alpha(t)\displaystyle\int_0^t g(t-s)\Delta u(s)ds + \mu_0 u_t + \mu_1(x, t-\tau(t)) = 0, \text{ in } \Omega\times(0,+\infty), \\ u(x,t) = 0, \text{ on } \partial\Omega\times(0,+\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \text{ in } \Omega, \\ u_t(x,t) = g_0(x,t), \text{ in } \Omega\times(-\tau(0),0), \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with sufficiently smooth boundary $\partial\Omega$, a, b, σ, μ_0, μ_1 are fixed positive constants, g and f are given functions, $\tau(t)$ represents the time delay.

In [16], the authors proved a local existence of solutions for the following strong damped wave equation with nonlinear integral term (memory term)

$$\begin{aligned} &u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu_1\left(x, t, u(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right) u_x\right] \\ &\quad + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2\left(x, s, u(x,s), \|u(s)\|^2, \|u_x(s)\|^2\right) u_x(x,s)\right] ds \\ &\quad = f\left(x, t, u, u_x, u_t, \|u(t)\|^2, \|u_x(t)\|^2\right), \ 0 < x < 1, \ t > 0, \end{aligned} \tag{1.9}$$

associated with Robin-Dirichlet boundary conditions and initial conditions, where $\lambda > 0$ is a constant, μ_1, μ_2, g, f are given functions which satisfy some certain conditions. Moreover, the authors established an asymptotic expansion in small parameter of solutions for the equation (1.9) perturbed by replacing f with $f\left(x, t, u, u_x, u_t, \|u(t)\|^2, \|u_x(t)\|^2\right) + \varepsilon f_1\left(x, t, u, u_x, u_t, \|u(t)\|^2, \|u_x(t)\|^2\right)$. For more recent studies of Kirchhoff-Carrier type equation, we refer to the results of asymptotic expansion of solutions for Kirchhoff–Love equation [21] and the results of existence, blow-up and exponential decay estimates for Kirchhoff-Carrier wave equation in an annular [17].

Motivated by the above works, we consider the existence, uniqueness and asymptotic expansion of solutions for the problem (1.1)-(1.3). The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, by using the linear approximate method, the Faedo-Galerkin method and the arguments of compactness, we prove the existence and uniqueness of weak solution for the problem (1.1)-(1.3). In Section 4, we establish the $(N+1)^{th}$-order asymptotic expansion in a small parameter ε for the solutions of the following perturbed problem

$$u_{tt} - \frac{\partial}{\partial x}\left[\mu_\varepsilon[u](x,t) u_x\right] = f_\varepsilon[u](x,t), \ 0 < x < 1, \ 0 < t < T, \tag{1.10}$$

associated with (1.2) and (1.3), where

$$\begin{cases} f_\varepsilon[u](x,t) = f\left(x, t, u, u_x, u_t, \int_0^1 g[u](x,y,t)dy\right) + \varepsilon f_1\left(x, t, u, u_x, u_t, \int_0^1 g_1[u](x,y,t)dy\right), \\ \mu_\varepsilon[u](x,t) = \mu\left(x, t, \int_0^1 \sigma[u](x,y,t)dy\right) + \varepsilon\mu_1\left(x, t, \int_0^1 \sigma_1[u](x,y,t)dy\right), \\ g[u](x,y,t) = g(x, y, t, u(y,t), u_x(y,t), u_t(y,t)), \\ g_1[u](x,y,t) = g_1(x, y, t, u(y,t), u_x(y,t), u_t(y,t)), \\ \sigma[u](x,y,t) = \sigma(x, y, t, u(y,t), u_x(y,t)), \\ \sigma_1[u](x,y,t) = \sigma_1(x, y, t, u(y,t), u_x(y,t)). \end{cases} \tag{1.11}$$

These results regard a relative generalization of [9], [11], [13]-[16], [20].

2 Preliminaries

Put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty, \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess \sup_{0<t<T} \|u(t)\|_X \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \triangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $g \in C^k([0,1]^2 \times \mathbb{R}_+ \times \mathbb{R}^3)$, $g = g(x,y,t,z_1,z_2,z_3)$, we put $D_1 g = \frac{\partial g}{\partial x}$, $D_2 g = \frac{\partial g}{\partial y}$, $D_3 g = \frac{\partial g}{\partial t}$, $D_{i+3} g = \frac{\partial g}{\partial z_i}$, with $i = 1,2,3$ and $D^\beta g = D_1^{\beta_1} \cdots D_6^{\beta_6} g$; $\beta = (\beta_1, \cdots, \beta_6) \in \mathbb{Z}_+^6$, $|\beta| = \beta_1 + \cdots + \beta_6 = k$, $D^{(0,\cdots,0)} g = g$.

Similarly, with $\mu = \mu(x,t,z)$, we also put $D_1 \mu = \frac{\partial \mu}{\partial x}$, $D_2 \mu = \frac{\partial \mu}{\partial t} = \mu'$, $D_3 \mu = \frac{\partial \mu}{\partial z}$.

We shall use the following norm on H^1

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}. \tag{2.1}$$

We put

$$V = \{ v \in H^1 : v(1) = 0 \}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + h_0 u(0) v(0), \ \forall u, v \in V. \tag{2.3}$$

V is a closed subspace of H^1 and on V three norms $\|v\|_{H^1}$, $\|v_x\|$ and $\|v\|_a = \sqrt{a(v,v)}$ are equivalent norms.

Lemma 2.1. (see [1]) *The imbedding $H^1 \hookrightarrow C^0(\bar{\Omega})$ is compact and*

$$\|v\|_{C^0(\bar{\Omega})} \le \sqrt{2} \|v\|_{H^1} \ \text{for all } v \in H^1, \tag{2.4}$$

where $\|v\|_{C^0(\bar{\Omega})} = \sup_{x \in [0,1]} |v(x)|$.

Lemma 2.2. *Let $h_0 \ge 0$. The imbedding $V \hookrightarrow C^0(\bar{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\bar{\Omega})} \le \|v_x\| \le \|v\|_a, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \le \|v_x\| \le \|v\|_a \le \sqrt{1+h_0} \|v\|_{H^1}, \end{cases} \tag{2.5}$$

for all $v \in V$.

Lemma 2.3. *Let $h_0 \ge 0$. There is an orthonormal base $\{\tilde{w}_j\}_{j=1}^\infty$ in L^2 that contains eigenvectors of $-\Delta$ operator corresponding to eigenvalues $\{\lambda_j\}_{j=1}^\infty$*, and satisfies $\tilde{w}_{jx}(0) - h_0 \tilde{w}_j(0) = \tilde{w}_j(1) = 0$ and

$$\begin{cases} 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \ \lim_{j \to +\infty} \lambda_j = +\infty \\ a(\tilde{w}_j, \ v) = \lambda_j \langle \tilde{w}_j, \ v \rangle \ \text{for all } v \in V, \ j = 1,2,\cdots. \end{cases} \tag{2.6}$$

Moreover, $\{\tilde{w}_j / \sqrt{\lambda_j}\}_{j=1}^\infty$ is also an orthonormal base of V with respect to the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.3).

The proof of Lemma 2.3 can be found in [[18]; Theorem 7.7, page 87], with $H = L^2$ and V, $a(\cdot,\cdot)$ as defined by (*2.2*), (*2.3*).

Definition 2.4. *A weak solution of the initial-boundary value problem (1.1)-(1.3) is a function $u \in \widetilde{W} = \{u \in L^\infty(0,T; V \cap H^2) : u' \in L^\infty(0,T;V),\ u'' \in L^\infty(0,T;L^2)\}$, and satisfies the following variational equation*

$$\langle u''(t), w \rangle + A[u](t; u(t), w) = \langle f[u](t), w \rangle, \tag{2.7}$$

for all $w \in V$, a.e., $t \in (0,T)$, *together with initial conditions*

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \tag{2.8}$$

where, for each $w \in \widetilde{W}$, $\{A[w](t;\cdot,\cdot)\}_{0\le t\le T}$ *is a family of symmetric bilinear forms on* $V\times V$ *defined by*

$$A[w](t;u,v) = \langle \mu[w](t)u_x, v_x\rangle + h_0\mu[w](0,t)u(0)v(0),\ \forall u,v\in V,\ 0\le t\le T, \tag{2.9}$$

with $h_0\ge 0$ *is a given constant, and*

$$\begin{aligned} \mu[w](x,t) &= \mu\left(x,t,\int_0^1 \sigma[w](x,y,t)dy\right), \\ \sigma[w](x,y,t) &= \sigma(x,y,t,w(y,t),w_x(y,t)), \\ f[u](x,t) &= f\left(x,t,u,u_x,u_t,\int_0^1 g[u](x,y,t)dy\right), \\ g[u](x,y,t) &= g(x,y,t,u(y,t),u_x(y,t),u_t(y,t)). \end{aligned} \tag{2.10}$$

3 Existence and uniqueness

In order to study the existence and uniqueness of weak solution of the problem (1.1)-(1.3), we make the following assumptions:

(H_1) $(\tilde{u}_0,\tilde{u}_1)\in \left(V\cap H^2\right)\times V$ satisfy $\tilde{u}_{0x}(0)-h_0\tilde{u}_0(0)=0$;
(H_2) $g\in C^1([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^3)$;
(H_3) $\sigma\in C^2([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2)$;
(H_4) $f\in C^1([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$;
(H_5) $\mu\in C^2([0,1]\times\mathbb{R}_+\times\mathbb{R})$ and there is a constant $\mu_0>0$ such that
$\mu(x,t,z)\ge\mu_0$, for all $(x,t,z)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}$;

Fix $T^*>0$. For each $M>0$ given, we put $\bar{H}_M(\sigma)$, $H_M(g)$, $\tilde{K}_M(\mu)$, $K_M(f)$ as follows

$$\begin{cases} \tilde{K}_M(\mu) = \sum_{|\alpha|\le 2}\tilde{K}_0(M,D^\alpha\mu,\sigma), \\ K_M(f) = K_0(M,f,g)+\sum_{i=1}^6 K_0(M,D_if,g), \\ \bar{H}_M(\sigma) = \sum_{|\alpha|\le 2}\bar{H}_0(M,D^\alpha\sigma), \\ H_M(g) = H_0(M,g)+\sum_{i=1}^6 H_0(M,D_ig), \end{cases} \tag{3.1}$$

where

$$\begin{cases} \bar{H}_0(M,\sigma) = \sup_{(x,y,t,y_1,y_2)\in A_1(M)} |\sigma(x,y,t,y_1,y_2)|, \\ H_0(M,g) = \sup_{(x,y,t,z_1,z_2,z_3)\in A_2(M)} |g(x,y,t,z_1,z_2,z_3)|, \\ \tilde{K}_0(M,\mu,\sigma) = \sup_{(x,t,z)\in A_3(M)} |\mu(x,t,z)|, \\ K_0(M,f,g) = \sup_{(x,t,v_1,v_2,v_3,v_4)\in A_4(g,M)} |f(x,t,v_1,v_2,v_3,v_4)|, \\ A_1(M) = \{(x,t,y,y_1,y_2): 0\le y\le x\le 1,\ 0\le t\le T^*,\ \max_{1\le i\le 2}|y_i|\le M\}, \\ A_2(M) = \{(x,t,y,z_1,z_2,z_3): 0\le y\le x\le 1,\ 0\le t\le T^*,\ \max_{1\le i\le 3}|z_i|\le M\}, \\ A_3(\sigma,M) = \{(x,t,z): 0\le x\le 1,\ 0\le t\le T^*,\ |z|\le \bar{H}_0(M,\sigma)\}, \\ A_4(g,M) = \{(x,t,v_1,v_2,v_3,v_4): 0\le x\le 1,\ 0\le t\le T^*,\ \max_{1\le i\le 3}|v_i|\le M,\ \ |v_4|\le H_0(M,g)\}. \end{cases} \tag{3.2}$$

For each $T\in(0,T^*]$ and $M>0$, we put

$$\begin{cases} W(M,T) = \{v\in L^\infty(0,T;V\cap H^2): v_t\in L^\infty(0,T;V),\ v_{tt}\in L^2(Q_T), \\ \qquad\qquad \text{with } \|v\|_{L^\infty(0,T;V\cap H^2)},\ \|v_t\|_{L^\infty(0,T;V)},\ \|v_{tt}\|_{L^2(Q_T)}\le M\}, \\ W_1(M,T) = \{v\in W(M,T): v_{tt}\in L^\infty(0,T;L^2)\}, \end{cases} \tag{3.3}$$

in which $Q_T = \Omega \times (0,T)$.

Now, we shall establish a recurrent sequence $\{u_m\}$ that the first term u_0 is chosen by $u_0 \equiv \tilde{u}_0$, and suppose that

$$u_{m-1} \in W_1(M,T). \tag{3.4}$$

Then, we find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), v\rangle + A_m(t; u_m(t), v) = \langle F_m(t), v\rangle, \ \forall v \in V, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} A_m(t;u,v) &= A[u_{m-1}](t;u,v) = \langle \mu_m(t)u_x, v_x\rangle + h_0\mu_m(0,t)u(0)v(0), \ \forall u, \ v \in V, \qquad (3.6)\\ \mu_m(x,t) &= \mu\left(x,t,\int_0^1 \sigma[u_{m-1}](x,y,t)dy\right), \\ \sigma[u_{m-1}](x,y,t) &= \sigma(x,y,t,u_{m-1}(y,t),\nabla u_{m-1}(y,t)), \\ F_m(x,t) &= f\left(x,t,u_{m-1},\nabla u_{m-1},u_{m-1}',\int_0^1 g[u_{m-1}](x,y,t)dy\right), \\ g[u_{m-1}](x,y,t) &= g(x,y,t,u_{m-1}(y,t),\nabla u_{m-1}(y,t),u_{m-1}'(y,t)). \end{aligned}$$

Theorem 3.1. Suppose that $(H_1)-(H_5)$ *hold. Then, there are positive constants* M, T *such that there exists the recurrent sequence* $\{u_m\}$ *defined by* (3.4)-(3.6).

Proof . The proof Theorem 3.1 consists of several steps as follows.

Step 1. Faedo–Galerkin approximation (see Lions [10]). The Galerkin approximate solution of the problem (3.4)-(3.6) is found in form

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)w_j, \tag{3.7}$$

where $c_{mj}^{(k)}(t)$ satisfies the following system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + A_m(t; u_m^{(k)}(t), w_j) = \langle F_m(t), w_j\rangle, \ 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.8}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } V. \end{cases} \tag{3.9}$$

The system (3.8) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \sum_{i=1}^{k} A_{ij}^{(m)}(t)c_{mi}^{(k)}(t) = F_{mj}(t), \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \ 1 \leq j \leq k, \end{cases} \tag{3.10}$$

where

$$A_{ij}^{(m)}(t) = A_m(t; w_i, w_j), \ F_{mj}(t) = \langle F_m(t), w_j\rangle, \ 1 \leq i,j \leq k. \tag{3.11}$$

By using the arguments of ordinary differential equation theory, we can easily prove that the system (3.10)-(3.11) has a unique solution $c_{mj}^{(k)}(t)$, $1 \leq j \leq k$ on $[0,T]$.

Step 2. A priori estimates.

First, we need the following lemma such that its proof is easy, hence we omit the details.

Lemma 3.2. Put $\mu^* = \tilde{K}_M(\mu)\left[1 + (1+2M)\bar{H}_M(\sigma)\right]$, *we get that*

$$\begin{array}{ll} (i) & |A_m(t;u,v)| \le \tilde{K}_M(\mu)\left\|u\right\|_a \left\|v\right\|_a \text{ for all } u,\, v \in V,\, 0 \le t \le T^*, \\ (ii) & A_m(t;v,v) \ge \mu_0 \left\|v\right\|_a^2 \text{ for all } v \in V,\, 0 \le t \le T^*, \\ (iii) & \dfrac{\partial A_m}{\partial t}(t;u,v) = \langle \mu_m'(t)u_x, v_x\rangle + h_0\mu_m'(0,t)u(0)v(0), \text{ for all } u, v \in V, \\ (iv) & \left|\dfrac{\partial A_m}{\partial t}(t;v,v)\right| \le \mu^* \left\|v\right\|_a^2 \text{ for all } v \in V,\, 0 \le t \le T^*, \\ (v) & \dfrac{d}{dt}A_m(t;u_m^{(k)}(t), u_m^{(k)}(t)) = 2A_m(t;u_m^{(k)}(t), \dot{u}_m^{(k)}(t)) + \dfrac{\partial A_m}{\partial t}(t;u_m^{(k)}(t), u_m^{(k)}(t)). \end{array} \tag{3.12}$$

Next, we put

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds, \tag{3.13}$$

where

$$\begin{cases} X_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|^2 + A_m(t;u_m^{(k)}(t), u_m^{(k)}(t)), \\ Y_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|\sqrt{\mu_m(t)}\Delta u_m^{(k)}(t)\right\|^2. \end{cases} \tag{3.14}$$

Then, it follows from (3.8), $(3.12)_{(iii),\ (v)}$, (3.13), (3.14), that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\langle \mu_{mx}(0)\tilde{u}_{0kx}, \triangle\tilde{u}_{0k}\rangle + 2\langle F_m(0), \triangle\tilde{u}_{0k}\rangle \\ &\quad + \int_0^t ds \int_0^1 \dot{\mu}_m(x,s)|\triangle u_m^{(k)}(x,s)|^2 dx + \int_0^t \frac{\partial A_m}{\partial s}(s;u_m^{(k)}(s), u_m^{(k)}(s))ds \\ &\quad + 2\int_0^t \langle \frac{\partial}{\partial s}\left[\mu_{mx}(s)u_{mx}^{(k)}(s)\right], \triangle u_m^{(k)}(s)\rangle ds - 2\langle \mu_{mx}(t)u_{mx}^{(k)}(t), \triangle u_m^{(k)}(t)\rangle \\ &\quad + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t \langle F_m'(s), \triangle u_m^{(k)}(s)\rangle ds \\ &\quad - 2\langle F_m(t), \triangle u_m^{(k)}(t)\rangle + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\ &\equiv S_m^{(k)}(0) + 2\left\langle \mu_{mx}(0)\tilde{u}_{0kx}, \Delta\tilde{u}_{0k}\right\rangle + 2\langle F_m(0), \triangle\tilde{u}_{0k}\rangle + \sum_{j=1}^{8} I_j. \end{aligned} \tag{3.15}$$

We shall estimate $I_j,\ j = 1, ..., 8$ on the right-hand side of (3.15) as follows.

First term I_1. We note that

$$\mu_m'(x,t) = D_2\mu[u_{m-1}] + D_3\mu[u_{m-1}] \int_0^1 \frac{\partial\sigma[u_{m-1}]}{\partial t}(x,y,t)dy, \tag{3.16}$$

where

$$\begin{aligned} D_i\mu[u_{m-1}] &= D_i\mu\left(x, t, \int_0^1 \sigma(x,y,t,u_{m-1}(y,t), \nabla u_{m-1}(y,t))dy\right),\ i = 1,2,3, \\ \frac{\partial\sigma[u_{m-1}]}{\partial t}(x,y,t) &= D_3\sigma[u_{m-1}] + D_4\sigma[u_{m-1}]u_{m-1}'(y,t) + D_5\sigma[u_{m-1}]\nabla u_{m-1}'(y,t), \\ D_i\sigma[u_{m-1}](x,y,t) &= D_i\sigma(x,y,t,u_{m-1}(y,t), \nabla u_{m-1}(y,t)),\ i = 1, \cdots, 5. \end{aligned}$$

Then, by (3.1), (3.2) and (3.16), we obtain

$$|\mu_m'(x,t)| \le \mu^*. \tag{3.17}$$

Hence

$$I_1 = \int_0^t ds \int_0^1 \mu_m'(x,s)|\triangle u_m^{(k)}(x,s)|^2 dx \le \frac{\mu^*}{\mu_0}\int_0^t S_m^{(k)}(s)ds. \tag{3.18}$$

Second term I_2. By Lemma 3.2 (ii) and (iv), we have

$$|I_2| = \left| \int_0^t \frac{\partial A_m}{\partial s}(s; u_m^{(k)}(s), u_m^{(k)}(s))ds \right| \le \mu^* \int_0^t \left\| u_m^{(k)}(s) \right\|_a^2 ds \le \frac{\mu^*}{\mu_0} \int_0^t S_m^{(k)}(s)ds. \tag{3.19}$$

Third term I_3. By using Cauchy - Schwartz inequality, we get that

$$|I_3| = 2 \left| \int_0^t \left\langle \frac{\partial}{\partial s} \left[\mu_{mx}(s) u_{mx}^{(k)}(s) \right], \triangle u_m^{(k)}(s) \right\rangle ds \right| \le \frac{2}{\sqrt{\mu_0}} \int_0^t J_m^{(k)}(s) \sqrt{S_m^{(k)}(s)} ds, \tag{3.20}$$

where $J_m^{(k)}(s) = \left\| \frac{\partial}{\partial s} \left[\mu_{mx}(s) u_{mx}^{(k)}(s) \right] \right\|$. By the fact that $S_m^{(k)}(t) \ge \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| u_{mx}^{(k)}(t) \right\|^2$, we have

$$\begin{aligned} J_m^{(k)}(s) &= \left\| \frac{\partial}{\partial s} \left[\mu_{mx}(s) u_{mx}^{(k)}(s) \right] \right\| \\ &\le \| \mu_{mx}(s) \|_{C^0(\bar{\Omega})} \left\| \dot{u}_{mx}^{(k)}(s) \right\| + \| \dot{\mu}_{mx}(s) \| \left\| u_{mx}^{(k)}(s) \right\|_{C^0(\bar{\Omega})} \\ &\le \left(\| \mu_{mx}(s) \|_{C^0(\bar{\Omega})} + \sqrt{\frac{1}{\mu_0}} \| \dot{\mu}_{mx}(s) \| \right) \sqrt{S_m^{(k)}(s)}. \end{aligned} \tag{3.21}$$

On the other hand, by $\frac{\partial \mu_m}{\partial x}(x,t) = D_1 \mu[u_{m-1}] + D_3 \mu[u_{m-1}] \int_0^1 D_1 \sigma[u_{m-1}](x,y,t)dy$, it implies that

$$\| \mu_{mx}(s) \|_{C^0(\bar{\Omega})} \le \tilde{K}_M(\mu) \left(1 + \bar{H}_M(\sigma) \right). \tag{3.22}$$

Similarly, by the following equality

$$\begin{aligned} \mu'_{mx}(x,t) &= \frac{\partial}{\partial s} \left[\frac{\partial \mu_m}{\partial x}(x,t) \right] \\ &= D_2 D_1 \mu[u_{m-1}] + D_3 D_1 \mu[u_{m-1}] \int_0^1 \frac{\partial \sigma[u_{m-1}]}{\partial t}(x,y,t)dy \\ &\quad + \left(D_2 D_3 \mu[u_{m-1}] + D_3^2 \mu[u_{m-1}] \int_0^1 \frac{\partial \sigma[u_{m-1}]}{\partial t}(x,y,t)dy \right) \int_0^1 D_1 \sigma[u_{m-1}](x,y,t)dy \\ &\quad + D_3 \mu[u_{m-1}] \int_0^1 \frac{\partial D_1 \sigma[u_{m-1}]}{\partial t}(x,y,t)dy; \frac{\partial \sigma[u_{m-1}]}{\partial t}(x,y,t) \\ &= D_3 \sigma[u_{m-1}](x,y,t) + D_4 \sigma[u_{m-1}](x,y,t) u'_{m-1}(y,t) \\ &\quad + D_3 \sigma[u_{m-1}](x,y,t) \nabla u'_{m-1}(y,t); \frac{\partial D_1 \sigma[u_{m-1}]}{\partial t}(x,y,t) \\ &= D_3 D_1 \sigma[u_{m-1}](x,y,t) + D_4 D_1 \sigma[u_{m-1}](x,y,t) u'_{m-1}(y,t) \\ &\quad + D_5 D_1 \sigma[u_{m-1}](x,y,t) \nabla u'_{m-1}(y,t), \end{aligned}$$

hence, we obtain

$$\begin{aligned} \| \dot{\mu}_{mx}(t) \| &\le \tilde{K}_M(\mu) \left[1 + \bar{H}_M(\sigma) \int_0^1 \left(1 + \left\| u'_{m-1}(t) \right\| + \left\| \nabla u'_{m-1}(t) \right\| \right) dy \right] \\ &\quad + \tilde{K}_M(\mu) \bar{H}_M(\sigma) \left[1 + \bar{H}_M(\sigma) \int_0^1 \left(1 + \left\| u'_{m-1}(t) \right\| + \left\| \nabla u'_{m-1}(t) \right\| \right) dy \right] \\ &\quad + \tilde{K}_M(\mu) \bar{H}_M(\sigma) \int_0^1 \left(1 + \left\| u'_{m-1}(t) \right\| + \left\| \nabla u'_{m-1}(t) \right\| \right) dy \\ &\le \tilde{K}_M(\mu) \left[1 + 2(1+M) \bar{H}_M(\sigma) + 2(1+2M) \bar{H}_M^2(\sigma) \right]. \end{aligned} \tag{3.23}$$

By (3.22) and (3.23), it follows from (3.21) that

$$J_m^{(k)}(s) \le \zeta_1(M) \sqrt{S_m^{(k)}(s)}, \tag{3.24}$$

where

$$\zeta_1(M)=\tilde{K}_M(\mu)\left(1+\bar{H}_M(\sigma)+\sqrt{\frac{1}{\mu_0}}\left[1+2(1+M)\bar{H}_M(\sigma)+2(1+2M)\bar{H}_M^2(\sigma)\ \right]\right). \tag{3.25}$$

Therefore, we derive from (3.19) and (3.23) that

$$I_3\le\frac{2}{\sqrt{\mu_0}}\zeta_1(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.26}$$

Fourth term I_4. Using Cauchy - Schwartz inequality again, we have

$$|I_4|=\left|-2\langle\mu_{mx}(t)u_{mx}^{(k)}(t),\triangle u_m^{(k)}(t)\rangle\right|\le\frac{1}{\beta}\left\|\mu_{mx}(t)u_{mx}^{(k)}(t)\right\|^2+\frac{\beta}{\mu_0}S_m^{(k)}(t), \tag{3.27}$$

for all $\beta>0$. On the other hand, it follows from (3.24) that

$$\begin{aligned}\left\|\mu_{mx}(t)u_{mx}^{(k)}(t)\right\|&=\left\|\mu_{mx}(0)\nabla\tilde{u}_{0k}+\int_0^t\frac{\partial}{\partial s}\left[\mu_{mx}(s)u_{mx}^{(k)}(s)\right]ds\right\| \\ &\le\|\mu_{mx}(0)\|_{C^0(\bar{\Omega})}\|\nabla\tilde{u}_{0k}\|+\int_0^t J_m^{(k)}(s)ds \\ &\le\|\mu_{mx}(0)\|_{C^0(\bar{\Omega})}\|\nabla\tilde{u}_{0k}\|+\zeta_1(M)\int_0^t\sqrt{S_m^{(k)}(s)}ds.\end{aligned} \tag{3.28}$$

Hence, we deduce from (3.27) and (3.28) that

$$|I_4|\le\frac{\beta}{\mu_0}S_m^{(k)}(t)+\frac{2}{\beta}\|\mu_{mx}(0)\|_{C^0(\bar{\Omega})}^2\|\nabla\tilde{u}_{0k}\|^2+\frac{2}{\beta}T\zeta_1^2(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.29}$$

Fifth term I_5.

$$|I_5|=2\left|\int_0^t\langle F_m(s),\dot{u}_m^{(k)}(s)\rangle ds\right|\le TK_M^2(f)+\int_0^t S_m^{(k)}(s)ds. \tag{3.30}$$

Sixth term I_6. Using Cauchy - Schwartz inequality, we have

$$|I_6|=\left|2\int_0^t\langle F_m'(s),\triangle u_m^{(k)}(s)\rangle ds\right|\le\int_0^t\|F_m'(s)\|^2ds+\frac{1}{\mu_0}\int_0^t S_m^{(k)}(s)ds. \tag{3.31}$$

Note that

$$\begin{aligned}F_m'(t)&=D_2f[u_{m-1}]+D_3f[u_{m-1}].u_{m-1}'+D_4f[u_{m-1}].\nabla u_{m-1}'+D_5f[u_{m-1}]u_{m-1}'' \\ &\quad+D_6f[u_{m-1}].\int_0^1\frac{\partial g[u_{m-1}]}{\partial t}(x,y,t)dy; \\ \frac{\partial g[u_{m-1}]}{\partial t}(x,y,t)&=D_3g[u_{m-1}](x,y,t)+D_4g[u_{m-1}](x,y,t)u_{m-1}'(y,t) \\ &\quad+D_5g[u_{m-1}](x,y,t)\nabla u_{m-1}'(y,t)+D_6g[u_{m-1}](x,y,t)u_{m-1}''(y,t),\end{aligned}$$

hence we get that

$$\|F_m'(t)\|=K_M(f)\left(1+3M\right)\left[1+H_M(g)\right]. \tag{3.32}$$

Then, we deduece from (3.31) and (3.32) that

$$|I_6|\le TK_M^2(f)\left(1+3M\right)^2\left[1+H_M(g)\right]^2+\frac{1}{\mu_0}\int_0^t S_m^{(k)}(s)ds. \tag{3.33}$$

Seventh term I_7.We have

$$\begin{aligned}|I_7|&=\left|-2\langle F_m(t),\triangle u_m^{(k)}(t)\rangle\right|\le\frac{1}{\beta}||F_m(t)||^2+\beta\left\|\Delta u_m^{(k)}(t)\right\|^2 \\ &\le\frac{2}{\beta}\left(\|F_m(0)\|^2+T\int_0^t\|F_m'(s)\|^2ds\right)+\frac{\beta}{\mu_0}S_m^{(k)}(t) \\ &=\frac{2}{\beta}\left(\|F_m(0)\|^2+TK_M^2(f)\left(1+3M\right)^2\left[1+H_M(g)\right]^2\right)+\frac{\beta}{\mu_0}S_m^{(k)}(t),\ \text{for all }\beta>0.\end{aligned} \tag{3.34}$$

Eighth term I_8. We note that the equation $(3.8)_1$ can be rewritten as follows

$$\left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle - \left\langle \frac{\partial}{\partial x}\left(\mu_m(t) u_{mx}^{(k)}(t)\right), w_j \right\rangle = \langle F_m(t), w_j \rangle, 1 \le j \le k. \tag{3.35}$$

After replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating, we get that

$$\begin{aligned} I_8 = \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds &\le 2\int_0^t \left\| \frac{\partial}{\partial x}\left(\mu_m(s) u_{mx}^{(k)}(s)\right) \right\|^2 ds + 2\int_0^t \|F_m(s)\|^2 ds \\ &\le 2\int_0^t \left\| \frac{\partial}{\partial x}\left(\mu_m(s) u_{mx}^{(k)}(s)\right) \right\|^2 ds + 2T K_0^2(M,f). \end{aligned} \tag{3.36}$$

By (3.22), we have

$$\begin{aligned} \left\| \frac{\partial}{\partial x}\left(\mu_m(s) u_{mx}^{(k)}(s)\right) \right\|^2 &\le \left(\left\| \mu_{mx}(s) u_{mx}^{(k)}(s) \right\| + \left\| \mu_m(s) \Delta u_m^{(k)}(s) \right\| \right)^2 \\ &\le 2\tilde{K}_M(\mu) \left(\tilde{K}_M(\mu) \left(1 + \bar{H}_M(\sigma)\right)^2 \left\| u_{mx}^{(k)}(s) \right\|^2 + \left\| \sqrt{\mu_m(s)} \Delta u_m^{(k)}(s) \right\|^2 \right) \\ &\le 2\tilde{K}_M(\mu) \left(1 + \tilde{K}_M(\mu)\left(1 + \bar{H}_M(\sigma)\right)^2\right) \left(\left\| u_{mx}^{(k)}(s) \right\|^2 + \left\| \sqrt{\mu_m(s)} \Delta u_m^{(k)}(s) \right\|^2 \right) \\ &\le 2\tilde{K}_M(\mu) \left(1 + \tilde{K}_M(\mu)\left(1 + \bar{H}_M(\sigma)\right)^2\right) \left(\frac{1+\mu_0}{\mu_0} \right) S_m^{(k)}(s). \end{aligned} \tag{3.37}$$

Therefore, we deduce from (3.36) and (3.37) that

$$I_8 \le 2T K_0^2(M,f) + \zeta_2(M) \int_0^t S_m^{(k)}(s) ds, \tag{3.38}$$

where

$$\zeta_2(M) = 4\tilde{K}_M(\mu) \left(\frac{1+\mu_0}{\mu_0} \right) \left[1 + \tilde{K}_M(\mu)\left(1 + \bar{H}_M(\sigma)\right)^2 \right]. \tag{3.39}$$

Choosing $\beta > 0$, with $\frac{2\beta}{\mu_0} \le \frac{1}{2}$, it follows from (3.15), (3.18) - (3.20), (3.26), (3.29) - (3.30), (3.33) - (3.34) and (3.38), that

$$\begin{aligned} S_m^{(k)}(t) \le \tilde{C}_0^{(k)} &+ 2T \left[K_M^2(f) \left(1 + \frac{2}{\beta}(1+3M)^2 (1 + H_M(g))^2 \right) + K_0^2(M,f) \right] \\ &+ \tilde{C}_1(M,T) \int_0^t S_m^{(k)}(s) ds, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} \tilde{C}_0^{(k)} = \tilde{C}_0^{(k)}(\mu, \sigma, f, g, \tilde{u}_{0k}, \tilde{u}_{1k}) &= 2S_m^{(k)}(0) + 4\langle \mu_{mx}(0) \tilde{u}_{0kx}, \Delta \tilde{u}_{0k} \rangle \\ &+ 4\langle F_m(0), \triangle \tilde{u}_{0k} \rangle + \frac{4}{\beta} \|\mu_{mx}(0)\|_{C^0(\bar{\Omega})}^2 \|\nabla \tilde{u}_{0k}\|^2 + \frac{4}{\beta}\|F_m(0)\|^2, \\ \tilde{C}_1(M,T) = 2\left[1 + \frac{1+2\mu^*}{\mu_0} + \frac{2}{\beta} T \zeta_1^2(M) + \frac{2}{\sqrt{\mu_0}} \zeta_1(M) + \zeta_2(M) \right]&. \end{aligned} \tag{3.41}$$

Due to the convergences given in (3.9), there is a constant $M > 0$ independent of k and m such that

$$\tilde{C}_0^{(k)}(\mu, \sigma, f, g, \tilde{u}_{0k}, \tilde{u}_{1k}) \le \frac{1}{2} M^2. \tag{3.42}$$

So, from (3.40) and (3.42), we can choose $T \in (0, T^*]$ such that

$$\left[\frac{1}{2}M^2 + 2T\left(K_M^2(f) \left(1 + \frac{2}{\beta}(1+3M)^2 (1 + H_M(g))^2 \right) + K_0^2(M,f) \right) \right] \exp\left(T \tilde{C}_1(M,T) \right) \le M^2, \tag{3.43}$$

and

$$k_T = 2\sqrt{T}\left(1+\frac{1}{\sqrt{\mu_0}}\right)\sqrt{M^2\tilde{K}_M^2(\mu)\bar{H}_M^2(\sigma)\left(1+\sqrt{2}(2+\bar{H}_M(\sigma))\right)^2+K_M^2(f)\left(1+H_M(g)\right)^2} \times \exp\left[T\left(\frac{2\mu_0+\mu^*}{2\mu_0}\right)\right] < 1. \tag{3.44}$$

Finally, it follows from (3.40), (3.42) and (3.43) that

$$S_m^{(k)}(t) \le M^2 \exp\left(-T\tilde{C}_1(M,T)\right) + \tilde{C}_1(M,T)\int_0^t S_m^{(k)}(s)ds. \tag{3.45}$$

By using Gronwall's Lemma, we deduce from (3.45) that

$$S_m^{(k)}(t) \le M^2 \exp\left(-T\tilde{C}_1(M,T)\right)\exp\left(t\tilde{C}_1(M,T)\right) \le M^2, \tag{3.46}$$

for all $t \in [0,T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k. \tag{3.47}$$

Step 3. Limiting process. By (3.47), there is a subsequence of $\{u_m^{(k)}\}$ which is denoted by the same symbol such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weak*}, \\ \dot{u}_m^{(k)} \to \dot{u}_m & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to \ddot{u}_m & \text{in} \quad L^2(Q_T) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.48}$$

By taking the limitations in (3.8), we have u_m satisfying (3.5) and (3.6) in $L^2(0,T)$.

On the other hand, it follows from $(3.5)_1$ and $(3.48)_4$ that $u_m'' = \frac{\partial}{\partial x}\left(\mu_m(t)u_{mx}\right) + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$. Theorem 3.1 is proved completely. □

Using Theorem 3.1 and the arguments of compactness, we shall prove the existence and uniqueness of weak solution for the problem (1.1)-(1.3) which is obtained in the following theorem.

Theorem 3.3. Let $(H_1)-(H_5)$ *hold. The recurrent sequence* $\{u_m\}$ *defined by* (3.4)-(3.5) *converges strongly in*

$$W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}, \tag{3.49}$$

to a function u that is a unique weak solution of the problem (1.1)-(1.3). Furthermore, we have the following estimate

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.50}$$

where $k_T \in [0,1)$ *is defined by* (3.44) *and* C_T *is a constant depending only on* T, h_0, f, g, μ, σ, $\tilde{u}_0$, $\tilde{u}_1$ *and* k_T.

Proof . (a) *Existence of solution.* First, we note that $W_1(T)$ is a Banach space with the corresponding norm (see Lions [10]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)}. \tag{3.51}$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + A_{m+1}(t, w_m(t), w) = -A_{m+1}(t, u_m(t), w) + A_m(t, u_m(t), w) \\ \qquad\qquad + \langle F_{m+1}(t) - F_m(t), w\rangle, \ \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.52}$$

Note that

$$\begin{cases} \dfrac{d}{dt}A_{m+1}(t, w_m(t), w_m(t)) = 2A_{m+1}(t, w_m(t), w_m'(t)) + \dfrac{\partial A_{m+1}}{\partial t}(t, w_m(t), w_m(t)), \\ A_{m+1}(t, u_m(t), w_m'(t)) - A_m(t, u_m(t), w_m'(t)) = -\left\langle \dfrac{\partial}{\partial x}\left[(\mu_{m+1}(t) - \mu_m(t))\, u_{mx}(t)\right], w_m'(t)\right\rangle. \end{cases} \tag{3.53}$$

Taking $w = w'_m$ in $(3.52)_1$, after integrating in t, we get

$$Z_m(t) = \int_0^t \frac{\partial A_{m+1}}{\partial t}(s, w_m(s), w_m(s))ds + 2\int_0^t \left\langle \frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s) - \mu_m(s)\right) u_{mx}(s)\right], w'_m(s)\right\rangle ds \tag{3.54}$$
$$+ 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w'_m(s)\right\rangle ds$$
$$\equiv J_1 + J_2 + J_3,$$

where

$$Z_m(t) = \|w'_m(t)\|^2 + A_{m+1}(t, w_m(t), w_m(t)) \geq \|w'_m(t)\|^2 + \mu_0 \|w_m(t)\|_a^2. \tag{3.55}$$

Next, we estimate the integrals on the right-hand side of (3.54) as follows.

First integral J_1. By $(3.12)_{(iv)}$ and (3.55), we have

$$|J_1| \leq \int_0^t \left|\frac{\partial A_{m+1}}{\partial t}(s, w_m(s), w_m(s))\right| ds \leq \frac{\mu^*}{\mu_0}\int_0^t Z_m(s)ds. \tag{3.56}$$

Second integral J_2. By the following inequalities

$$\begin{cases} \|\Delta u_m(s)\| \leq \|u_m(s)\|_{H^2} \leq M, \\ \|u_{mx}(s)\|_{C^0(\bar{\Omega})} \leq \sqrt{2}\,\|u_{mx}(s)\|_{H^1} \leq \sqrt{2}\,\|u_m(s)\|_{H^2} \leq \sqrt{2}M, \\ \|D_i\mu[u_m](s)\|_{C^0(\bar{\Omega})} \leq \tilde{K}_M(\mu), \quad i = 1,\ 3, \\ \|D_1\sigma[u_m](s)\|_{C^0(\bar{\Omega})} \leq \bar{H}_M(\sigma), \\ \|\mu_{m+1}(s) - \mu_m(s)\|_{C^0(\bar{\Omega})} \leq 2\tilde{K}_M(\mu)\bar{H}_M(\sigma)\,\|\nabla w_{m-1}(s)\| \leq 2\tilde{K}_M(\mu)\bar{H}_M(\sigma)\,\|w_{m-1}\|_{W_1(T)}, \\ \|D_i\mu[u_m](s) - D_i\mu[u_{m-1}](s)\|_{C^0(\bar{\Omega})} \leq 2\tilde{K}_M(\mu)\bar{H}_M(\sigma)\,\|w_{m-1}\|_{W_1(T)},\ i = 1,\ 3, \\ \|D_1g[u_m](s) - D_1g[u_{m-1}](s)\|_{C^0(\bar{\Omega})} \leq 2\bar{H}_M(\sigma)\,\|w_{m-1}\|_{W_1(T)}, \end{cases} \tag{3.57}$$

and the equality

$$\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s) - \mu_m(s)\right)\nabla u_m(s)\right] = \left(\mu_{m+1}(s) - \mu_m(s)\ \right)\Delta u_m(s) + \left(D_1\mu[u_m](s) - D_1\mu[u_{m-1}](s)\ \right)u_{mx}(s) \tag{3.58}$$
$$+ \left[\left(D_3\mu[u_m](s) - D_3\mu[u_{m-1}](s)\ \right)\int_0^1 D_1g[u_m](x,y,s)dy\right]u_{mx}(s)$$
$$+ \left[D_3\mu[u_{m-1}](s)\int_0^1 \left(D_1g[u_m] - D_1g[u_{m-1}]\ \right)dy\right]u_{mx}(s),$$

we obtain that

$$\left\|\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s) - \mu_m(s)\right)\nabla u_m(s)\right]\right\| \leq 2M\tilde{K}_M(\mu)\bar{H}_M(\sigma)\left[1 + \sqrt{2}(2 + \bar{H}_M(\sigma))\ \right]\|w_{m-1}\|_{W_1(T)}. \tag{3.59}$$

This implies that

$$|J_2| \leq 2\left|\int_0^t \left\langle \frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s) - \mu_m(s)\right)\nabla u_m(s)\right], w'_m(s)\right\rangle ds\right| \tag{3.60}$$
$$\leq 4TM^2\tilde{K}_M^2(\mu)\bar{H}_M^2(\sigma)\left[1 + \sqrt{2}(2 + \bar{H}_M(\sigma))\ \right]^2\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds.$$

Third integral J_3.

$$|J_3| \leq 2\left|\int_0^t \left\langle F_{m+1}(s) - F_m(s), w'_m(s)\right\rangle ds\right| \tag{3.61}$$
$$\leq 2\int_0^t \|F_{m+1}(s) - F_m(s)\|\,\|w'_m(s)\|\,ds$$
$$\leq \int_0^t \|F_{m+1}(s) - F_m(s)\|^2 ds + \int_0^t \|w'_m(s)\|^2 ds.$$

By (H_2) and (H_4), we have

$$\begin{aligned}\|F_{m+1}(t)-F_m(t)\| &\leq K_M(f)\left(\|w_{m-1}(t)\|+\|\nabla w_{m-1}(t)\|+\left\|w_{m-1}'(t)\right\|\right) \\ &\quad + K_M(f)H_M(g)\int_0^1\left(|w_{m-1}(y,t)|+|\nabla w_{m-1}(y,t)|+\left|w_{m-1}'(y,t)\right|\right)dy \\ &\leq K_M(f)\left(2\|\nabla w_{m-1}(t)\|+\left\|w_{m-1}'(t)\right\|\right)+K_M(f)H_M(g)\left(2\|\nabla w_{m-1}(t)\|+\left\|w_{m-1}'(t)\right\|\right) \\ &\leq 2K_M(f)(1+H_M(g))\|w_{m-1}\|_{W_1(T)}.\end{aligned} \tag{3.62}$$

Hence

$$\int_0^t\|F_{m+1}(s)-F_m(s)\|^2 ds \leq 4TK_M^2(f)[1+H_M(g)]^2\|w_{m-1}\|_{W_1(T)}^2. \tag{3.63}$$

Then, we deduce from (3.61) and (3.63) that

$$|J_3| \leq 4TK_M^2(f)[1+H_M(g)]^2\|w_{m-1}\|_{W_1(T)}^2+\int_0^t Z_m(s)ds. \tag{3.64}$$

Combining (3.54), (3.56), (3.60) and (3.64), we obtain

$$\begin{aligned}Z_m(t) &\leq 4T\left[M^2\tilde{K}_M^2(\mu)\bar{H}_M^2(\sigma)\left(1+\sqrt{2}(2+\bar{H}_M(\sigma))\right)^2+K_M^2(f)\left(1+H_M(g)\right)^2\right]\|w_{m-1}\|_{W_1(T)}^2 \\ &\quad + \frac{2\mu_0+\mu^*}{\mu_0}\int_0^t Z_m(s)ds.\end{aligned} \tag{3.65}$$

By using Gronwall's lemma, we derive from (3.65) that

$$\|w_m\|_{W_1(T)} \leq k_T\|w_{m-1}\|_{W_1(T)}, \ \forall m\in\mathbb{N}, \tag{3.66}$$

where $k_T\in(0,1)$ is defined as in (3.44).

The estimate (3.66) implies that

$$\|u_m-u_{m+p}\|_{W_1(T)} \leq \|u_0-u_1\|_{W_1(T)}(1-k_T)^{-1}k_T^m, \ \forall m,p\in\mathbb{N}. \tag{3.67}$$

This follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then, there exists $u\in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.68}$$

Due to $u_m\in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j}\to u & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weak*}, \\ u_{m_j}'\to u' & \text{in} \quad L^\infty(0,T;V) \text{ weak*}, \\ u_{m_j}''\to u'' & \text{in} \quad L^2(Q_T) \text{ weak}, \\ u\in W(M,T). \end{cases} \tag{3.69}$$

We also note that

$$|\mu_m(x,t)-\mu[u](x,t)| \leq 2\tilde{K}_M(\mu)\bar{H}_M(\sigma)\|u_{m-1}-u\|_{W_1(T)}, \text{ a.e. } (x,t)\in Q_T. \tag{3.70}$$

Hence, from (3.68) and (3.70), we obtain

$$\mu_m \to \mu[u] \text{ strongly in } L^\infty(Q_T), \tag{3.71}$$

On the other hand, for all $v\in V$, we have

$$|A_m(t;u_m,v)-A[u](t;u,v)| \leq (1+h_0)\tilde{K}_M(\mu)\left[2\bar{H}_M(\sigma)M\|u_{m-1}-u\|_{W_1(T)}+\|u_m-u\|_{W_1(T)}\right]\|v_x\|. \tag{3.72}$$

Hence

$$\int_0^T (A_m(t;u_m,v) - A[u](t;u,v))\,\phi(t)dt \to 0,\ \forall v \in V,\ \forall \phi \in L^1(0,T). \tag{3.73}$$

Moreover, we aslo have

$$\|F_m(t) - f[u](t)\|_{L^\infty(0,T;L^2)} \le 2K_M(f)\,(1 + H_M(g))\,\|u_{m-1} - u\|_{W_1(T)}\,. \tag{3.74}$$

Therefore, it implies from (3.68) and (3.74) that

$$F_m(t) \to f[u](t) \text{ strong in } L^\infty(0,T;L^2). \tag{3.75}$$

Finally, taking the limitaions in (3.5)–(3.6) as $m = m_j \to \infty$, it implies from (3.68), $(3.69)_{1,3}$, (3.73) and (3.75) that there exists $u \in W(M,T)$ satisfying

$$\langle u''(t), w\rangle + A[u](t;u(t),w) = \langle f[u](t), w\rangle\,, \tag{3.76}$$

for all $w \in V$ and

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \tag{3.77}$$

On the other hand, by the assumptions $(H_2) - (H_5)$, we obtain from $(3.69)_4$, (3.75) and (3.76) that

$$u'' = \frac{\partial}{\partial x}\,(\mu[u](t)u_x) + f[u](t) \in L^\infty(0,T;L^2). \tag{3.78}$$

Thus, we have $u \in W_1(M,T)$. Then, the existence of solution is confirmed.

(b) *Uniqueness of solution.*

Let $u_1,\ u_2 \in W_1(M,T)$ be two weak solutions of (1.1) - (1.3). Then $u = u_1 - u_2$ satisfies the following variational problem

$$\begin{cases} \langle u''(t), w\rangle + A[u_1](t;u(t),w) = -A[u_1](t;u_2(t),w) + A[u_2](t;u_2(t),w) + \langle F_1(t) - F_2(t), w\rangle\,, \forall w \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.79}$$

where

$$\begin{aligned} A[u_i](t;u,w) &= \langle \mu[u_i](t)u_x, w_x\rangle + h_0\mu[u_i](0,t)u(0)w(0),\ u,\ w \in V, \\ \mu[u_i](x,t) &= \mu\left(x,t,\int_0^1 \sigma\left(x,y,t,u_i(y,t),\nabla u_i(y,t)dy\right)\right),\ i = 1,2, \\ F_i(x,t) = f[u_i](x,t) &= f\left(x,t,u_i,\nabla u_i,u_i',\int_0^1 g[u_i](x,y,t)dy\right),\ i = 1,2, \\ g[u_i](x,y,t) &= g\left(x,y,t,u_i(y,t),\nabla u_i(y,t),u_i'(y,t)\right),\ i = 1,2. \end{aligned} \tag{3.80}$$

Taking $w = u'$ in $(3.79)_1$ and integrating in t, we get

$$\begin{aligned} Z(t) = &\int_0^t \frac{\partial A[u_1]}{\partial t}(s;u(s),u(s))ds + 2\int_0^t \left\langle \frac{\partial}{\partial x}\left[(\mu[u_1](s) - \mu[u_2](s))\,\nabla u_2(s)\right], u'(s)\right\rangle ds \\ &+ 2\int_0^t \langle F_1(s) - F_2(s), u'(s)\rangle\, ds, \end{aligned} \tag{3.81}$$

where $Z(t) = \|u'(t)\|^2 + A[u_1](t;u(t),u(t))$.

By making similarly the above estimates, we derive from (3.81) that

$$Z(t) \le \tilde{Z}_M \int_0^t Z(s)ds, \tag{3.82}$$

where $\tilde{Z}_M = \dfrac{\mu^*}{\mu_0} + \dfrac{1}{\sqrt{\mu_0}}TM\tilde{K}_M(\mu)\bar{H}_M(\sigma)\left[1 + \sqrt{2}(2 + \bar{H}_M(\sigma))\ \right] + 4K_M(f)\,[1 + H_M(g)]\left(1 + \dfrac{1}{\sqrt{\mu_0}}\right)$.

Finally, using Gronwall's lemma, we deduce from (3.82) that $Z(t) = 0$, i.e., $u_1 \equiv u_2$. Therefore, Theorem 3.3 is proved completely. □

4 Asymptotic expansion of solution

In this section, we suppose that $(H_1)-(H_5)$ hold. In order to establish an asymptotic expansion of weak solution of perturbed problem in a small parameter, we need the following additional assumptions:

(H_6) $f_1 \in C^1([0,1]\times\mathbb{R}_+\times\mathbb{R}^4),\ g_1 \in C^1([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^3)$;

(H_7) $\mu_1 \in C^2([0,1]\times\mathbb{R}_+\times\mathbb{R}),\ \sigma_1 \in C^2([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2)$ and $\mu_1(x,t,z)\geq 0,\ \forall(x,t,z)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}$.

Then, we consider the following perturbed problem in a small parameter ε

$$(P_\varepsilon)\begin{cases} u_{tt}-\dfrac{\partial}{\partial x}\left[\mu_\varepsilon[u](x,t)u_x\right]=f_\varepsilon[u](x,t),\ 0<x<1,\ 0<t<T,\\ u_x(0,t)-h_0u(0,t)=u(1,t)=0,\\ u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x),\end{cases}$$

where

$$\begin{aligned}\mu_\varepsilon[u](x,t)&=\mu\left(x,t,\int_0^1\sigma[u](x,y,t)dy\right)+\varepsilon\mu_1\left(x,t,\int_0^1\sigma_1[u](x,y,t)dy\right),\\ f_\varepsilon[u](x,t)&=f\left(x,t,u,u_x,u_t,\int_0^1 g[u](x,y,t)dy\right)+\varepsilon f_1\left(x,t,u,u_x,u_t,\int_0^1 g_1[u](x,y,t)dy\right),\\ g_1[u](x,y,t)&=g_1(x,y,t,u(y,t),u_x(y,t),u_t(y,t)),\\ \sigma_1[u](x,y,t)&=\sigma_1(x,y,t,u(y,t),u_x(y,t)).\end{aligned}$$

We note that, by Theorem 3.3, (P_ε) has a unique weak solution u_ε depending on ε, satisfying $u_\varepsilon\in W_1(M,T)$, in which $M,\ T$ are independent of ε, these constants are chosen as in (3.38), (3.40) and (*3.41*), with $K_M(f)+K_M(f_1)$, $\tilde{K}_M(\mu)+\tilde{K}_M(\mu_1)$, $H_M(g)+H_M(g_1)$, $\bar{H}_M(\sigma)+\bar{H}_M(\sigma_1)$ stand for $K_M(f)$, $\tilde{K}_M(\mu)$, $H_M(g)$, $\bar{H}_M(\sigma)$ respectively.

Moreover, we can prove that the limitation u_0 in suitable function spaces of the family $\{u_\varepsilon\}$ as $\varepsilon\to 0$ is a unique weak solution of the problem (P_0) (corresponding to $\varepsilon=0$) also satisfying $u_0\in W_1(M,T)$.

In what follows, we shall study the asymptotic expansion of the solution of the problem (P_ε) with respect to a small parameter ε.

For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)\in\mathbb{Z}_+^N$, and $x=(x_1,\cdots,x_N)\in\mathbb{R}^N$, we put

$$\begin{cases}|\alpha|=\alpha_1+\cdots+\alpha_N,\quad \alpha!=\alpha_1!\cdots\alpha_N!,\\ \alpha,\ \beta\in\mathbb{Z}_+^N,\quad \alpha\leq\beta\Longleftrightarrow\alpha_i\leq\beta_i\quad\forall i=1,\cdots,N,\\ x^\alpha=x_1^{\alpha_1}\cdots x_N^{\alpha_N}.\end{cases}$$

First, we need the following lemma.

Lemma 4.1. Let $m,\ N\in\mathbb{N}$ *and* $x=(x_1,\cdots,x_N)\in\mathbb{R}^N$, $\varepsilon\in\mathbb{R}$. *Then*

$$\left(\sum_{i=1}^N x_i\varepsilon^i\right)^m=\sum_{k=m}^{mN}P_k^{(m)}[N,x]\varepsilon^k, \tag{4.1}$$

where the coefficients $P_k^{(m)}[N,x]$, $m\leq k\leq mN$ *depending on* $x=(x_1,\cdots,x_N)$ *defined by the formulas*

$$\begin{cases}P_k^{(m)}[N,x]=\begin{cases}u_k, & 1\leq k\leq N,\ m=1,\\ \displaystyle\sum_{\alpha\in A_k^{(m)}(N)}\frac{m!}{\alpha!}x^\alpha, & m\leq k\leq mN,\ m\geq 2,\end{cases}\\ A_k^{(m)}(N)=\{\alpha\in\mathbb{Z}_+^N:|\alpha|=m,\ \displaystyle\sum_{i=1}^N i\alpha_i=k\}.\end{cases} \tag{4.2}$$

The proof of Lemma 4.1 is easy, hence we omit the details. □

Now, we assume that

(H_8) $f \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4),\ f_1 \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4),$
$g \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4),\ g_1 \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^4);$

(H_9) $\mu \in C^{N+2}([0,1]\times\mathbb{R}_+\times\mathbb{R}),\ \mu_1 \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}),$
$\mu \geq \mu_0 > 0$ and $\mu_1 \geq 0$, for all $(x,t,z) \in [0,1]\times\mathbb{R}_+\times\mathbb{R}$,
$\sigma \in C^{N+2}([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2),\ \sigma_1 \in C^{N+1}([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2).$

For simplicity in presentation, we use the following notations

$$\begin{aligned}
&\mu[u](x,t) = \mu\left(x,t,\int_0^1 \sigma[u](x,y,t)dy\right),\\
&f[u](x,t) = f\left(x,t,u,u_x,u_t,\int_0^1 g[u](x,y,t)dy\right),\\
&g[u](x,y,t) = g(x,y,t,u(y,t),u_x(y,t),u_t(y,t)),\\
&\sigma[u](x,y,t) = \sigma(x,y,t,u(y,t),u_x(y,t)).
\end{aligned}$$

Let u_0 be a unique weak solution of the problem (P_0) corresponding to $\varepsilon = 0$, *i.e.*,

$$(P_0)\begin{cases}
u_0'' - \dfrac{\partial}{\partial x}\left(\mu[u_0])u_{0x}\right) = f[u_0],\ 0<x<1,\ 0<t<T,\\
u_{0x}(0,t) - h_0u_0(0,t) = u_0(1,t) = 0,\\
u_0(x,0) = \tilde{u}_0(x),\ u_0'(x,0) = \tilde{u}_1(x),\\
u_0 \in W_1(M,T).
\end{cases}$$

Let us consider the sequence of the weak solutions u_k, $1\leq k\leq N$, defined by the following problems:

$$(\tilde{P}_k)\begin{cases}
u_k'' - \dfrac{\partial}{\partial x}\left(\mu[u_0]u_{kx}\right) = \tilde{F}_k\left[u_k\right],\ 0<x<1,\ 0<t<T,\\
u_{kx}(0,t) - h_0u_k(0,t) = u_k(1,t) = 0,\\
u_k(x,0) = u_k'(x,0) = 0,\\
u_k \in W_1(M,T),
\end{cases}$$

where $\tilde{F}_k\left[u_k\right]$, $1\leq k\leq N$, are defined by

$$\tilde{F}_k\left[u_k\right] = \begin{cases}
f[u_0], & k=0,\\
\pi_k\left[N,f,g\right] + \pi_{k-1}\left[N-1,f_1,g_1\right] + \sum_{i=1}^{k}\dfrac{\partial}{\partial x}\left[\left(\rho_i\left[N,\mu,\sigma\right] + \rho_{i-1}\left[N-1,\mu_1,\sigma_1\right]\right)\nabla u_{k-i}\right], & 1\leq k\leq N,
\end{cases} \tag{4.3}$$

with $\pi_k\left[N,f,g\right]$ and $\rho_k\left[N,\mu,\sigma\right]$ are respectively defined by

$$a/\ \pi_k\left[N,f,g\right] = \begin{cases}
f[u_0], & k=0,\\
\sum_{|\gamma|=1}^{k}\frac{1}{\gamma!}D^\gamma f[u_0]\Phi_k[\gamma,N,g,u_0,\vec{u}], & 1\leq k\leq N,
\end{cases} \tag{4.4}$$

where $\vec{u} = (u_1,\cdots,u_N)$ and

$$\Phi_k[\gamma,N,g,u_0,\vec{u}] = \sum_{\substack{(k_1,k_2,k_3,k_4)\in\tilde{A}(\gamma,N)\\ k_1+k_2+k_3+k_4=k}} P_{k_1}^{(\gamma_1)}[N,\vec{u}]P_{k_2}^{(\gamma_2)}[N,\nabla\vec{u}]P_{k_3}^{(\gamma_3)}[N,\vec{u}']P_{k_4}^{(\gamma_4)}[N,\vec{\kappa}[N,g,u_0,\vec{u}]], \tag{4.5}$$

with

$$\begin{aligned}
&\tilde{A}(\gamma,N) = \{(k_1,\cdots,k_4)\in\mathbb{Z}_+^4 : \gamma_i \leq k_i \leq N\gamma_i,\ \forall i=1,2,3,4\},\\
&\gamma = (\gamma_1,\cdots,\gamma_4)\in\mathbb{Z}_+^4,\ 1\leq|\gamma|\leq N,
\end{aligned} \tag{4.6}$$

and $\vec{\kappa}[N, g, u_0, \vec{u}] = (\bar{\kappa}_1[N, g, u_0, \vec{u}], \cdots, \bar{\kappa}_N[N, g, u_0, \vec{u}])$ is defined by

$$\begin{aligned} \bar{\kappa}_k[N, g, u_0, \vec{u}] &= \sum_{1 \le |\beta| \le k} \frac{1}{\beta!} \int_0^1 D^\beta g[u_0] \Psi_k[\beta, N, \vec{u}] ds, \\ \Psi_k[\beta, N, \vec{u}] &= \sum_{\substack{(k_1, k_2, k_3) \in \widetilde{A}(\beta, N), \\ k_1 + k_2 + k_3 = k}} P_{k_1}^{(\beta_1)}[N, \vec{u}] P_{k_2}^{(\beta_2)}[N, \nabla \vec{u}] P_{k_3}^{(\beta_3)}[N, \vec{u}'], \\ \widetilde{A}(\beta, N) &= \{(k_1, k_2, k_3) \in \mathbb{Z}_+^3 : \beta_i \le k_i \le N\beta_i, \ \forall i = 1, 2, 3\}. \end{aligned} \tag{4.7}$$

$$b/ \ \rho_k[N, \mu, \sigma] = \begin{cases} \mu[u_0], & k = 0, \\ \sum_{j=1}^{k} \frac{1}{j!} D^j \mu[u_0] \Re_k[j, N, \sigma, u_0, \vec{u}], & 1 \le k \le N, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} \Re_k[j, N, \sigma, u_0, \vec{u}] &= P_k^{(j)}[N, \vec{\chi}[N, \sigma, u_0, \vec{u}]] \\ &= \begin{cases} \bar{\chi}_k[N, \sigma, u_0, \vec{u}], & j = 1, \\ \sum_{\alpha \in A_k^{(j)}(N)} \frac{j!}{\alpha!} \vec{\chi}^\alpha[N, \sigma, u_0, \vec{u}], & j \le k \le jN, \ j \ge 2, \end{cases} \end{aligned} \tag{4.9}$$

with $\vec{\chi}[N, \sigma, u_0, \vec{u}] = (\bar{\chi}_1[N, \sigma, u_0, \vec{u}], \cdots, \bar{\chi}_N[N, \sigma, u_0, \vec{u}]\ )$ is defined by

$$\begin{cases} \bar{\chi}_k[N, \sigma, u_0, \vec{u}] = \sum_{1 \le |\beta| \le k} \frac{1}{\beta!} \int_0^1 D^\beta \sigma[u_0] \tilde{\Phi}_k[\beta, N, \vec{u}] dy, \ 1 \le k \le N, \\ \tilde{\Phi}_k[\beta, N, \vec{u}] = \sum_{\substack{(i,j) \in \widetilde{B}(\beta, N), \\ i + j = k}} P_i^{(\beta_1)}[N, \vec{u}] P_j^{(\beta_2)}[N, \nabla \vec{u}], \\ \widetilde{B}(\beta, N) = \{(i, j) \in \mathbb{Z}_+^2 : \beta_1 \le i \le N\beta_1, \ \beta_2 \le j \le N\beta_2\}. \end{cases} \tag{4.10}$$

Then, we have the following theorem.

Theorem 4.2. Let (H_1), (H_8) *and* (H_9) *hold. Then there are positive constants* M *and* T *such that, for every* $0 \le \varepsilon < 1$, *the problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M, T)$ *satisfying an asymptotic expansion up to* $(N+1)^{th}$ *order as follows*

$$\left\| u_\varepsilon - \sum_{k=0}^{N} u_k \varepsilon^k \right\|_{W_1(T)} \le C_T \varepsilon^{N+1}, \tag{4.11}$$

where u_k, $0 \le k \le N$ *are the weak solutions of the problems* (P_0), $(\tilde{P}_k)$, $1 \le k \le N$, *respectively, and* C_T *is a constant depending only on* N, T, μ, μ_1, σ, σ_1, f, f_1, g, g_1, u_k, $0 \le k \le N$.

In order to prove Theorem 4.2, we need the following Lemmas.

Lemma 4.3. Let $\pi_k[N, f, g]$, $\rho_k[N, \mu, \sigma]$, $1 \le k \le N$, *be the functions are defined by the formulas* (4.4), (4.8). *Put* $h = \sum_{k=0}^{N} u_k \varepsilon^k$, *then we have*

$$\begin{aligned} f[h] &= \sum_{k=0}^{N} \pi_k[N, f, g] \varepsilon^k + \varepsilon^{N+1} \hat{R}_N^{(1)}[f, g, u_0, \vec{u}, \varepsilon], \\ \mu[h] &= \sum_{k=0}^{N} \rho_k[N, \mu, \sigma] \varepsilon^k + \varepsilon^{N+1} \hat{R}_N^{(2)}[\mu, \sigma, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{4.12}$$

with $\left\| \hat{R}_N^{(1)}[f, g, u_0, \vec{u}, \varepsilon] \right\|_{L^\infty(0,T;L^2)} + \left\| \hat{R}_N^{(2)}[\mu, \sigma, u_0, \vec{u}, \varepsilon] \right\|_{L^\infty(0,T;L^2)} \le C$, *where* C *is a constant depending only on* N, T, μ, μ_1, σ, σ_1, f, f_1, g, g_1, u_k, $0 \le k \le N$.

Proof . In the case of $N=1$, the proof of (4.12) is easy, hence we omit the details. We shall prove (4.12) in the case $N \geq 2$. Putting $h = u_0 + \sum_{k=1}^{N} u_k \varepsilon^k \equiv u_0 + h_1$, we have

$$f[h] = f\left(x,t,h(x,t),h_x(x,t),h_t(x,t),\int_0^1 g(x,t,y,h(y,t),h_x(y,t),h_t(y,t))dy\right) \tag{4.13}$$
$$= f\left(x,t,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1',\int_0^1 g[u_0](x,y,t)dy+\xi\right),$$
$$g[h](x,y,t) = g(x,t,y,h(y,t),h_x(y,t),h_t(y,t)),$$

where $\xi = \int_0^1 \left(g[u_0+h_1](x,y,t)dy - g[u_0](x,y,t)\right) dy.$

By using Taylor's expansion of the function $f\left(x,t,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1',\int_0^1 g[u_0](x,y,t)dy+\xi\right)$ around the point $[u_0] \equiv \left(x,t,u_0,\nabla u_0,u_0',\int_0^1 g[u_0](x,y,t)dy\right)$ up to $(N+1)^{th}$ order, we obtain

$$f[h] = f\left(x,t,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1',\int_0^1 g[u_0](x,y,t)dy+\xi\right) \tag{4.14}$$
$$= f[u_0] + \sum_{1\leq|\gamma|\leq N} \frac{1}{\gamma!} D^\gamma f[u_0] h_1^{\gamma_1} (\nabla h_1)^{\gamma_2} (h_1')^{\gamma_3} \xi^{\gamma_4} + R_N[f,u_0,h_1,\xi],$$

where

$$R_N[f,u_0,h_1,\xi] = \sum_{|\gamma|=N+1} \frac{N+1}{\gamma!} h_1^{\gamma_1} (\nabla h_1)^{\gamma_2} (h_1')^{\gamma_3} \xi^{\gamma_4} \int_0^1 (1-\theta)^N D^\gamma f(x,t,\theta)\, d\theta \tag{4.15}$$
$$= \varepsilon^{N+1} R_N^{(1)}[f,u_0,h_1,\xi,\varepsilon],$$

$\gamma = (\gamma_1,\cdots,\gamma_4) \in \mathbb{Z}_+^4$, $|\gamma| = \gamma_1+\cdots+\gamma_4$, $\gamma! = \gamma_1!\cdots\gamma_4!$, $D^\gamma f = D_3^{\gamma_1} D_4^{\gamma_2} D_5^{\gamma_3} D_6^{\gamma_4} f$,

$$D^\gamma f[u_0] = D^\gamma f\left(x,t,u_0(x,t),\nabla u_0(x,t),u_0'(x,t),\int_0^1 g[u_0](x,y,t)dy\right), \tag{4.16}$$
$$D^\gamma f(x,t,\theta) = D^\gamma f\left(x,t,u_0+\theta h_1,\nabla u_0+\theta\nabla h_1,u_0'+\theta h_1',\int_0^1 g[u_0](x,y,t)dy+\theta\xi\right).$$

Using the formula (4.1), we have

$$h_1^{\gamma_1} = \left(\sum_{k=1}^{N} u_k \varepsilon^k\right)^{\gamma_1} = \sum_{k=\gamma_1}^{N\gamma_1} P_k^{(\gamma_1)}[N,\vec{u}]\varepsilon^k,\ \ \vec{u} = (u_1,\cdots,u_N). \tag{4.17}$$

Similarly, with $(\nabla h_1)^{\gamma_2}$, $(h_1')^{\gamma_3}$, we also have

$$(\nabla h_1)^{\gamma_2} = \left(\sum_{k=1}^{N} \nabla u_k \varepsilon^k\right)^{\gamma_2} = \sum_{k=\gamma_2}^{N\gamma_2} P_k^{(\gamma_2)}[N,\nabla\vec{u}]\varepsilon^k, \tag{4.18}$$

$$(h_1')^{\gamma_3} = \left(\sum_{k=1}^{N} u_k' \varepsilon^k\right)^{\gamma_3} = \sum_{k=\gamma_3}^{N\gamma_3} P_k^{(\gamma_3)}[N,\vec{u}']\varepsilon^k, \tag{4.19}$$

where $\vec{u}' = (u_1',\cdots,u_N')$, $\nabla\vec{u} = (\nabla u_1,\cdots,\nabla u_N)$.

Hence, we deduce from (4.17)-(4.19), that

$$(h_1)^{\beta_1} (\nabla h_1)^{\beta_2} (h_1')^{\beta_3} = \sum_{k=|\beta|}^{N} \Psi_k[\beta,N,\vec{u}]\varepsilon^k + \sum_{k=N+1}^{N|\beta|} \Psi_k[\beta,N,\vec{u}]\varepsilon^k, \tag{4.20}$$

where $\Psi_k[\beta, N, \vec{u}]$, $1 \le k \le N\,|\beta|$, are defined by (4.7).

By using Taylor's expansion of the function $g[h](x,y,t) = g(x,t,y,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1')$ around the point $[u_0] \equiv (x,t,y,u_0,\nabla u_0,u_0')$ up to $(N+1)^{th}$ order, we obtain

$$\begin{aligned} g[h](x,y,t) &= g(x,t,y,(u_0+h_1)(y,t),(\nabla u_0+\nabla h_1)(y,t),(u_0'+h_1')(y,t)) \\ &= g[u_0] + \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0](h_1)^{\beta_1}(\nabla h_1)^{\beta_2}(h_1')^{\beta_3} + R_N[g,u_0,h_1,\varepsilon], \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} R_N[g,u_0,h_1,\varepsilon] &= \sum_{|\beta|=N+1} \frac{N+1}{\beta!} h_1^{\beta_1}(\nabla h_1)^{\beta_2}(h_1')^{\beta_3} \int_0^1 (1-\theta)^N D^\beta g\,(x,t,\theta)\,d\theta \\ &= \varepsilon^{N+1} R_N^{(1)}[\varepsilon,g,u_0,h_1], \end{aligned} \tag{4.22}$$

$\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{Z}_+^3$, $|\beta| = \beta_1+\beta_2+\beta_3$, $\beta! = \beta_1!\beta_2!\beta_3!$, $D^\beta g = D_4^{\beta_1} D_5^{\beta_2} D_6^{\beta_3} g$,

$$\begin{aligned} D^\beta g[u_0] &= D^\beta g\,(x,t,y,u_0,\nabla u_0,u_0')\,, \\ D^\beta g\,(x,t,\theta) &= D^\beta g(x,t,y,u_0+\theta h_1,\nabla u_0+\theta\nabla h_1,u_0'+\theta h_1'). \end{aligned} \tag{4.23}$$

Hence, it follows from (4.21), (4.22) that

$$\begin{aligned} g[h] &= g[u_0] + \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=|\beta|}^{N} \Psi_k[\beta,N,\vec{u}]\varepsilon^k \\ &\quad + \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=N+1}^{N|\beta|} \Psi_k[\beta,N,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(1)}[\varepsilon,g,u_0,h_1] \\ &= g[u_0] + \sum_{k=1}^{N} \sum_{1\le|\beta|\le k} \frac{1}{\beta!} \Psi_k[\beta,N,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(2)}[\varepsilon,g,u_0,h_1], \end{aligned} \tag{4.24}$$

where

$$\varepsilon^{N+1} R_N^{(2)}[\varepsilon,g,u_0,h_1] = \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=N+1}^{N|\beta|} \Psi_k[\beta,N,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(1)}[\varepsilon,g,u_0,h_1]. \tag{4.25}$$

Therefore

$$\begin{aligned} \xi &= \int_0^1 (g[h](x,y,t) - g[u_0](x,y,t))\,dy \\ &= \sum_{k=1}^{N} \left(\sum_{1\le|\beta|\le k} \frac{1}{\beta!} \int_0^1 D^\beta g[u_0]\Psi_k[\beta,N,\vec{u}]dy \right) \varepsilon^k + |\varepsilon|^{N+1} \int_0^1 R_N^{(2)}[\varepsilon,g,u_0,h_1,\vec{u}]dy \\ &= \sum_{k=1}^{N} \bar{\kappa}_k[N,g,u_0,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(3)}[\varepsilon,g,u_0,h_1,\vec{u}], \end{aligned} \tag{4.26}$$

where $\bar{\kappa}_k[N,g,u_0,\vec{u}]$, $1 \le k \le N$, are defined by (4.7) and

$$\varepsilon^{N+1} R_N^{(3)}[\varepsilon,g,u_0,h_1,\vec{u}] = \varepsilon^{N+1} \int_0^1 R_N^{(2)}[\varepsilon,g,u_0,h_1,\vec{u}]dy. \tag{4.27}$$

On the other hand, we also have

$$\begin{aligned}\xi^{\gamma_4} &= \left(\sum_{k=1}^{N} \bar{\kappa}_k[N,g,u_0,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(3)}[\varepsilon,g,u_0,h_1,\vec{u}]\right)^{\gamma_4} \\ &= \left(\sum_{k=1}^{N} \bar{\kappa}_k[N,g,u_0,\vec{u}]\varepsilon^k\right)^{\gamma_4} + \varepsilon^{N+1} R_N^{(4)}[\varepsilon,\gamma_4,g,u_0,h_1,\vec{u}] \\ &= \sum_{k=\gamma_4}^{\gamma_4 N} P_k^{(\gamma_4)}\left[N,\vec{\kappa}[N,g,u_0,\vec{u}]\right]\varepsilon^k + \varepsilon^{N+1} R_N^{(4)}[\varepsilon,\gamma_4,g,u_0,h_1,\vec{u}],\end{aligned} \tag{4.28}$$

where

$$\begin{aligned}&P_k^{(\gamma_4)}\left[N,\vec{\kappa}[N,g,u_0,\vec{u}]\right] \\ &= \begin{cases} \bar{\kappa}_k[N,g,u_0,\vec{u}], & \gamma_4 = 1,\ 1 \le k \le N, \\ \displaystyle\sum_{\alpha \in A_k^{(\gamma_4)}(N)} \frac{\gamma_4!}{\alpha!} \bar{\kappa}_1^{\alpha_1}[N,g,u_0,\vec{u}] \cdots \bar{\kappa}_N^{\alpha_N}[N,g,u_0,\vec{u}], & \gamma_4 \le k \le \gamma_4 N,\ \gamma_4 \ge 2,\end{cases}\end{aligned} \tag{4.29}$$

and

$$A_k^{(\gamma_4)}(N) = \{\alpha \in \mathbb{Z}_+^N : |\alpha| = \gamma_4, \sum_{i=1}^{N} i\alpha_i = k\}, \tag{4.30}$$

with $\vec{\kappa}[N,g,u_0,\vec{u}] = (\bar{\kappa}_1[N,g,u_0,\vec{u}], \cdots, \bar{\kappa}_N[N,g,u_0,\vec{u}])$ is defined by (4.7).

Thus, combining (4.17) - (4.19), (4.29), it leads to

$$\begin{aligned}&h_1^{\gamma_1}(\nabla h_1)^{\gamma_2}(h_1')^{\gamma_3}\xi^{\gamma_4} \\ &= h_1^{\gamma_1}(\nabla h_1)^{\gamma_2}(h_1')^{\gamma_3}\left(\sum_{k=\gamma_4}^{\gamma_4 N} P_k^{(\gamma_4)}\left[N,\vec{\kappa}[N,g,u_0,\vec{u}]\right]\varepsilon^k + \varepsilon^{N+1} R_N^{(4)}[\varepsilon,\gamma_4,g,u_0,h_1,\vec{u}]\right) \\ &= h_1^{\gamma_1}(\nabla h_1)^{\gamma_2}(h_1')^{\gamma_3}\sum_{k=\gamma_4}^{\gamma_4 N} P_k^{(\gamma_4)}\left[N,\vec{\kappa}[N,g,u_0,\vec{u}]\right]\varepsilon^k + \varepsilon^{N+1} R_N^{(5)}[\varepsilon,\gamma,g,u_0,\vec{u}] \\ &= \sum_{k=|\gamma|}^{N|\gamma|} \Phi_k[\gamma,N,g,u_0,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(5)}[\varepsilon,\gamma,g,u_0,\vec{u}],\end{aligned} \tag{4.31}$$

where $\Phi_k[\gamma,N,g,u_0,\vec{u}]$ is defined by (4.5) and (4.6).

$$\varepsilon^{N+1} R_N^{(5)}[\varepsilon,\gamma,g,u_0,\vec{u}] = \varepsilon^{N+1} h_1^{\gamma_1}(\nabla h_1)^{\gamma_2}(h_1')^{\gamma_3} R_N^{(4)}[\varepsilon,\gamma_4,g,u_0,h_1,\vec{u}]. \tag{4.32}$$

Separating $\displaystyle\sum_{k=|\gamma|}^{N|\gamma|}$ into $\displaystyle\sum_{k=|\gamma|}^{N}$ and $\displaystyle\sum_{k=N+1}^{N|\gamma|}$, we deduce from (4.31) that

$$h_1^{\gamma_1}(\nabla h_1)^{\gamma_2}(h_1')^{\gamma_3}\xi^{\gamma_4} = \sum_{k=|\gamma|}^{N} \Phi_k[\gamma,N,g,u_0,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(6)}[\varepsilon,\gamma,g,u_0,\vec{u}], \tag{4.33}$$

with

$$\varepsilon^{N+1} R_N^{(6)}[\varepsilon,\gamma,g,u_0,\vec{u}] = \sum_{k=N+1}^{N|\gamma|} \Phi_k[\gamma,N,g,u_0,\vec{u}]\varepsilon^k + \varepsilon^{N+1} R_N^{(5)}[\varepsilon,\gamma,g,u_0,\vec{u}]. \tag{4.34}$$

By (4.14) and (4.33), we get

$$
\begin{aligned}
f[h] &= f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] h_1^{\gamma_1} (\nabla h_1)^{\gamma_2} (h_1')^{\gamma_3} \xi^{\gamma_4} + R_N[f, u_0, h_1, \xi] \\
&= f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] \sum_{k=|\gamma|}^{N} \Phi_k[\gamma, N, g, u_0, \vec{u}]\varepsilon^k \\
&\quad + \varepsilon^{N+1} \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] R_N^{(6)}[\varepsilon, \gamma, g, u_0, \vec{u}] + \varepsilon^{N+1} R_N^{(1)}[f, u_0, h_1, \xi] \\
&= f[u_0] + \sum_{k=1}^{N} \left(\sum_{1\le|\gamma|\le k} \frac{1}{\gamma!} D^\gamma f[u_0] \Phi_k[\gamma, N, g, u_0, \vec{u}] \right) \varepsilon^k + \varepsilon^{N+1} \hat{R}_N[f, g, u_0, h_1, \xi] \\
&= f[u_0] + \sum_{k=1}^{N} \pi_k \left[N, f, g\right] \varepsilon^k + \varepsilon^{N+1} \hat{R}_N^{(1)}[f, g, u_0, \vec{u}, \varepsilon],
\end{aligned}
\tag{4.35}
$$

where $\pi_k \left[N, f, g\right]$, $1 \le k \le N$, are defined by (4.4)-(4.7) and

$$
\hat{R}_N^{(1)}[f, g, u_0, \vec{u}, \varepsilon] = \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] R_N^{(6)}[\varepsilon, \gamma, g, u_0, \vec{u}] + R_N^{(1)}[f, u_0, h_1, \xi]. \tag{4.36}
$$

By the boundedness of the functions u_k, ∇u_k, u_k', $1 \le k \le N$ in $L^\infty(0, T; H^1)$, we obtain from (4.15), (4.22), (4.25), (4.27), (4.32), (4.34) and (4.36) that $\left\| \hat{R}_N^{(1)}[f, g, u_0, \vec{u}, \varepsilon] \right\|_{L^\infty(0,T;L^2)} \le C$, where C is a positive constant depending only on N, T, f, g, u_k, $1 \le k \le N$. Hence, $(4.12)_1$ is proved.

Similarly, using (4.4)-(4.7) and $(4.12)_1$ for $f = f(x, t, y_1, y_2, y_3, y_4) = \mu(x, t, y_4)$, $D_3 f = D_4 f = D_5 f = 0$, $D_6 f = D_4 \mu$ and $\pi_k \left[N, f, g\right] = \rho_k \left[N, \mu, \sigma\right]$, we obtain $(4.12)_2$, where $\rho_k \left[N, \mu, \sigma\right]$, $1 \le k \le N$ which is defined by (4.8)-(4.10). Therefore, Lemma 4.3 is proved completely. □□

Let $u = u_\varepsilon \in W_1(M, T)$ be the unique weak solution of the problem (P_ε). Then $v = u_\varepsilon - \sum_{k=0}^{N} u_k \varepsilon^k \equiv u_\varepsilon - h$ satisfies the following problem

$$
\begin{cases}
v'' - \dfrac{\partial}{\partial x} \left(\mu_\varepsilon \left[v + h\right] v_x \right) = F_\varepsilon \left[v + h\right] - F_\varepsilon \left[h\right] \\
\qquad + \dfrac{\partial}{\partial x} \left[\left(\mu_\varepsilon \left[v + h\right] - \mu_\varepsilon \left[h\right] \right) h_x \right] + E_\varepsilon(x, t),\ 0 < x < 1,\ 0 < t < T, \\
v_x(0, t) - h_0 v(0, t) = v(1, t) = 0, \\
v(x, 0) = v'(x, 0) = 0,
\end{cases}
\tag{4.37}
$$

where

$$
E_\varepsilon(x, t) = f\left[h\right] - f\left[u_0\right] + \varepsilon f_1\left[h\right] + \frac{\partial}{\partial x} \left[\left(\mu\left[h\right] - \mu\left[u_0\right] + \varepsilon \mu_1\left[h\right] \right) h_x \right] - \sum_{k=1}^{N} \tilde{F}_k \varepsilon^k, \tag{4.38}
$$

and

$$
\begin{cases}
F_\varepsilon\left[v\right] = f\left(x, t, v, v_x, v_t, \displaystyle\int_0^1 g[v](x, y, t) dy \right) + \varepsilon f_1 \left(x, t, v, v_x, v_t, \displaystyle\int_0^1 g_1[v](x, y, t) dy \right), \\
\mu_\varepsilon\left[v\right] = \mu\left(x, t, \displaystyle\int_0^1 \sigma[v](x, y, t) dy \right) + \varepsilon \mu_1 \left(x, t, \displaystyle\int_0^1 \sigma_1[v](x, y, t) dy \right), \\
g_1[v](x, y, t) = g_1(x, y, t, v(y, t), v_x(y, t), v_t(y, t)), \\
\sigma_1[v](x, y, t) = \sigma_1(x, y, t, v(y, t), v_x(y, t)).
\end{cases}
\tag{4.39}
$$

Then, we have the following lemma.

Lemma 4.4. Let (H_1), (H_8) *and* (H_9) *hold. Then there is a positive constant* C_* *depending only on* N, T, μ, μ_1, σ, σ_1, f, f_1, g, g_1, u_k, $1 \le k \le N$ *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_* \varepsilon^{N+1}. \tag{4.40}$$

Proof . In the case $N = 1$, the proof of Lemma 4.4 is easy, hence we omit the details. We shall prove (4.40) in the case $N \ge 2$.

By using (4.12) for $f_1[h]$ and $\mu_1[h]$, we obtain

$$f_1[h] = f_1[u_0] + \sum_{k=1}^{N-1} \pi_k\left[N-1, f_1, g_1\right]\varepsilon^k + \varepsilon^N R_{N-1}^{(1)}[f_1, g_1, u_0, \vec{u}, \varepsilon], \tag{4.41}$$
$$\mu_1[h] = \mu_1[u_0] + \sum_{k=1}^{N-1} \rho_k\left[N-1, \mu_1, \sigma_1\right]\varepsilon^k + \varepsilon^N R_{N-1}^{(2)}[\mu_1, \sigma_1, u_0, \vec{u}, \varepsilon],$$

where $\left\|R_{N-1}^{(1)}[f_1, g_1, u_0, \vec{u}, \varepsilon]\right\|_{L^\infty(0,T;L^2)} + \left\|R_{N-1}^{(2)}[\mu_1, \sigma_1, u_0, \vec{u}, \varepsilon]\right\|_{L^\infty(0,T;L^2)} \le C$, with C is a constant depending only on N, T, f_1, g_1, μ_1, σ_1, u_k, $0 \le k \le N$.

By (4.41), we rewrite $\varepsilon f_1[h]$ and $\varepsilon\mu_1[h]$ as follows

$$\varepsilon f_1[h] = \varepsilon f_1[u_0] + \sum_{k=2}^{N} \pi_{k-1}\left[N-1, f_1, g_1\right]\varepsilon^k + \varepsilon^{N+1} R_{N-1}^{(1)}[f_1, g_1, u_0, \vec{u}, \varepsilon], \tag{4.42}$$
$$\varepsilon\mu_1[h] = \varepsilon\mu_1[u_0] + \sum_{k=2}^{N} \rho_{k-1}\left[N-1, \mu_1, \sigma_1\right]\varepsilon^k + \varepsilon^{N+1} R_{N-1}^{(2)}[\mu_1, \sigma_1, u_0, \vec{u}, \varepsilon].$$

Hence, we deduce from (4.12) and (4.42) that

$$\begin{aligned} f[h] - f[u_0] + \varepsilon f_1[h] &= \left(\pi_1[N, f, g, u_0, \vec{u}] + f_1[u_0]\right)\varepsilon \\ &+ \sum_{k=2}^{N} \left[\pi_k[N, f, g, u_0, \vec{u}] + \pi_{k-1}[N-1, f_1, g_1, u_0, \vec{u}]\right]\varepsilon^k \\ &+ \varepsilon^{N+1} \tilde{R}_N^{(1)}[f, g, f_1, g_1, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{4.43}$$

where

$$\varepsilon^{N+1} \tilde{R}_N^{(1)}[f, g, f_1, g_1, u_0, \vec{u}, \varepsilon] = \varepsilon^{N+1}\left(\hat{R}_N[f, g, u_0, \vec{u}, \varepsilon] + R_{N-1}^{(1)}[f_1, g_1, u_0, \vec{u}, \varepsilon]\right), \tag{4.44}$$

and

$$\begin{aligned} \left(\mu\left[h\right] - \mu\left[u_0\right] + \varepsilon\mu_1\left[h\right]\right) h_x &= \sum_{k=1}^{N} \nabla u_0 \left(\rho_k\left[N, \mu, \sigma\right] + \rho_{k-1}\left[N-1, \mu_1, \sigma_1\right]\right)\varepsilon^k \\ &+ \sum_{k=2}^{2N} \left(\sum_{\substack{i,j=1,\\ i+j=k}}^{N} \left(\rho_i\left[N, \mu, \sigma\right] + \rho_{i-1}\left[N-1, \mu_1, \sigma_1\right]\right)\nabla u_j \right)\varepsilon^k \\ &+ \varepsilon^{N+1} \tilde{R}_N^{(2)}[\mu, \sigma, \mu_1, \sigma_1, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{4.45}$$

where

$$\tilde{R}_N^{(2)}[\mu, \sigma, \mu_1, \sigma_1, u_0, \vec{u}, \varepsilon] = \left(\hat{R}_N[\mu, \sigma, u_0, \vec{u}, \varepsilon] + R_{N-1}^{(2)}[\mu_1, \sigma_1, u_0, \vec{u}, \varepsilon]\right) h_x. \tag{4.46}$$

Separating $\sum_{k=2}^{2N}$ into $\sum_{k=2}^{N}$ and $\sum_{k=N+1}^{2N}$, we deduce from (4.45) that

$$\begin{aligned}(\mu[h]-\mu[u_0]+\varepsilon\mu_1[h])\,h_x &= \left(\sum_{k=1}^{N}\nabla u_0\left(\rho_k[N,\mu,\sigma]+\rho_{k-1}[N-1,\mu_1,\sigma_1]\right)\varepsilon^k\right) \\ &\quad+\sum_{k=2}^{N}\left(\sum_{\substack{i,j=1,\\ i+j=k}}^{N}\left(\rho_i[N,\mu,\sigma]+\rho_{i-1}[N-1,\mu_1,\sigma_1]\right)\nabla u_j\right)\varepsilon^k \\ &\quad+\sum_{k=N+1}^{2N}\left(\sum_{\substack{i,j=1,\\ i+j=k}}^{N}\left(\rho_i[N,\mu,\sigma]+\rho_{i-1}[N-1,\mu_1,\sigma_1]\right)\nabla u_j\right)\varepsilon^k \\ &\quad+\varepsilon^{N+1}\tilde{R}_N^{(2)}[\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon] \\ &= \sum_{k=1}^{N}\left[\sum_{i=1}^{k}\left(\rho_i[N,\mu,\sigma]+\rho_{i-1}[N-1,\mu_1,\sigma_1]\right)\nabla u_{k-i}\right]\varepsilon^k+\varepsilon^{N+1}\tilde{R}_N^{(3)}[\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon],\end{aligned} \tag{4.47}$$

where

$$\begin{aligned}\varepsilon^{N+1}\tilde{R}_N^{(3)}[\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon] &= \sum_{k=N+1}^{2N}\left(\sum_{\substack{i,j=1,\\ i+j=k}}^{N}\left(\rho_i[N,\mu,\sigma]+\rho_{i-1}[N-1,\mu_1,\sigma_1]\right)\nabla u_j\right)\varepsilon^k \\ &\quad+\varepsilon^{N+1}\tilde{R}_N^{(2)}[\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon].\end{aligned} \tag{4.48}$$

Combining (4.3), (4.38), (4.43) and (4.47), we get that

$$\begin{aligned}E_\varepsilon(x,t) &= f[h]-f[u_0]+\varepsilon f_1[h]+\frac{\partial}{\partial x}\left[(\mu[h]-\mu[u_0]+\varepsilon\mu_1[h])\,h_x\right]-\sum_{k=1}^{N}\tilde{F}_k\varepsilon^k \\ &\quad+\sum_{k=1}^{N}\left[\pi_k[N,f,g,u_0,\vec{u}]+\pi_{k-1}[N-1,f_1,g_1,u_0,\vec{u}]\right]\varepsilon^k \\ &\quad+\sum_{k=1}^{N}\left[\sum_{i=1}^{k}\frac{\partial}{\partial x}\left[\left(\rho_i[N,\mu,\sigma]+\rho_{i-1}[N-1,\mu_1,\sigma_1]\right)\nabla u_{k-i}\right]\right]\varepsilon^k-\sum_{k=1}^{N}\tilde{F}_k\varepsilon^k \\ &\quad+\varepsilon^{N+1}\tilde{R}_N^{(4)}[f,g,f_1,g_1,\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon] \\ &= \varepsilon^{N+1}\tilde{R}_N^{(4)}[f,g,f_1,g_1,\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon],\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}&\varepsilon^{N+1}\tilde{R}_N^{(4)}[f,g,f_1,g_1,\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon] \\ &= \varepsilon^{N+1}\left(\tilde{R}_N^{(1)}[f,g,f_1,g_1,u_0,\vec{u},\varepsilon]+\frac{\partial}{\partial x}\tilde{R}_N^{(3)}[\mu,\sigma,\mu_1,\sigma_1,u_0,\vec{u},\varepsilon]\right).\end{aligned} \tag{4.50}$$

By the boundedness of u_k, ∇u_k, $1\le k\le N$ in $L^\infty(0,T;H^1)$, we obtain from (4.12), (4.44), (4.48), and (4.50) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)}\le C_*\varepsilon^{N+1}, \tag{4.51}$$

where C_* is a constant depending only on N, T, μ, μ_1, σ, σ_1, f, f_1, g, g_1, u_k, $1\le k\le N$.

Lemma 4.4 is proved.□ □

Proof of Theorem 4.2.

We consider a sequence $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ v_m'' - \dfrac{\partial}{\partial x}\left(\mu_\varepsilon[v_{m-1}+h]v_{mx}\right) = F_\varepsilon\left[v_{m-1}+h\right] - F_\varepsilon\left[h\right] + \dfrac{\partial}{\partial x}\left[\left(\mu_\varepsilon\left[v_{m-1}+h\right] - \mu_\varepsilon\left[h\right]\right)h_x\right] \\ \qquad\qquad +E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\ v_{mx}(0,t) - h_0 v_m(0,t) = v_m(1,t) = 0, \\ v_m(x,0) = v_m'(x,0) = 0,\ m \geq 1, \end{cases} \tag{4.52}$$

where

$$\begin{cases} F_\varepsilon\left[v\right] = f\left(x,t,v,v_x,v_t,\displaystyle\int_0^1 g[v](x,y,t)dy\right) + \varepsilon f_1\left(x,t,v,v_x,v_t,\displaystyle\int_0^1 g_1[v](x,y,t)dy\right), \\ \mu_\varepsilon\left[v\right] = \mu\left(x,t,\displaystyle\int_0^1 \sigma[v](x,y,t)dy\right) + \varepsilon\mu_1\left(x,t,\displaystyle\int_0^1 \sigma_1[v](x,y,t)dy\right), \\ g_1[v](x,y,t) = g_1(x,y,t,v(y,t),v_x(y,t),v_t(y,t)), \\ \sigma_1[v](x,y,t) = \sigma_1(x,y,t,v(y,t),v_x(y,t)). \end{cases} \tag{4.53}$$

We shall prove that there exists a constant C_T, independent of m and ε, such that

$$\|v_m\|_{W_1(T)} \leq C_T \varepsilon^{N+1},\ \text{with}\ |\varepsilon| \leq 1,\ \text{for all}\ m. \tag{4.54}$$

Indeed, by multiplying both sides of $(4.52)_2$ with v_m' and after integrating in t, we have

$$\begin{aligned} Z_m(t) = {} & 2\int_0^t \langle E_\varepsilon(s), v_m'(s)\rangle ds + \int_0^t \frac{\partial A_{m,\varepsilon}}{\partial t}(s; v_m(s), v_m(s))ds \\ & + 2\int_0^t \langle \frac{\partial}{\partial x}\left[\left(\mu_\varepsilon[v_{m-1}+h] - \mu_\varepsilon[h]\right)h_x\right], v_m'(s)\rangle ds \\ & + 2\int_0^t \left\langle F_\varepsilon\left[v_{m-1}+h\right] - F_\varepsilon\left[h\right], v_m'(s)\right\rangle ds, \end{aligned} \tag{4.55}$$

where

$$\begin{aligned} Z_m(t) &= \|v_m'(t)\|^2 + A_{m,\varepsilon}(t; v_m(t), v_m(t)) \geq ||v_m'(t)||^2 + \mu_0 \|v_m(t)\|_a^2, \\ A_{m,\varepsilon}(t;u,v) &= \langle \mu_{m,\varepsilon}(t)u_x, v_x\rangle + h_0\mu_{m,\varepsilon}(0,t)u(0)v(0),\ \forall u,v \in V, \\ \mu_{m,\varepsilon}(x,t) &= \mu\left(x,t,\int_0^1 g[v_{m-1}+h](x,y,t)dy\right) + \varepsilon\mu_1\left(x,t,\int_0^1 g_1[v_{m-1}+h](x,y,t)dy\right). \end{aligned} \tag{4.56}$$

By using Lemmas 4.4, we deduce from (4.55) that

$$\begin{aligned} Z_m(t) \leq {} & TC_*^2\varepsilon^{2N+2} + \int_0^t \|v_m'(s)\|^2 ds + \int_0^t \frac{\partial A_{m,\varepsilon}}{\partial t}(s; v_m(s), v_m(s))ds \\ & + 2\int_0^t \left\|\frac{\partial}{\partial x}\left[\left(\mu_\varepsilon[v_{m-1}+h] - \mu_\varepsilon[h]\right)h_x\right]\right\| \|v_m'(s)\| ds \\ & + 2\int_0^t \left\|F_\varepsilon\left[v_{m-1}+h\right] - F_\varepsilon\left[h\right]\right\| \|v_m'(s)\| ds \\ = {} & TC_*^2\varepsilon^{2N+2} + \int_0^t \|v_m'(s)\|^2 ds + \hat{J}_1 + \hat{J}_2 + \hat{J}_3. \end{aligned} \tag{4.57}$$

We estimate the integrals on the right-hand side of (4.57) as follows.

Estimation of $\hat{J}_1$. Note that, we are easy to estimate that

$$\frac{\partial \mu_{m,\varepsilon}}{\partial t}(x,t) \leq \bar{\zeta}_1, \tag{4.58}$$

with $\bar{\zeta}_1 = \tilde{K}_{M_*}(\mu)\left[1 + (1+2M_*)\bar{H}_{M_*}(\sigma)\right] + \tilde{K}_{M_*}(\mu_1)\left[1 + (1+2M_*)\bar{H}_{M_*}(\sigma_1)\right],\ M_* = (N+2)M.$

Then, it follows from (4.57) that

$$\left|\hat{J}_1\right| \leq \int_0^t \left|\frac{\partial A_{m,\varepsilon}}{\partial t}(s; v_m(s), v_m(s))\right| ds \leq \bar{\zeta}_1 \int_0^t \|v_m(s)\|_a^2 \, ds. \tag{4.59}$$

Estimation of $\hat{J}_2$. First, we need to estimate $\left\| \frac{\partial}{\partial x} \left[(\mu[v_{m-1}+h] - \mu[h])\, h_x \right] \right\|$.

Note that

$$\begin{aligned} &\|\mu[v_{m-1}+h] - \mu[h]\|_{C^0(\bar{\Omega})} \leq 2\tilde{K}_{M_*}(\mu)\bar{H}_{M_*}(\sigma)\, \|v_{m-1}\|_{W_1(T)}, \\ &\|D_i\mu[v_{m-1}+h] - D_i\mu[h]\| \leq 2\tilde{K}_{M_*}(\mu)\bar{H}_{M_*}(\sigma)\, \|v_{m-1}\|_{W_1(T)}, \ i = 1, 3, \\ &\|D_1\sigma[v_{m-1}+h] - D_1\sigma[h]\| \leq 2\bar{H}_{M_*}(\sigma)\, \|v_{m-1}\|_{W_1(T)}, \end{aligned} \tag{4.60}$$

then, due to the following equality

$$\begin{aligned} \tfrac{\partial}{\partial x} \left[(\mu[v_{m-1}+h] - \mu[h])\, h_x \right] = {} & (\mu[v_{m-1}+h] - \mu[h])\, h_{xx} + (D_1\mu[v_{m-1}+h] - D_1\mu[h])\, h_x \\ & + D_3\mu[v_{m-1}+h] \left(\int_0^1 (D_1\sigma[v_{m-1}+h] - D_1\sigma[h])\, dy \right) h_x \\ & + (D_3\mu\,[v_{m-1}+h] - D_3\mu[h]) \left(\int_0^1 D_1\sigma[h] dy \right) h_x, \end{aligned} \tag{4.61}$$

we have that

$$\left\| \frac{\partial}{\partial x} \left[(\mu[v_{m-1}+h] - \mu[h])\, h_x \right] \right\| \leq d(\mu, \sigma, M_*)\, \|v_{m-1}\|_{W_1(T)}, \tag{4.62}$$

where $d(\mu, \sigma, M_*) = 2M_*\tilde{K}_{M_*}(\mu)\bar{H}_{M_*}(\sigma) \left[1 + \sqrt{2} \left(2 + \bar{H}_{M_*}(\sigma) \right) \right]$.

Using the same estimations above for $\mu_\varepsilon = \mu + \varepsilon\mu_1$, we have

$$\left\| \tfrac{\partial}{\partial x} \left[(\mu_\varepsilon[v_{m-1}+h] - \mu_\varepsilon[h])\, h_x \right] \right\| \leq \bar{\zeta}_2\, \|v_{m-1}\|_{W_1(T)}, \tag{4.63}$$

where $\bar{\zeta}_2 = d(\mu, \sigma, M_*) + d(\mu_1, \sigma_1, M_*)$.

We derive from (4.63) that

$$\begin{aligned} \hat{J}_2 &= 2\int_0^t \left\| \frac{\partial}{\partial x} \left[(\mu_\varepsilon[v_{m-1}+h] - \mu_\varepsilon[h])\, h_x \right] \right\| \|v_m'(s)\|\, ds \\ &\leq T\bar{\zeta}_2^2\, \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \|v_m'(s)\|^2\, ds. \end{aligned} \tag{4.64}$$

Estimation of $\hat{J}_3$.By

$$\begin{aligned} \|F_\varepsilon\,[v_{m-1}+h] - F_\varepsilon\,[h]\| &\leq \|f\,[v_{m-1}+h] - f\,[h]\| + \|f_1\,[v_{m-1}+h] - f_1\,[h]\| \\ &\leq [2K_{M_*}(f)(1 + H_{M_*}(g)) + 2K_{M_*}(f_1)(1 + H_{M_*}(g_1))]\, \|v_{m-1}\|_{W_1(T)}, \end{aligned} \tag{4.65}$$

it follows from (4.57) that

$$\hat{J}_3 = 2\int_0^t \|F_\varepsilon\,[v_{m-1}+h] - F_\varepsilon\,[h]\|\, \|v_m'(s)\|\, ds \leq T\bar{\zeta}_3^2\, \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \|v_m'(s)\|^2\, ds, \tag{4.66}$$

with $\bar{\zeta}_3 = [2K_{M_*}(f)(1 + H_{M_*}(g)) + 2K_{M_*}(f_1)(1 + H_{M_*}(g_1))]^2$.

Combining (4.57), (4.59), (4.64) and (4.66), it leads to

$$Z_m(t) \leq TC_*^2\varepsilon^{2N+2} + T\bar{\zeta}_3^2\, \|v_{m-1}\|_{W_1(T)}^2 + \left(3 + \frac{\bar{\zeta}_1}{\mu_0} \right) \int_0^t Z_m(s) ds. \tag{4.67}$$

Using Gronwall's lemma, we deduce from (4.67) that

$$\|v_m\|_{W_1(T)} \leq \sigma_T\, \|v_{m-1}\|_{W_1(T)} + \delta_T(\varepsilon), \ \forall m \geq 1, \tag{4.68}$$

where $\sigma_T = \eta_T \bar{\zeta}_3$, $\delta_T(\varepsilon) = C_* \eta_T \varepsilon^{N+1}$, $\eta_T = \left(1 + \frac{1}{\sqrt{\mu_0}}\right) \sqrt{T \exp\left[\left(3 + \frac{\bar{\zeta}_1}{\mu_0}\right) T\right]}$.

Due to the dependence of η_T on T as above, we can assume that

$$\sigma_T < 1, \text{ with a sufficiently small constant } T. \tag{4.69}$$

Then, to close the proof of Theorem 4.2, we need the following lemma of which the proof is easy.

Lemma 4.5. Let $\{\gamma_m\}$ *is a sequence that satisfies*

$$\gamma_m \leq \sigma \gamma_{m-1} + \delta \textit{ for all } m \geq 1, \ \gamma_0 = 0, \tag{4.70}$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ *are given constants. Then*

$$\gamma_m \leq \delta/(1-\sigma) \textit{ for all } m \geq 1. \ \square \tag{4.71}$$

Applying Lemma 4.5 to $\gamma_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T = \eta_T \bar{\zeta}_3 < 1$, $\delta = \delta_T(\varepsilon) = C_* \eta_T \varepsilon^{N+1}$, it follows from (4.68) that

$$\|v_m\|_{W_1(T)} \leq \frac{\delta_T(\varepsilon)}{1 - \sigma_T} = C_T \varepsilon^{N+1}, \tag{4.72}$$

where $C_T = \dfrac{C_* \eta_T}{1 - \eta_T \bar{\zeta}_3}$.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.52) converges strongly in $W_1(T)$ to the solution v of the problem (4.37). Hence, taking the limitation as $m \to +\infty$ in (4.72), we get

$$\|v\|_{W_1(T)} \leq C_T \varepsilon^{N+1}. \tag{4.73}$$

This implies (4.11).

The proof of Theorem 4.2 is proved completely. $\square$

5 Conclusions

In this work, we have studied an initial-boundary value problem for a class of wave equations with nonlinear integral terms. After linearizing the nonlinear integral terms, the Feado-Galerkin method has been used to find the finite dimensional approximate solution. Then, the existence and uniqueness have been established by constructing a recurrent sequence that converges to the weak solution of the proposed problem. In addition, a high-order asymptotic expansion of solutions for the perturbed problem in a small parameter has also been considered, in which the necessary lemmas of expanding multivariable polynomials have been used to get the desired results.

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