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Strong convergence theorem for split variational inclusion problem and finite family of fixed point problems

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Abstract

The main objective of this paper is to introduce and study a new type of iterative method to approximate a common solution of split variational inclusion problem and a finite family of fixed point problems in real Hilbert spaces. Furthermore, we show that the sequence generated by the proposed iterative method converges strongly to a common solution to these problems. The method and results presented in this paper extend and unify some recent known results in this field. Finally, a numerical example is used to demonstrate the convergence analysis of the sequences generated by the iterative method.

Keywords: Split variational inclusion problem, Nonexpansive mapping, Averaged mapping, Fixed point problem, Strong convergence, Iterative method
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1 Introduction

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hibert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ and let C be a nonempty closed convex subset of $\mathbb{H}_1$. In 2011, Moudafi [1] introduced the following split monotone variational inclusion problem (in short, SMVIP): Find $\tilde{x} \in \mathbb{H}_1$ such that

$$0 \in g_1(\tilde{x}) + B_1(\tilde{x}), \tag{1.1}$$

and $w^* = \mathbb{A}\tilde{x} \in \mathbb{H}_2$ solves

$$0 \in g_2(w^*) + B_2(w^*), \tag{1.2}$$

where, $B_1 : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$, $B_2 : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ are multi-valued maximal monotone mappings and $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator.

The split feasibility problem, split zero problem, split fixed point problem, split variational inequality problem, see [1, 3, 4, 5, 6, 16] are special cases of the split monotone variatianal inclusion problem (1.1)-(1.2). They have been investigated by numerous authors and solve real life problems essentially in modelling of inverse problems, sensor networks in computerized tomography, data compression and radiation therapy; for details, see [2, 4, 5, 7].

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If $g_1 \equiv 0$ and $g_2 \equiv 0$, then SMVIP (1.1)-(1.2) reduces to the following split variational inclusion problem (in short, SVIP): Find $\tilde{x} \in \mathbb{H}_1$ such that

$$0 \in B_1(\tilde{x}), \tag{1.3}$$

and $w^* = \mathbb{A}\tilde{x} \in \mathbb{H}_2$ solves

$$0 \in B_2(w^*). \tag{1.4}$$

When we looked at it independently, (1.3) is the variational inclusion problem and its solution set represented by SolVIP(B_1). The SVIP (1.3)-(1.4) constitutes a pair of variational inclusion problems must be solved so that the image $w^* = \mathbb{A}\tilde{x}$ under a given bounded linear operator $\mathbb{A}$, of the solution $\tilde{x}$ of SVIP (1.3) in $\mathbb{H}_1$ is the solution of another SVIP (1.4) in another space $\mathbb{H}_2$, we denote the solution set of SVIP (1.4) by SolVIP(B_2). The solution set of SVIP (1.3)-(1.4) is denoted by $\Omega = \{\tilde{x} \in \mathbb{H}_1 : \tilde{x} \in \text{SolVIP}(B_1) \text{ and } \mathbb{A}\tilde{x} \in \text{SolVIP}(B_2)\}$. In 2012, Byrne et al.[3] introduced and studied the following iterative algorithm: For a given $x_0 \in \mathbb{H}_1$, the sequence $\{x_m\}$ generated by the following iterative algorithm:

$$x_{m+1} = J_{\lambda_1}^{B_1}(x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m), \quad \lambda_1 > 0.$$

Under some appropriate conditions, they obtained weak and strong convergence theorems solving for SVIP (1.3)-(1.4).

In 2013, Kazmi and Rizvi [12] introduced and studied the following iterative algorithm:

$$\begin{aligned} v_m &= J_{\lambda_1}^{B_1}(x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m), \\ x_{m+1} &= \alpha_m g(x_m) + (1 - \alpha_m)Tv_m. \end{aligned}$$

Under some appropriate conditions, they proved that the sequence $\{x_m\}$ generated by algorithm converges strongly to the common solution of fixed point problem and SVIP (1.3)-(1.4). On the other hand, A fixed point problem for a nonexpansive mapping $T : \mathbb{H}_1 \to \mathbb{H}_1$ is defined as: Find $x \in \mathbb{H}_1$ such that

$$Tx = x. \tag{1.5}$$

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on $\mathbb{H}_1$ such that $\mathcal{S} = \bigcap_{i=1}^N Fix(T_i) \neq \phi$, where $Fix(T_i) = \{x \in \mathbb{H}_1 : T_i x = x\}$. Zhou and Wang [20] proposed an explicit iterative algorithm for approximating a common solution of the variatioanal inequality over the set of common fixed points of a finite family of nonexpansive mappings. They introduced the following iterative algorithm: $x_1 \in \mathbb{H}_1$ and

$$x_{m+1} = (1 - \lambda_n \mu F)T_N^m T_{N-1}^m \dots T_1^m, \quad m \geq 1,$$

where $F : \mathbb{H}_1 \to \mathbb{H}_1$ is a k-Lipschitzian and η-strongly monotone mapping and $T_i^m = (1-\delta_m^i)I+\delta_m^i T_i$, for $i = 1, 2, \dots N$. Under some appropriate conditions, they proved that the sequence $\{x_m\}$ converges to a common fixed point of the above mapping.

In this paper, we approximate a common solution of split variational inclusion problem and a finite family of fixed point problems for nonexpaansive mappings in real Hilbert space: Find $\tilde{x} \in \mathbb{H}_1$ such that

$$\tilde{x} \in \Omega \cap \bigcap_{i=1}^N Fix(T_i).$$

Recently, in past few years, there have been many authors who have been interested in finding a common solution of the fixed point problem and split variational inclusion problem (1.3)-(1.4), see [9, 12, 13, 14].

Motivated by the work of Moudafi [1], Byrne et al. [3], Kazmi and Rizvi [12], P. Majee and Nahak [14], Zhou and Wang [20] and by the continuing work in this direction, we suggest and analyze an iterative method for approximating a common solution of SVIP (1.3)-(1.4) and a finite family of fixed point problems for nonexpansive mappings in the real Hilbert spaces. Furthermore, we show that the sequences generated by our iterative method converges strongly to a common solution of SVIP (1.3)-(1.4) and a finite family of fixed point problems in real Hilbert spaces. The iterative method and results discussed in this article are new and can be considered as a generalization and refinement of the previously published work in this field.

2 Preliminaries

In order to prove our main results, we need to review some basic definitions and lemmas, which will be needed in the following section. A mapping $T : \mathbb{H}_1 \to \mathbb{H}_1$ is said to be
(i) monotone if

$$\langle Tx - Tw, x - w\rangle \geq 0, \quad \forall\, x, w \in \mathbb{H}_1.$$

(ii) α-strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Tw, x - w\rangle \geq \alpha\|x - w\|^2, \quad \forall\, x, w \in \mathbb{H}_1.$$

(iii) nonexpansive if

$$\|Tx - Tw\| \leq \|x - w\|, \quad \forall\, x, w \in \mathbb{H}_1.$$

(iv) firmly nonexpansive if

$$\langle Tx - Tw, x - w\rangle \geq \|Tx - Tw\|^2, \quad \forall\, x, w \in \mathbb{H}_1.$$

It is well known that every nonexpansive mapping $T : \mathbb{H}_1 \to \mathbb{H}_1$ satisfy the inequality:

$$\langle (x - Tx) - (w - Tw), Tw - Tx\rangle \;\leq\; \frac{1}{2}\|(Tx - x) - (Tw - w)\|^2, \quad \forall\, (x, w) \in \mathbb{H}_1 \times \mathbb{H}_2.$$

Therefore for all $(x, w) \in \mathbb{H}_1 \times\; Fix\,(T)$, we get

$$\langle x - Tx, w - Tx\rangle \leq \frac{1}{2}\|(Tx - x)\|^2. \tag{2.1}$$

A mapping $g : \mathbb{H}_1 \to \mathbb{H}_1$ is called ρ-Lipschitzian if there exists a constant $\rho > 0$ such that

$$\|g(x) - g(w)\| \leq \rho\|x - w\|, \quad \forall\, x, w \in \mathbb{H}_1.$$

A multi-valued mapping $B_1 : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ is called monotone if for all $x_1, x_2 \in \mathbb{H}_1$ there exist $v_1 \in B_1x_1$ and $v_2 \in B_1x_2$ such that

$$\langle x_1 - x_2, v_1 - v_2\rangle \geq 0.$$

A monotone mapping B_1 is maximal if $G(B_1)$, the graph of B_1 defined as

$$G(B_1) = \{(x_1, v_1) : v_1 \in B_1x_1\},$$

is not contained properly in the graph of any other monotone mapping. **Remark**: It is also well known that a monotone mapping B_1 is maximal if and only if for
$(x_1, v_1) \in \mathbb{H}_1 \times \mathbb{H}_1, \langle x_1 - x_2, v_1 - v_2\rangle \geq 0$ for each $(x_2, v_2) \in G(B_1)$ implies that $v_1 \in B_1x_1$.
Let $B_1 : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ be a multi-valued maximal monotone mapping. Then, the resolvent operator $J^{B_1}_{\lambda_1} : \mathbb{H}_1 \to \mathbb{H}_1$ associated with B_1, is defined by

$$J^{B_1}_{\lambda_1}(x_1) = (I + \lambda_1 B_1)^{-1}(x_1), \quad \forall\, x_1 \in \mathbb{H}_1, \quad \lambda_1 > 0,$$

where I is the identity operator on $\mathbb{H}_1$. We noticed that the resolvent operator $J^{B_1}_{\lambda_1}$ is single-valued, nonexpansive and firmly nonexpansive.
Also, in the Hilbert space $\mathbb{H}_1$, following properties hold:

$$(a) \quad \|x + w\|^2 \leq \|x\|^2 + 2\langle w, x + w\rangle, \qquad \forall\, x, w \in \mathbb{H}_1. \tag{2.2}$$

$$(b) \quad \|\alpha x + (1 - \alpha)w\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|w\|^2 - \alpha(1 - \alpha)\|x - w\|^2, \qquad \forall\, x, w \in \mathbb{H}_1. \tag{2.3}$$

A mapping $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_1$ is said to be strongly positive mapping with cofficient δ, if there exists a constant $\delta > 0$ such that

$$\langle \mathbb{A}x, x\rangle \geq \delta\|x\|^2, \quad \forall\, x \in \mathcal{H}_1.$$

Definition 2.1. *[18] A mapping $T : \mathbb{H}_1 \to \mathbb{H}_1$ is called averaged mapping if there exists some number $\alpha \in (0, 1)$ such that $T = (1 - \alpha)I + \alpha S$, where $I : \mathbb{H}_1 \to \mathbb{H}_1$ is the identity mapping and $S : \mathbb{H}_1 \to \mathbb{H}_1$ is a nonexpansive mapping. An averaged mapping is also a nonexpansive mapping and $Fix(S) = Fix(T)$.*

Lemma 2.2. *([4], [8]) If the mapping $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\cap_{i=1}^n Fix(T_i) = Fix(T_1T_2 \dots T_N).$$

In particular, for $N = 2$, $Fix(T_1) \cap Fix(T_2) = Fix(T_1T_2) = Fix(T_2T_1)$.

Lemma 2.3 (Demiclosedness Principle). *[10] Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}_1$ and $T : C \to C$ be a nonexpansive mapping. If $\{x_m\}$ is a sequence in C weakly converge to $x \in C$ and $\{(I-T)x_m\}$ converges strongly to $w \in C$, then $(I - T)x = w$. In particular, if $w = 0$, then $x \in Fix(T)$.*

Lemma 2.4. *[15] Assume that $\mathbb{A}$ is a strongly positive bounded linear operator on a Hilbert space $\mathbb{H}_1$ with coefficient $\bar{\delta} > 0$ and $0 < \rho < \|\mathbb{A}\|^{-1}$, then $\|I - \rho\mathbb{A}\| \leq 1 - \rho\bar{\delta}$.*

Lemma 2.5. *[19] Assume that S is a k-strictly pseudocontractive mapping on a Hilbert space $\mathbb{H}_1$. Define a mapping T by $Tx = \alpha x + (1-\alpha)Sx$ for all $x \in \mathbb{H}_1$, where $\alpha \in [k, 1)$. Then, T is nonexpansive mapping with $Fix(T) = Fix(S)$.*

Lemma 2.6. *[16] Suppose $\mathbb{H}_1$ is a Hilbert space. Let $g : C \to C$ be a ρ-Lipschitzian mapping and $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_1$ be a strongly positive bounded linear operator with coefficient $\gamma > 0$. If $\mu\gamma > \beta\rho$, then*

$$\langle(\mu\mathbb{A} - \beta g)x - (\mu\mathbb{A} - \beta g)w, x - w\rangle \geq (\mu\gamma - \beta\rho)\|x - w\|^2, \qquad \forall\, x, w \in \mathbb{H}_1 \tag{2.4}$$

That is, $\mu\mathbb{A} - \beta g$ is strongly monotone with coefficient $\mu\gamma - \beta\rho$.

Lemma 2.7. *[11] Let $\{\alpha_m\}$ be a sequence of non negative real numbers such that*

$$\alpha_{m+1} \leq (1 - \delta_m)\alpha_m + \gamma_m.$$

where $\{\delta_m\}$ is a sequence in $(0,1)$ and $\{\gamma_m\}$ is a sequence in $\mathbb{R}$ such that (i) $\sum_{m=1}^\infty \delta_m = \infty$, (ii) $\lim_{m\to\infty} \sup \frac{\gamma_m}{\delta_m} \leq 0$ or $\sum_{m=1}^\infty |\gamma_m| < \infty$ then $\lim_{m\to\infty} \alpha_m = 0$.

Lemma 2.8. *[17] Let $\{x_m\}$ and $\{t_m\}$ be two bounded sequences in a Banach space X and let $\{\sigma_m\}$ be a sequence in [0, 1] which satisfy the following conditions:*

$$0 < \lim_{m\to\infty} \inf \sigma_m \leq \lim_{m\to\infty} \sup \sigma_m < 1.$$

Suppose $x_{m+1} = (1 - \sigma_m)t_m + \sigma_m x_m$, for all integers $m \geq 0$ and

$$\lim_{m\to\infty} \sup(\|t_{m+1} - t_m\| - \|x_{m+1} - x_m\|) \leq 0 \quad \text{then} \quad \lim_{m\to\infty} \|t_m - x_m\| = 0.$$

3 Main Result

In this section, we prove a strong convergence theorem based on the proposed itrative method for approximating a common solution of SVIP (1.3)-(1.4) and a finite family of fixed point problems in real Hilbert spaces.

Theorem 3.1. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and C be a nonempty closed convex subset of $\mathbb{H}_1$. Let $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator. Assume that $B_1 : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ and $B_2 : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ are two maximal monotone operators. Let $\{S_i\}_{i=1}^N$ is a finite family of nonexpansive mappings on $\mathbb{H}_1$ such that $\mathcal{J} = \Omega \bigcap \mathcal{S} \neq \phi$, where $\mathcal{S} = \bigcap_{i=1}^N Fix(S_i)$. Let $g : \mathbb{H}_1 \to \mathbb{H}_1$ be a ρ-Lipschitzian mapping with coefficient $\rho > 0$ and $\mathbb{D} : \mathbb{H}_1 \to \mathbb{H}_1$ be a strongly positive bounded linear operator with coefficient $\gamma > 0$. Define a sequence $\{x_m\}$ as follows: $x_1 \in \mathbb{H}_1$ and

$$\begin{cases} v_m = J_{\lambda_1}^{B_1}(x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)Ax_m), \\ w_m = \gamma_m x_m + (1 - \gamma_m)S_N^m S_{N-1}^m \dots S_1^m v_m, \\ x_{m+1} = \alpha_m\beta g(x_m) + \sigma_m x_m + ((1 - \sigma_m)I - \alpha_m\mu\mathbb{D})w_m, \qquad m \geq 1, \end{cases} \tag{3.1}$$

where $\lambda_1, \beta, \mu > 0$, the sequence $\{\alpha_m\}, \{\sigma_m\}, \{\gamma_m\} \subset [0,1], \delta_m^i \in (0,1)$ for $i = 1, 2, \dots N$, $S_i^m = (1 - \delta_m^i)I + \delta_m^i S_i$, $\delta \in (0, \frac{1}{L})$, L is the spectral radius of the operator $\mathbb{A}^*\mathbb{A}$, $\mathbb{A}^*$ is the adjoint operator of $\mathbb{A}$. Also the following conditions are satisfied:

(i) $\mu\gamma > \beta\rho, 0 < \alpha_m \leq \min\left\{1, \dfrac{1}{\mu\|\mathbb{D}\|}\right\}$, $0 \leq \sigma_m \leq b < 1$ for some b $\in (0,1)$ and $0 < \lim\limits_{m\to\infty} \inf \sigma_m \leq \lim\limits_{m\to\infty} \sup \sigma_m < 1$;

(ii) $\lim\limits_{m\to\infty} \alpha_m = 0$ and $\sum_{m=0}^{\infty} \alpha_m = \infty$;

(iii) $\lim\limits_{m\to\infty} |\delta^i_{m+1} - \delta^i_m| = 0$ for $i = 1, 2, \ldots, N$;

(iv) $0 < c \leq \gamma_m \leq d < 1$ and $\lim\limits_{m\to\infty} |\gamma_{m+1} - \gamma_m| = 0$ for some $c, d \in \mathbb{R}$.

Then the sequence $\{x_m\}$ converges strongly to $\tilde{x} \in \mathcal{J}$, which is a unique solution of the subsequent variational inequality:

$$\langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x\rangle \leq 0, \quad \forall\, x \in \mathcal{J}. \tag{3.2}$$

Proof . We prove the theorem for $N = 2$. The method is easily adaptable to the general case. The proof is divided into five steps.

Step 1. From the condition (i) and (ii), we may consider without loss of generality that

$$\alpha_m\mu \leq (1 - \sigma_m)\|\mathbb{D}\|^{-1} \qquad \forall\, m \geq 0.$$

Since $\mathbb{D}$ is strongly positive bounded linear operator on $\mathbb{H}_1$, then

$$\|\mathbb{D}\| = \sup\{|\langle \mathbb{D}v, v\rangle\| : v \in \mathbb{H}_1, \|v\| = 1\}.$$

We observe that

$$\begin{aligned}\langle ((1 - \sigma_m)I - \alpha_m\mu\mathbb{D})v, v\rangle &= 1 - \sigma_m - \alpha_m\mu\langle \mathbb{D}v, v\rangle \\ &\geq 1 - \sigma_m - \alpha_m\mu\|\mathbb{D}\| \\ &\geq 0.\end{aligned}$$

This shows that $((1 - \sigma_m)I - \alpha_m\mu\mathbb{D})$ is positive. It follows that

$$\begin{aligned}\|((1 - \sigma_m)I - \alpha_m\mu\mathbb{D})\| &= \sup\{\langle ((1 - \sigma_m)I - \alpha_m\mu\mathbb{D})v, v\rangle : v \in \mathbb{H}_1, \|v\| = 1\} \\ &= \sup\{1 - \sigma_m - \alpha_m\mu\langle \mathbb{D}v, v\rangle : v \in \mathbb{H}_1, \|v\| = 1\} \\ &\leq 1 - \sigma_m - \alpha_m\mu\gamma.\end{aligned}$$

Next, we show that the sequence $\{x_m\}$ is bounded. Let $\tilde{p} \in \mathcal{J}$, we have $\tilde{p} = J^{B_1}_{\lambda_1}\tilde{p}, \mathbb{A}\tilde{p} = J^{B_2}_{\lambda_1}(\mathbb{A}\tilde{p})$ and $S\tilde{p} = \tilde{p}$. We estimate

$$\begin{aligned}\|v_m - \tilde{p}\|^2 &= \|J^{B_1}_{\lambda_1}(x_m + \delta\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m) - \tilde{p}\|^2 \\ &= \|J^{B_1}_{\lambda_1}(x_m + \delta\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m) - J^{B_1}_{\lambda_1}\tilde{p}\|^2 \\ &\leq \|(x_m + \delta\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m) - \tilde{p}\|^2 \\ &= \|x_m - \tilde{p}\|^2 + \delta^2\|\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\|^2 \\ &+ 2\delta\langle x_m - \tilde{p}, \mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\rangle.\end{aligned} \tag{3.3}$$

Thus, we have

$$\begin{aligned}\|v_m - \tilde{p}\|^2 = \|x_m - \tilde{p}\|^2 &+ \delta^2\langle (J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m, \mathbb{A}\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\rangle \\ &+ 2\delta\langle x_m - \tilde{p}, \mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\rangle.\end{aligned} \tag{3.4}$$

Now, we have

$$\begin{aligned}\delta^2\langle (J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m, \mathbb{A}\mathbb{A}^*(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\rangle &\leq L\delta^2\langle (J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m, (J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\rangle \\ &= L\delta^2\|(J^{B_2}_{\lambda_1} - I)\mathbb{A}x_m\|^2.\end{aligned} \tag{3.5}$$

Denoting $\Delta = 2\delta\langle x_m - \tilde{p}, \mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle$ and using (2.1), we have

$$\begin{aligned}\Delta &= 2\delta\langle x_m - \tilde{p}, \mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle \\ &= 2\delta\langle \mathbb{A}(x_m - \tilde{p}), (J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle \\ &= 2\delta\langle \mathbb{A}(x_m - \tilde{p}) + (J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m - (J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m, (J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle \\ &= 2\delta\{\langle J_{\lambda_1}^{B_2}\mathbb{A}x_m - \mathbb{A}\tilde{p}, (J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle - \|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\} \\ &\leq 2\delta\ \{\frac{1}{2}\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\ - \|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\} \\ &\leq -\delta\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\}. \end{aligned} \tag{3.6}$$

Using (3.4) (3.5) and (3.6), we obtain

$$\|v_m - \tilde{p}\|^2 \leq \|x_m - \tilde{p}\|^2 + \delta(L\delta - 1)\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2. \tag{3.7}$$

Since $\delta \in (0, \frac{1}{L})$, we obtain

$$\|v_m - \tilde{p}\|^2 \leq \|x_m - \tilde{p}\|^2. \tag{3.8}$$

It follows from (3.1) and (3.8) that

$$\begin{aligned}\|w_m - \tilde{p}\| &= \|\sigma_m x_m + (1-\sigma_m)S_2^m S_1^m v_m - \tilde{p}\| \\ &\leq \sigma_m\|x_m - \tilde{p}\| + (1-\sigma_m)\|S_2^m S_1^m v_m - \tilde{p}\| \\ &\leq \sigma_m\|x_m - \tilde{p}\| + (1-\sigma_m)\|v_m - \tilde{p}\ | \\ &\leq \sigma_m\|x_m - \tilde{p}\| + (1-\sigma_m)\|x_m - \tilde{p}\| \\ &= \sigma_m\|x_m - \tilde{p}\|. \end{aligned} \tag{3.9}$$

Since $0 < \alpha_m\mu < \|\mathbb{D}\|^{-1}$, then by Lemma 2.4, we get $\|I - \alpha_m\mu\mathbb{D}\| \leq 1 - \alpha\mu\gamma$. It follows that

$$\begin{aligned}\|x_{m+1} - \tilde{p}\| &= \|\alpha_m\beta g(x_m) + ((1-\sigma_m)I - \alpha\mu\mathbb{D}))w_m - \tilde{p}\| \\ &= \|\alpha_m(\beta g(x_m) - \mu\mathbb{D}\tilde{p}) + \sigma_m(x_m - \tilde{p}) + ((1-\sigma_m)I - \alpha_m\mu\mathbb{D}))(w_m - \tilde{p})\| \\ &\leq \|\alpha_m(\beta g(x_m) - \mu\mathbb{D}\tilde{p})\| + \sigma_m\|x_m - \tilde{p}\| + \|((1-\sigma_m)I - \alpha_m\mu\mathbb{D}))(w_m - \tilde{p})\| \\ &\leq \alpha_m\|\beta g(x_m) - \beta g(\tilde{p})\| + \alpha_m\|\beta g(\tilde{p}) - \mu\mathbb{D}\tilde{p}\| + \sigma_m\|x_m - \tilde{p}\| + (1-\sigma_m - \alpha_m\mu\gamma)\|(w_m - \tilde{p})\| \\ &\leq \alpha_m\beta\rho\|x_m - \tilde{p}\| + \sigma_m\|x_m - \tilde{p}\| + (1-\sigma_m - \alpha_m\mu\gamma)\|(x_m - \tilde{p})\| + \alpha_m\|\beta g(\tilde{p}) - \mu\mathbb{D}\tilde{p}\| \\ &= [1 - \alpha_m(\mu\gamma - \beta\rho)]\|x_m - \tilde{p}\| + \alpha_m(\mu\gamma - \beta\rho)\left(\frac{\|\beta g(\tilde{p}) - \mu\mathbb{D}\tilde{p}\|}{\mu\gamma - \beta\rho}\right) \\ &\leq \max\left(\|x_m - \tilde{p}\|, \frac{\|\beta g(\tilde{p}) - \mu\mathbb{D}\tilde{p}\|}{\mu\gamma - \beta\rho}\right) \\ &\leq \cdots \leq \max\left(\|x_0 - \tilde{p}\|, \frac{\|\beta g(\tilde{p}) - \mu\mathbb{D}\tilde{p}\|}{\mu\gamma - \beta\rho}\right). \end{aligned}$$

Therefore, $\{x_m\}$ is bounded, and so are $\{v_m\}, \{w_m\}, \{S_1^m v_m\}, \{S_2^m S_1^m v_m\}, \{g(x_m)\}$ and $\{\mathbb{D}(w_m)\}$.

Step2. We show that $\lim_{m\to\infty}\|x_{m+1} - x_m\| = 0$. Let us consider $t_m = \frac{x_{m+1} - \sigma_m x_m}{(1-\sigma_m)}$, then

$$x_{m+1} = (1-\sigma_m)t_m + \sigma_m x_m. \tag{3.10}$$

Now,

$$\begin{aligned}t_{m+1} - t_m &= \frac{x_{n+2} - \sigma_{m+1}x_{m+1}}{1-\sigma_{m+1}} - \frac{x_{m+1} - \sigma_m x_m}{1-\sigma_m} \\ &= \frac{\alpha_{m+1}\beta g(x_{m+1}) + ((1-\sigma_{m+1})I - \alpha_{m+1}\mu\mathbb{D})w_{m+1}}{1-\sigma_{m+1}} \\ &\quad - \frac{\alpha_m\beta g(x_m) + ((1-\sigma_m)I - \alpha_m\mu\mathbb{D})w_m}{1-\sigma_m} \\ &= \frac{\alpha_{m+1}(\beta g(x_{m+1}) - \mu\mathbb{D}w_{m+1})}{1-\sigma_{n+1}} - \frac{\alpha_m(\beta g(x_m) - \mu\mathbb{D}w_m)}{1-\sigma_m} + w_{m+1} - w_m. \end{aligned} \tag{3.11}$$

So,

$$\begin{aligned}\|t_{m+1}-t_m\| &\le \frac{\alpha_{m+1}}{1-\sigma_{n+1}}\|\beta g(x_{m+1})-\mu\mathbb{D}w_{m+1}\| + \frac{\alpha_m}{1-\sigma_m}\|\beta g(x_m)-\mu\mathbb{D}w_m\| + \|w_{m+1}-w_m\| \\ &\le \frac{\alpha_{m+1}}{1-b}\|\beta g(x_{m+1})-\mu\mathbb{D}w_{m+1}\| + \frac{\alpha_m}{1-b}\|\beta g(x_m)-\mu\mathbb{D}w_m\| + \|w_{m+1}-w_m\|. \qquad (3.12)\end{aligned}$$

Now,

$$\begin{aligned}\|w_{m+1}-w_m\| &= \|\gamma_{m+1}x_{m+1}+(1-\gamma_{m+1})S_2^{m+1}S_1^{m+1}v_{m+1}-\gamma_m x_m+(1-\gamma_m)S_2^mS_1^mv_m\| \\ &\le \|(1-\gamma_{m+1})S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^mS_1^mv_m)-(\gamma_{m+1}-\gamma_m)S_2^mS_1^mv_m \\ &\quad +\gamma_{m+1}(x_{m+1}-x_m)+(\gamma_{m+1}-\gamma_m)x_m\| \\ &\le \gamma_{m+1}\|x_{m+1}-x_m\|+(1-\gamma_{m+1})\|S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^mS_1^mv_m\| \\ &\quad +(\gamma_{m+1}-\gamma_m)\|S_2^mS_1^mv_m-x_m\|. \qquad (3.13)\end{aligned}$$

In addition, we have

$$\begin{aligned}\|S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^mS_1^mv_m\| &\le \|S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^mS_1^mv_{m+1}\|+\|S_2^mS_1^mv_{m+1}-S_2^mS_1^mv_m\| \\ &\le \|S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^{m+1}S_1^mv_{m+1}\| \\ &\quad +\|S_2^{m+1}S_1^mv_{m+1}-S_2^mS_1^mv_{m+1}\|+\|v_{m+1}-v_m\| \\ &\le \|S_1^{m+1}v_{m+1}-S_1^mv_{m+1}\|+\|S_2^{m+1}S_1^mv_{m+1}-S_2^mS_1^mv_{m+1}\| \\ &\quad +\|v_{m+1}-v_m\|. \qquad (3.14)\end{aligned}$$

Since, for $\delta \in (0, \frac{1}{L})$, the mapping $J_{\lambda_1}^{B_1}(x_m+\delta\mathbb{A}^*(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m)$ is averaged and hence nonexpansive, then we obtain

$$\begin{aligned}\|v_{m+1}-v_m\| &= \|J_{\lambda_1}^{B_1}(x_{m+1}+\delta\mathbb{A}^*(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_{m+1})-J_{\lambda_1}^{B_1}(x_m+\delta\mathbb{A}^*(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m)\| \\ &\le \|J_{\lambda_1}^{B_1}(I+\delta\mathbb{A}^*(J_{\lambda_1}^{B_2}-I)\mathbb{A})x_{m+1})-J_{\lambda_1}^{B_1}(I+\delta\mathbb{A}^*(J_{\lambda_1}^{B_2}-I)\mathbb{A})x_m)\| \\ &\le \|x_{n+1}-x_m\|. \qquad (3.15)\end{aligned}$$

Using (3.14), (3.15) in (3.12), we get

$$\begin{aligned}\|t_{m+1}-t_m\|-\|x_{m+1}-x_m\| &\le \frac{\alpha_{m+1}}{1-b}\|\beta g(x_{m+1})-\mu\mathbb{D}w_{m+1}\|+\frac{\alpha_m}{1-b}\|\beta g(x_m)-\mu\mathbb{D}w_m\| \\ &\quad +(1-\gamma_{m+1})(\|S_1^{m+1}v_{m+1}-S_1^mv_{m+1}\|+\|S_2^{m+1}S_1^mv_{m+1}-S_2^mS_1^mv_{m+1}\|) \\ &\quad +(\gamma_{m+1}-\gamma_m)\|S_2^mS_1^mv_m-x_m\|. \qquad (3.16)\end{aligned}$$

It follows from the definition S_i^m that

$$\begin{aligned}\|S_1^{m+1}v_{m+1}-S_1^mv_{m+1}\| &= \|(1-\delta_{m+1}^1)u_{n+1}+\delta_{m+1}^1S_1v_{m+1}-(1-\delta_m^1)v_{m+1}+\delta_m^1S_1v_{m+1}\| \\ &\le |\delta_{m+1}^1-\delta_m^1|(\|v_{m+1}\|+\|S_1v_{m+1}\|).\end{aligned}$$

Since $\lim_{m\to\infty}|\delta_{m+1}^i-\delta_m^i|=0$ for i = 1,2 and $\{v_m\}$ and $\{S_1v_m\}$ are bounded, we get

$$\lim_{m\to\infty}\|S_1^{m+1}v_{m+1}-S_1^mv_{m+1}\|=0. \qquad (3.17)$$

Similarly,

$$\|S_2^{m+1}S_1^mv_{m+1}-S_2^mS_1^mv_{m+1}\| \le |\delta_{m+1}^2-\delta_m^2|(\|S_1^mv_{m+1}\|+\|S_2^mS_1^mv_{m+1}\|),$$

from which it follows that

$$\lim_{m\to\infty}\|S_2^{m+1}S_1^{m+1}v_{m+1}-S_2^mS_1^mv_{m+1}\|=0. \qquad (3.18)$$

Hence, by using the conditions (ii) and (iv), from (3.16), (3.17) and (3.19), we get

$$\lim_{m\to\infty}\sup(\|t_{m+1}-t_m\|-\|x_{m+1}-x_m\|)\le 0.$$

Thus by (3.10) and Lemma 2.8, we conclude that $\lim_{m\to\infty}\|t_m - x_m\| = 0$, which implies that

$$\lim_{m\to\infty}\|x_{m+1} - x_m\| = 0.$$

Now,

$$\begin{aligned}\|x_m - w_m\| &\le \|x_m - x_{m+1}\| + \|x_{m+1} - w_m\|\\ &\le \|x_m - x_{m+1}\| + \|\alpha_m\beta g(x_m) + \sigma_m x_m + ((1-\sigma_m)I - \alpha\mu\mathbb{D}))w_m - w_m\|\\ &\le \|x_m - x_{m+1}\| + \alpha_m\|\beta g(x_m) - \mu\mathbb{D}w_m\| + \sigma_m\|x_m - w_m\|\\ &\le \|x_m - x_{m+1}\| + \alpha_m(\|\beta g(x_m)\| - \|\mu\mathbb{D}w_m\|) + \sigma_m\|x_m - w_m\|,\end{aligned}$$

that is

$$\|x_m - w_m\| \le \frac{1}{1-\sigma_m}\|x_m - x_{m+1}\| + \frac{\alpha_m}{1-\sigma_m}(\|\beta g(x_m)\| - \|\mu\mathbb{D}w_m\|),$$

which together with the condition (i) and (ii) implies that

$$\lim_{m\to\infty}\|x_m - w_m\| = 0. \tag{3.19}$$

Step 3: We will show that the $\lim_{m\to\infty}\|x_m - v_m\| = 0$. Using (3.4) and (3.8), we observe that

$$\begin{aligned}\|v_m - \tilde{p}\|^2 &= \|J_{\lambda_1}^{B_1}(x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m) - \tilde{p}\|^2\\ &= \|J_{\lambda_1}^{B_1}(x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m) - J_{\lambda_1}^{B_1}\tilde{p}\|^2\\ &\le \|v_m - \tilde{p}, x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m - \tilde{p}\|^2\\ &= \frac{1}{2}\Big\{\|v_m - \tilde{p}\|^2 + \|x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m - \tilde{p}\|^2 \ - \|(v_m - \tilde{p})\\ &\quad - [x_m + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m - \tilde{p}]\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|v_m - \tilde{p}\|^2 + \|x_m - \tilde{p}\|^2 + \delta(L\delta - 1)\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\\ &\quad - \|v_m - x_m - \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|v_m - \tilde{p}\|^2 + \|x_m - \tilde{p}\|^2 - [\|v_m - x_m\|^2 + \delta^2\|\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|^2\\ &\quad - 2\delta\langle v_m - x_m, \mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\rangle]\Big\}\\ &\le \frac{1}{2}\Big\{\|v_m - \tilde{p}\|^2 + \|x_m - \tilde{p}\|^2 - \|v_m - x_m\|^2 + 2\delta\|\mathbb{A}(v_m - x_m)\|\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|\Big\}.\end{aligned}$$

Hence, we obtain

$$\|v_m - \tilde{p}\|^2 \le \|x_m - \tilde{p}\|^2 - \|v_m - x_m\|^2 + 2\delta\|A(v_m - x_m)\|\|(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_m\|. \tag{3.20}$$

Also using (2.4) and (2.3), we get

$$\begin{aligned}\|x_{m+1} - \tilde{p}\|^2 &= \|\alpha_m\beta g(x_m) + \sigma_m x_m + ((1-\sigma_m)I - \alpha\mu\mathbb{D}))w_m - \tilde{p}\|^2\\ &= \|\alpha_m(\beta g(x_m) - \mu\mathbb{D}\tilde{p}) + \sigma_m(x_m - w_m) + (I - \alpha_m\mu\mathbb{D}))(w_m - \tilde{p})\|^2\\ &\le \|\sigma_m(x_m - w_m) + (I - \alpha_m\mu\mathbb{D})(w_m - \tilde{p})\|^2 + 2\alpha_m\langle\beta g(x_m) - \mu\mathbb{D}\tilde{p}, x_{m+1} - \tilde{p}\rangle\\ &\le \big[(1-\alpha_m\mu\gamma)\|w_m - \tilde{p}\| + \sigma_m\|x_m - w_m\|\big]^2 + 2\alpha_m\|\beta g(x_m) - \mu\mathbb{D}\tilde{p}\|\|x_{m+1} - \tilde{p}\|\\ &\le (1-\alpha_m\mu\gamma)^2\|w_m - \tilde{p}\|^2 + \sigma_m^2\|x_m - w_m\|^2 + 2\sigma_m(1-\alpha_m\mu\gamma)\|w_m - \tilde{p}\|\|x_m - w_m\|\\ &\quad + 2\alpha_m\|\beta g(x_m) - \mu\mathbb{D}\tilde{p}\|\|x_{m+1} - \tilde{p}\|\\ &\le (1-\alpha_m\mu\gamma)^2\big[\|\gamma_m(x_m - \tilde{p}) + (1-\gamma_m)(S_2^m S_1^m v_m - \tilde{p})\|\big]^2 + \sigma_m^2\|x_m - w_m\|^2\\ &\quad + 2\sigma_m(1-\alpha_m\mu\gamma)\|w_m - \tilde{p}\|\|x_m - w_m\| + 2\alpha_m\|\beta g(x_m) - \mu\mathbb{D}\tilde{p}\|\|x_{m+1} - \tilde{p}\|\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_m\mu\gamma)^2\big[\gamma_m\|x_m-\tilde{p}\|^2+(1-\gamma_m)\|S_2^mS_1^mv_m-\tilde{p}\|^2\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&\le (1-\alpha_m\mu\gamma)^2\big[\gamma_m\|x_m-\tilde{p}\|^2+(1-\gamma_m)\|v_m-\tilde{p}\|^2\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&\le (1-\alpha_m\mu\gamma)^2\big[\gamma_m\|x_m-\tilde{p}\|^2+(1-\gamma_m)\{\|x_m-\tilde{p}\|^2+\delta(L\delta-1)\|(J_{\lambda_1}^{B_2}-I)Ax_m\|^2\}\big]\\
&\quad+\sigma_m^2\|x_m-w_m\|^2+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&= (1-\alpha_m\mu\gamma)^2\big[\|x_m-\tilde{p}\|^2+(1-\gamma_m)\delta(L\delta-1)\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|^2\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|.
\end{aligned}$$

So,

$$\begin{aligned}
(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\delta(1-L\delta)\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|^2 &\le \sigma_m^2\|x_m-w_m\|^2+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|\\
&\quad+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&\quad+(1-\alpha_m\mu\gamma)^2\|x_m-\tilde{p}\|^2-\|x_{m+1}-\tilde{p}\|^2,
\end{aligned}$$

which gives

$$\begin{aligned}
&(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\delta(1-L\delta)\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|^2 \le (\alpha_m\mu\gamma)^2\|x_m-\tilde{p}\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|+\sigma_m^2\|x_m-w_m\|^2\\
&\quad-2\alpha_m\mu\gamma\|x_m-\tilde{p}\|^2+\|x_m-x_{m+1}\|(\|x_m-\tilde{p}\|+\|x_{m+1}-\tilde{p}\|). \qquad (3.21)
\end{aligned}$$

Since $(1-L\delta)>0$ using the condition (ii) and (3.19) in (3.21), we get

$$\lim_{m\to\infty}\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|=0. \qquad (3.22)$$

Again, using (3.20), we get

$$\begin{aligned}
\|x_{m+1}-\tilde{p}\|^2 &\le (1-\alpha_m\mu\gamma)^2\big[\gamma_m\|x_m-\tilde{p}\|^2+(1-\gamma_m)\|v_m-\tilde{p}\|^2\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&\le (1-\alpha_m\mu\gamma)^2\big[\gamma_m\|x_m-\tilde{p}\|^2+(1-\gamma_m)\big\{\|x_m-\tilde{p}\|^2-\|v_m-x_m\|^2\\
&\quad+2\delta\|\mathbb{A}(v_m-x_m)\|\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|\big\}\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|\\
&= (1-\alpha_m\mu\gamma)^2\|x_m-\tilde{p}\|^2-(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\|v_m-x_m\|^2\\
&\quad+2\delta(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\|\mathbb{A}(v_m-x_m)\|\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|\big\}\big]+\sigma_m^2\|x_m-w_m\|^2\\
&\quad+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|.
\end{aligned}$$

So,

$$\begin{aligned}
(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\|v_m-x_m\|^2 &\le 2\delta(1-\alpha_m\mu\gamma)^2(1-\gamma_m)\|\mathbb{A}(v_m-x_m)\|\|(J_{\lambda_1}^{B_2}-I)\mathbb{A}x_m\|\\
&\quad+\sigma_m^2\|x_m-w_m\|^2+2\sigma_m(1-\alpha_m\mu\gamma)\|w_m-\tilde{p}\|\|x_m-w_m\|\\
&\quad+2\alpha_m\|\beta g(x_m)-\mu\mathbb{D}\tilde{p}\|\|x_{m+1}-\tilde{p}\|+(\alpha_m\mu\gamma)^2\|x_m-\tilde{p}\|^2\\
&\quad-2\alpha_m\mu\gamma\|x_m-\tilde{p}\|^2+\|x_{m+1}-x_m\|(\|x_m-\tilde{p}\|+\|x_{m+1}-\tilde{p}\|).
\end{aligned}$$

Thus, from the conditions $(ii),(iv)$, (3.19) and (3.22), we get

$$\lim_{m\to\infty}\|v_m-x_m\|=0. \qquad (3.23)$$

From (3.23), we get

$$\|x_{m+1}-v_m\|\le\|x_{m+1}-x_m\|+\|x_m-v_m\|\to 0, \quad \text{as} \quad m\to\infty. \qquad (3.24)$$

From (3.19) and (3.24), we get

$$\|w_m - v_m\| \leq \|w_m - x_m\| + \|x_m - x_{m+1}\| + \|x_{m+1} - v_m\| \to 0, \quad \text{as} \quad m \to \infty. \tag{3.25}$$

Now,

$$\begin{aligned}\|S_2^m S_1^m v_m - v_m\| &\leq \|S_2^m S_1^m v_m - w_m\| + \|w_m - v_m\| \\ &= \gamma_m \|S_2^m S_1^m v_m - x_m\| + \|w_m - v_m\| \\ &\leq \gamma_m \|S_2^m S_1^m v_m - v_m\| + \gamma_m \|v_m - x_m\| + \|w_m - v_m\|.\end{aligned}$$

So, from the condition (iv), (3.23) and (3.25), we get

$$\|S_2^m S_1^m v_m - v_m\| \leq \frac{\gamma_m}{1-\gamma_m}\|v_m - x_m\| + \frac{1}{1-\gamma_m}\|w_m - v_m\| \to 0, \quad as \quad m \to \infty. \tag{3.26}$$

Step 4: We show that

$$\lim_{m\to\infty} \sup \langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_m\rangle \leq 0.$$

where $\tilde{x}$ is the unique solution of the variational inequality (3.2). Since $\{x_m\}$ is bounded, there exists a subsequence of $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \rightharpoonup \bar{x}$ as $j \to \infty$ and

$$\lim_{m\to\infty} \sup \langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_m\rangle = \lim_{j\to\infty} \langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m_j}\rangle.$$

Since $\|x_m - v_m\| \to 0$ as $m \to \infty$, so $v_{m_j} \rightharpoonup \bar{x}$. Noticing that $\{\delta_m^i\}$ is bounded for $i = 1, 2$, we can assume $\delta_{m_j}^i \to \delta_\infty^i$ as $j \to \infty$, where $0 < \delta_\infty^i < 1$ for $i = 1, 2$. Define $S_i^\infty = (1 - \delta_\infty^i)I + \delta_\infty^i S_i$(i = 1,2). Then we have $Fix(S_i^\infty) = Fix(S_i)$ for $i = 1, 2$. Furthermore, since $Fix(S_1^\infty) \cap Fix(S_2^\infty) = Fix(S_1) \cap Fix(S_2) = Fix(\mathcal{S}) \neq \phi$ and S_i^∞ is δ_∞^i-averaged for $i = 1, 2$, by Lemma 2.2, we get $Fix(S_2^\infty S_1^\infty) = Fix(S_1^\infty) \cap Fix(S_2^\infty) = Fix(\mathcal{S})$. Note that

$$\|S_i^{m_j} v - S_i^\infty v\| \leq |\delta_{m_j}^i - \delta_\infty^i|(\|v\| + \|S_j v\|),$$

hence, we get

$$\lim_{j\to\infty} \sup_{v\in\mathcal{D}} \|S_i^{m_j} v - S_i^\infty v\| = 0, \tag{3.27}$$

where $\mathcal{D}$ is a bounded subset of $\mathbb{H}_1$ that can be chosen at random. We also have

$$\begin{aligned}\|v_{m_j} - S_2^\infty S_1^\infty v_{m_j}\| &\leq \|v_{m_j} - S_2^{m_j} S_1^{m_j} v_{m_j}\| + \|S_2^{m_j} S_1^{m_j} v_{m_j} - S_2^\infty S_1^{m_j} v_{m_j}\| + \|S_2^\infty S_1^{m_j} v_{m_j} - S_2^\infty S_1^\infty v_{m_j}\| \\ &\leq \|v_{m_j} - S_2^{m_j} S_1^{m_j} v_{m_j}\| + \|S_2^{m_j} S_1^{m_j} v_{m_j} - S_2^\infty S_1^{m_j} v_{m_j}\| + \|S_1^{m_j} v_{m_j} - S_1^\infty v_{m_j}\| \\ &\leq \|v_{m_j} - S_2^{m_j} S_1^{m_j} v_{m_j}\| + \sup_{v\in\mathcal{D}'} \|S_2^{m_j} v - S_2^\infty v\| + \sup_{v\in\mathcal{D}''} \|S_1^{m_j} v - S_1^\infty v\|,\end{aligned} \tag{3.28}$$

where $\mathcal{D}'$ is bounded subset including $\{S_1^{m_j} v_{m_j}\}$ and $\mathcal{D}''$ is a bounded subset including $\{v_{m_j}\}$. It is as follows that from (3.26), (3.27) and (3.28) that $\lim_{j\to\infty} \|v_{m_j} - S_2^\infty S_1^\infty v_{m_j}\| = 0$. So by Lemma 2.3, we have $\bar{x} \in Fix(S_2^\infty S_1^\infty) = \mathcal{S}$.

On the other hand, $v_{m_k} = J_{\lambda_1}^{B_1}(x_{m_k} + \delta\mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_{m_k})$ can be written as

$$\frac{(x_{m_k} - v_{m_k}) + \mathbb{A}^*(J_{\lambda_1}^{B_2} - I)\mathbb{A}x_{m_k}}{\lambda_1} \quad \in \quad B_1 v_{m_k}. \tag{3.29}$$

Taking $k \to \infty$ in (3.29) and from (3.22) ,(3.23) and the fact that the graph of maximal monotone operator is weakly-strongly closed, $0 \in B_1(\bar{x})$, *i.e* $\bar{x} \in$ SolVIP(B_1). Because $\{x_m\}$ and $\{v_m\}$ exhibit the same asymptotical behaviour, $\{\mathbb{A}x_{m_k}\}$ weakly converges to $\mathbb{A}\bar{x}$. Again using (3.22) and the fact that the resolvent $J_{\lambda_1}^{B_2}$ is nonexpansive and Lemma 2.3, we get $\mathbb{A}\bar{x} \in B_2(\mathbb{A}\bar{x})$, i.e $\mathbb{A}\bar{x} \in$ SolVIP(B_2). Thus $\bar{x} \in \mathcal{J}$. So

$$\lim_{j\to\infty} \langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{n_j}\rangle = \langle (\mu\mathbb{D} - \beta g)\tilde{x}, \tilde{x} - \bar{x}\rangle \leq 0. \tag{3.30}$$

Step 5: Finally, we show that $x_m \to \tilde{x}$ as $m \to \infty$. From (2.4) and (3.1), we have

$$\begin{aligned}
\|x_{m+1} - x_m\|^2 &= \|\alpha_m \beta g(x_m) + \sigma_m x_m + ((I - \sigma_m)I - \alpha_m \mu \mathbb{D})w_m - \tilde{x}\|^2 \\
&= \|\alpha_m(\beta g(x_m) - \mu \mathbb{D}\tilde{x}) + \sigma_m(x_m - \tilde{x}) + ((I - \sigma_m)I - \alpha \mu \mathbb{D})(w_m - \tilde{x})\|^2 \\
&\le \|\sigma_m(x_m - \tilde{x}) + ((I - \sigma_m)I - \alpha_m \mu \mathbb{D})(w_m - \tilde{x})\|^2 \\
&+ 2\alpha_m \langle \beta g(x_m) - \mu \mathbb{D}\tilde{x}, x_{m+1} - \tilde{x}\rangle \\
&\le [(1 - \sigma_m - \alpha_m \mu\gamma)\|w_m - \tilde{x}\| + \sigma_m \|x_m - \tilde{x}\|]^2 + 2\alpha_m \langle \beta g(x_m) - \beta g(\tilde{x}), x_{m+1} - \tilde{x}\rangle \\
&\quad + 2\alpha_m \langle \beta g(\tilde{x}) - \mu \mathbb{D}\tilde{x}, x_{m+1} - \tilde{x}\rangle \\
&\le [(1 - \sigma_m - \alpha_m \mu\gamma)\|x_m - \tilde{x}\| + \sigma_m \|x_m - \tilde{x}\|]^2 + 2\alpha_m \beta\rho \|x_m - \tilde{x}\|\|x_{m+1} - \tilde{x}\| \\
&\quad + 2\alpha_m \langle \beta g(\tilde{x}) - \mu \mathbb{D}\tilde{x}, x_{m+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_m \mu\gamma)^2 \|x_m - \tilde{x}\|^2 + \alpha_m \beta\rho(\|x_m - \tilde{x}\|^2 + \|x_{m+1} - \tilde{x}\|^2) \\
&+ 2\alpha_m \langle (\mu \mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m+1}\rangle.
\end{aligned} \tag{3.31}$$

Since $\mu\gamma > \beta\rho$ and $0 < \alpha_m \le \frac{1}{\mu\|\mathbb{D}\|} \le \frac{1}{\mu\gamma}$, we get $1 - \alpha_m \mu\rho > 1 - \alpha_m \mu\rho \ge 0$. Hence, from (3.31) we get

$$\begin{aligned}
\|x_{m+1} - x_m\|^2 &\le \frac{(1 - \alpha_m \mu\gamma)^2 + \alpha_m \beta\rho}{1 - \alpha_m \beta\rho}\|x_m - \tilde{x}\|^2 + \frac{2\alpha_m}{1 - \alpha_m \beta\rho}\langle (\mu \mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m+1}\rangle \\
&\le \left[1 - \frac{2\alpha_m(\mu\gamma - \beta\rho)}{1 - \alpha_m \beta\rho}\right]\|x_m - \tilde{x}\|^2 + \frac{2\alpha_m}{1 - \alpha_m \beta\rho}\langle (\mu \mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m+1}\rangle \\
&+ \frac{\alpha_m^2 \mu^2 \gamma^2}{1 - \alpha_m \beta\rho}\|x_m - \tilde{x}\|^2 \\
&\le \left[1 - \frac{2\alpha_m(\mu\gamma - \beta\rho)}{1 - \alpha_m \beta\rho}\right]\|x_m - \tilde{x}\|^2 \\
&+ \frac{2\alpha_m(\mu\gamma - \beta\rho)}{1 - \alpha_m \beta\rho}\left[\frac{\langle (\mu \mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m+1}\rangle}{(\mu\gamma - \beta\rho)} + \alpha_m L\right],
\end{aligned} \tag{3.32}$$

where L is the constant satisfying $L \ge \sup_{m\ge 0}\left\{\frac{\mu^2\gamma^2}{2}\|x_m - \tilde{x}\|^2\right\}$. Now, using the condition (ii) and (3.30), we have

$$\sum_{m=0}^{\infty} \frac{2\alpha_m(\mu\gamma - \beta\rho)}{1 - \alpha_m \beta\rho} > \sum_{m=0}^{\infty} 2(\mu\gamma - \beta\rho)\alpha_m = \infty,$$

and

$$\lim_{m\to\infty} \sup \left(\frac{\langle (\mu \mathbb{D} - \beta g)\tilde{x}, \tilde{x} - x_{m+1}\rangle}{\mu\gamma - \beta\rho} + \alpha_m L\right) \le 0.$$

So, according to Lemma 2.7, we conclude that $\|x_m - \tilde{x}\| \to 0$, as $m \to \infty$. Which completes the proof. □

4 Numerical example

We give a numerical example which justify the theorem. Let $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{R}$, and $B_1 : \mathbb{H}_1 \to \mathbb{H}_1$ defined as $B_1(x) = 2x$ and $B_2 : \mathbb{H}_2 \to \mathbb{H}_2$ defined as $B_2(x) = -\frac{4}{5}x$. For $\lambda_1 = \frac{1}{4}$, we compute the resolvent of B_1 and B_2 as:

$$J_{\lambda_1}^{B_1}(x) = (1 + \lambda_1 B_1)^{-1}(x) = \frac{2}{3}(x),$$
$$J_{\lambda_1}^{B_2}(x) = (1 + \lambda_1 B_2)^{-1}(x) = \frac{5}{4}(x).$$

Define the mapping $\mathbb{A} : \mathbb{R} \to \mathbb{R}, \mathbb{D} : \mathbb{R} \to \mathbb{R}, S_1 : \mathbb{R} \to \mathbb{R}, S_2 : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $\mathbb{A}(x) = -x, \mathbb{D}(x) = 3x, S_1(x) = \frac{x}{3}, S_2(x) = \sin(x)$ and $g(x) = \frac{x}{3}\ \ \forall x \in \mathbb{R}$.

It is clear to observe that S_1 and S_2 are nonexpansive mappings, g is a Lipschitzian mapping with coefficient $\rho = \frac{1}{3}$ and $\mathbb{D}$ is a strongly positive bounded linear operator with coefficient $\gamma = 2$ and $\mathbb{A}$ is bounded linear operator on $\mathbb{R}$ with adjoint operator $\mathbb{A}^*$ such that $\|\mathbb{A}\| = \|\mathbb{A}^*\| = 1$. Now let us choose $\beta = 1$, $\mu = 2$, $\gamma_m = \frac{m+1}{m+2}$, $\sigma_m = \frac{m+1}{10(m+1)}$, and $\alpha_m = \frac{1}{m+6}$. Also, let us consider $\delta_m^1 = \frac{m+1}{m+2}$ and $\delta_m^1 = \frac{m+2}{m+3}$. It is simple to see that S_1, S_2 are nonexpansive with $Fix(S_1) = Fix(S_2) = \{0\}$ and hence $\Omega = \{0 \in \mathbb{H}_1 : 0 \in \text{SolVIP}(B_1)$ and $A(0) \in \text{SolVIP}(B_2)\}=\{0\}$. Therefore, $\mathcal{J} = \Omega \cap \mathcal{S} = \{0\} \ne \phi$. The stopping criterian for our proposed iterative method is $\|x_{m+1} - x_m\| \le 1 \times 10^{-6}$.

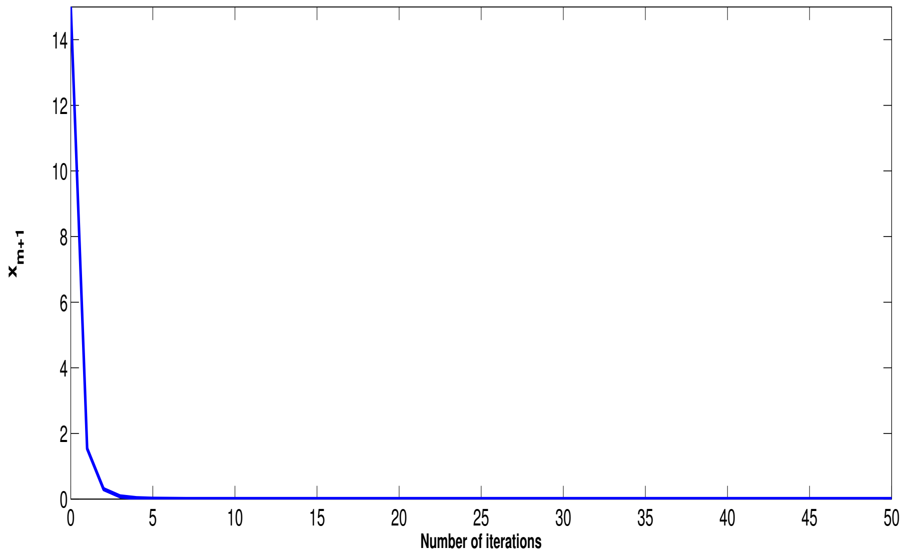

Figure 1: Convergence of $\{x_m\}$ for initial value $x_0 = 15$

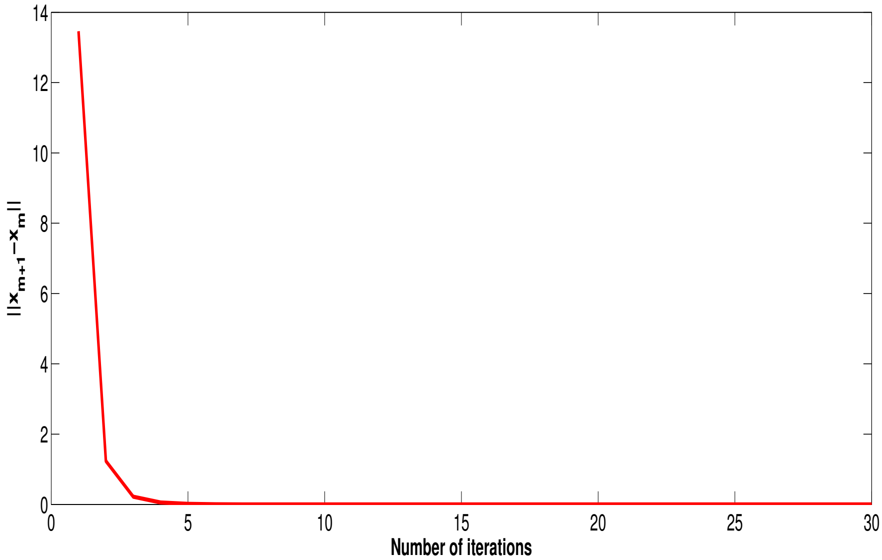

Figure 2: Error plotting of $\|x_{m+1} - x_m\|$.

Table 1: Numerical result of the iterative method with initial value $x_0 = 15$.

No. of iterations	x_m $x_0 = 15$	$\|x_{m+1} - x_m\|$ $x_0 = 15$
1	15.000000	13.461888
3	0.303597	$2.1967e^{-01}$
5	0.028717	$1.7303e^{-02}$
7	0.005074	$2.6130e^{-03}$
9	0.001279	$5.7500e^{-04}$
11	0.000407	$1.6200e^{-04}$
13	0.000152	$5.5000e^{-05}$
15	0.000065	$2.1000e^{-05}$
17	0.000030	$9.0000e^{-06}$
19	0.000015	$4.0000e^{-06}$
21	0.000008	$2.0000e^{-06}$
23	0.000005	$1.0000e^{-06}$

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