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Traveling wave solutions for systems of nonlinear advection-diffusion-reaction equations with delay and variable coefficients

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Abstract

This paper introduces the methods for constructing the exact solutions of systems of nonlinear Advection-Diffusion-Reaction (ADR) equations with delay and variable coefficients. ADR systems of equations are coupled models which can be used to describe a set of interacting processes. Precepts are given for reducing such systems of equations to simpler systems of delayed ordinary differential equations by using modified methods of functional constraints. New exact solutions are presented in the form of traveling wave solutions. Exact solutions are prescribed to particular nonlinear ADR systems of equations for illustration. Significant arbitrary functions are present in the solutions which justify the suitability of the solutions for solving various modelling problems, validating the potency of numeric, asymptotic, and approximate analytical methods. The range of applicability of the results in this paper is universal as the results involve variable coefficients and delay.

Keywords: Advection-diffusion-reaction, Exact solutions, Delay differential equations, Fundamental matrix
2020 MSC: Primary 35K55, 35B10; Secondary 35B40, 35K57

1 Introduction

Abbreviations
ODEs: Ordinary Differential Equations
PDEs: Partial Differential Equations
ADR: Advection-Diffusion-Reaction

Exact solutions of nonlinear Partial Differential Equations (PDEs) are imperative for a proper and accurate analysis of many phenomena and processes which occur in various fields of natural science. Exact solutions can serve as test problems to determine the validity of numerical and approximate analytical methods for solving PDEs. The confidence

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which can be placed on various numerical and approximate analytical methods and their range of applicability can be decided through exact solutions. How to construct the exact solutions has remained a pivotal study for researchers. Let $u = u(x,t)$ be the unknown function, $p > 0$ be the diffusion coefficient and $H(u)$ be the arbitrary function which accounts for local dynamics, how to construct the exact solutions of one-component RD equations of the form

$$u_t = au_{xx} + H(u), \tag{1.1}$$

and their various generalizations were considered in [18, 17, 26, 12, 13, 9, 11, 14, 2, 3, 24, 8, 23, 25, 5]. A reaction-diffusion system consists of two nonlinear diffusion equations such as

$$\begin{aligned} u_t &= a_1u_{xx} + H_1(u,v), \\ v_t &= a_2v_{xx} + H_2(u,v), \end{aligned} \tag{1.2}$$

where $u = u(x,t), v = v(x,t)$ are two unknown functions denoting the population densities of two interacting species. Systems of equations which are of the form (1.2) with constant coefficients were considered in [6, 7, 10, 15]. Generalized forms of (1.2) which involve delay in time and where the coefficients are also constants were considered in [21, 22, 16]. Recently, generalized forms of (1.2) with delays and where variable coefficients are associated with the time derivatives u_t and v_t, and also the kinetic functions H_1 and H_2, were considered in [1, 4].

This study considers systems of nonlinear Advection-Diffusion-Reaction (ADR) equations with delay and variable coefficients which are of the form

$$\begin{aligned} s_1(x)u_t &= p_1(x)u_{xx} + q_1(x)u_x + r_1(x)H_1(u,\bar{u},v,\bar{v}), \\ s_2(x)v_t &= p_2(x)v_{xx} + q_2(x)v_x + r_2(x)H_2(u,\bar{u},v,\bar{v}), \end{aligned} \tag{1.3}$$

with $u = u(x,t), \bar{u} = u(x,t-\tau), v = v(x,t), \bar{v} = v(x,t-\tau)$ and $\tau > 0$, which is the delay time. The u and v in (1.3) represent the state variables concentration at position $x \in \mathbb{R}$ and time t, e.g, densities of the prey and the predator. The diffusion coefficients are the functions $p_i(x) > 0$ and the advection rates are $q_i(x)$, which are real functions, where $i = 1,2$, (See e.g, [28]). The functions $s_i(x)$ are respectively associated with the time rate of change of concentration of state variables while the functions $r_i(x)$ are respectively associated with the arbitrary functions $H_i(u,\bar{u},v,\bar{v})$. Both $r_i(x)$ and $s_i(x)$ are real functions. The functions $H_i(u,\bar{u},v,\bar{v})$ account for local dynamics. New exact solutions that are in the form of traveling wave solutions are presented to systems of delay ADR equations with variable coefficients which are of the form (1.3). Significant arbitrary functions are present in the solutions which justify the suitability of the solutions for solving various modelling problems, validating the potency of numeric, asymptotic, and approximate analytical methods.

2 Preliminaries

Definition 2.1. Exact solution. In connection with nonlinear PDEs, exact solution signifies the case where the solution can be displayed in the following forms (see e.g, [12, 13, 26]):

(i) with respect to elementary functions;

(ii) in the form of definite or/and indefinite integrals;

(iii) with respect to solutions of Ordinary Differential Equations (ODEs) or systems of ODEs.

Two or more of the cases listed above could also be combined.

Definition 2.2. Linearly dependent or independent vector functions. A set of n vector functions $w_1(x)$, $w_2(x)$, ..., $w_n(x)$, are said to be linearly dependent on an interval $[\alpha,\beta]$ if there exists a set of numbers $\theta_1,\theta_2,\ldots,\theta_n$, with at least one nonzero number, such that

$$\theta_1w_1(x) + \theta_2w_2(x) + ... + \theta_nw_n(x) \equiv 0 \quad \forall\ x \in [\alpha,\beta]. \tag{2.1}$$

The given n vector functions are said to be linearly independent on the interval if (2.1) is only satisfied provided

$$\theta_1 = \theta_2 = ... = \theta_n = 0.$$

Definition 2.3. Commutative property of a matrix. A matrix $A(x)$ is said to be commutative with its integral provided

$$A(x) * \int_a^x A(\psi)d\psi = \int_a^x A(\psi)d\psi * A(x). \tag{2.2}$$

This property is readily satisfied by symmetric and indeed diagonal matrices.

Definition 2.4. Fundamental matrix. Consider the matrix differential equation which is given as

$$G'(x) = A(x)G(x). \tag{2.3}$$

A system of linearly independent solutions which satisfies (2.3) is referred to as a fundamental system of solutions. A square matrix whose columns is formed by linearly independent solutions is called a fundamental matrix. The fundamental matrix is nonsingular since its columns are formed by linearly independent solutions (See e.g, [27, 29]). Notice that the fundamental matrix is invertible since it is nonsingular. A system of matrix differential equations whose coefficient matrix is commutative with its integral has the fundamental matrix $\Gamma(x)$ which is given as

$$\Gamma(x) = e^{\int_{x_0}^x A(\omega)d\omega}. \tag{2.4}$$

Definition 2.5. Exponential matrix. Let $n \in \mathbb{N}$ and A be an $n \times n$ diagonal matrix with entries

$$A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix},$$

then

$$e^A = \begin{bmatrix} e^{a_1} & & & \\ & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{bmatrix}.$$

Two matrices A and B are said to commute if $AB = BA$. Commutable matrices A and B have the identity that

$$e^A e^B = e^{A+B}.$$

3 Construction of solutions for systems of nonlinear ADR equations

The exact solutions of (1.3) are to be constructed in the form

$$\begin{array}{ll} u = U(y), & y = t + \int h_1(x)dx, \\ v = V(z), & z = t + \int h_2(x)dx, \end{array} \tag{3.1}$$

where U and V can be explicitly determined if respectively the forms of u and v are known. The equations of the form (3.1) are conventionally called generalized traveling-wave solutions. The functions $h_1(x)$ and $h_2(x)$ may be given or are to be otherwise determined. Whether functions $h_1(x)$ and $h_2(x)$ are given or not, is determined by the goal in a particular situation. Substitute (3.1) into (1.3) to obtain

$$\begin{aligned} &p_1(x)h_1^2U''_{yy} + \{p_1(x)h'_1 + q_1(x)h_1 - s_1(x)\}\, U'_y + r_1(x)H_1(U, \bar{U}, V, \bar{V}) = 0, \\ &p_2(x)h_2^2V''_{zz} + \{p_2(x)h'_2 + q_2(x)h_2 - s_2(x)\}\, V'_z + r_2(x)H_2(U, \bar{U}, V, \bar{V}) = 0, \end{aligned} \tag{3.2}$$

with $h_1 = h_1(x)$ and $h_2 = h_2(x)$.

Let the coefficients in the system (3.2) fulfill the conditions

$$\begin{aligned} r_1(x) &= p_1(x)a_1h_1^2, \\ r_2(x) &= p_2(x)a_2h_2^2, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} p_1(x)h_1' + q_1(x)h_1 &= -b_1p_1(x)h_1^2 + s_1(x), \\ p_2(x)h_2' + q_2(x)h_2 &= -b_2p_2(x)h_2^2 + s_2(x), \end{aligned} \tag{3.4}$$

where a_i ($a_i \not\equiv 0$) and b_i are constants and $i = 1, 2$. The conditions (3.3) and (3.4) reduce (3.2) to a coupled ODEs

$$\begin{aligned} U''_{yy} - b_1U'_y + a_1H_1(U, \bar{U}, V, \bar{V}) = 0, \\ V''_{zz} - b_2V'_z + a_2H_2(U, \bar{U}, V, \bar{V}) = 0, \end{aligned} \tag{3.5}$$

with $\bar{U} = U(Y - \tau)$ and $\bar{V} = V(Z - \tau)$.

Remark 3.1. Let $V \equiv 0$ and $H_1(U, \bar{U}) = Uh(\bar{U}/U)$ in (3.5). Then $U = Ke^{\psi y}$ is admitted as the exact solution, where K is an arbitrary constant and ψ is determined by the transcendental equation (See e.g, [12])

$$\psi^2 - b_1\psi + a_1h\left(e^{-\tau\psi}\right) = 0. \tag{3.6}$$

The delay ODE in question can have different roots due to different roots from (3.6) .

Remark 3.2. Let $V \equiv 0, H_1(U, \bar{U}) = h(U)$ and $b_1 \equiv 0$ in (3.5). Then the general solution is given for any function $h(U)$ in the implicit form

$$\int \left[K_1 - 2\int h(U)dU\right]^{-1/2} dU = K_2 \pm y,$$

for arbitrary constants K_1 and K_2 with $a_1 \equiv 1$ (See e.g, [20]). An instantaneous system is characterized by the special case $H_1(U, \bar{U}) = h(U)$ and $b_i \equiv 0$ constitute a linear form of (3.4), where $i = 1, 2$.

The systems of equations (3.3) and (3.4) give the relation which the coefficients functions in (1.3) have with functions h_1 and h_2 which are present in (3.1). A system of differential equations in respect of h_1 and h_2 is formed by (3.4) and their respective algebraic relations are given by (3.3), in term of $p_i(x), q_i(x), r_i(x)$, and $s_i(x)$, where $i = 1, 2$.

3.1 Finding exact solutions when h_1 and h_2 are not given

Given the functions $p_i(x), q_i(x)$ as well as $s_i(x)$ and for $b_i \neq 0, i = 1, 2$ in (3.4), a system of differential equations of Riccati type

$$\begin{aligned} p_1(x)h_1' + b_1p_1(x)h_1^2 + q_1(x)h_1 - s_1(x) = 0, \\ p_2(x)h_2' + b_2p_1(x)h_2^2 + q_2(x)h_2 - s_2(x) = 0, \end{aligned} \tag{3.7}$$

is obtained. The degenerate and nondegenerate cases will be considered for the system (3.7).

3.1.1 Degenerate case

For $b_1 = b_2 = 0$ in (3.7), the corresponding degenerate system in vector form is

$$H'(x) = M(x)H(x) - s(x), \tag{3.8}$$

where

$$H(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \quad s(x) = \begin{bmatrix} s_1(x)/p_1(x) \\ s_2(x)/p_2(x) \end{bmatrix} \text{ and}$$

$$M(x) = \begin{bmatrix} q_1(x)/p_1(x) & 0 \\ 0 & q_2(x)/p_2(x) \end{bmatrix}. \tag{3.9}$$

The matrix $M(x)$ in (3.9) is observed to be symmetric and indeed, it is diagonal. The homogeneous part of (3.8) is given by

$$H'(x) = M(x)H(x). \tag{3.10}$$

Therefore the matrix differential equation (3.10) has the fundamental matrix which is given by

$$\Gamma(x) = e^{\int_{x_0}^{x} M(\omega)d\omega}. \tag{3.11}$$

Substitution of the fundamental matrix $\Gamma(x)$ into (3.10) results in

$$\Gamma'(x) = M(x)\Gamma(x). \tag{3.12}$$

Post-multiply (3.12) by $\Gamma^{-1}(x)$ (which is the inverse function of the fundamental matrix) to obtain

$$\begin{aligned}\Gamma'(x)\Gamma^{-1}(x) &= M(x)\Gamma(x)\Gamma^{-1}(x)\\ &\Rightarrow M(x) = \Gamma'(x)\Gamma^{-1}(x).\end{aligned}$$

Remark 3.3. Given a fundamental matrix $\Gamma(x)$, the coefficient matrix $M(x)$ of a homogeneous matrix differential equation is uniquely defined by

$$M(x) = \Gamma'(x)\Gamma^{-1}(x).$$

The general solution for the homogeneous matrix differential equation (3.10) is given by

$$H_C(x) = \Gamma(x)C, \tag{3.13}$$

which is in the term of fundamental matrix, where $C = (C_1, C_2)^T$, is a column vector that consists of arbitrary constants. $H_C(x)$ in (3.13) is called the complementary solution for the homogeneous part of (3.8). To solve the nonhomogeneous matrix differential equation (3.8), a method which is known as variation of constants (otherwise called Lagrange method) will be adopted. The constant vector C in (3.13) is replaced by $C(x)$, which is a continuously differentiable function with respect to independent variable x. Then the general solution for the nonhomogeneous matrix differential equation (3.8) takes the form

$$H(x) = \Gamma(x)C(x). \tag{3.14}$$

The unknown vector $C(x)$ can be found by substituting (3.14) into (3.8) to obtain

$$\begin{aligned}&\cancel{\Gamma'(x)}C(x) + \Gamma(x)C_x'(x) = \cancel{M(x)\Gamma(x)C(x)} - s(x)\\ \Rightarrow\ &\Gamma(x)C_x'(x) = -s(x).\end{aligned} \tag{3.15}$$

Pre-multiply (3.15) by $\Gamma^{-1}(x)$ to obtain

$$\begin{aligned}&\Gamma^{-1}(x)\Gamma(x)C_x'(x) = -\Gamma^{-1}(x)s(x)\\ \Rightarrow\ &C_x'(x) = -\Gamma^{-1}(x)s(x)\\ \Rightarrow\ &C(x) = C_0 - \int \Gamma^{-1}(x)s(x)dx,\end{aligned} \tag{3.16}$$

where C_0 is a column vector that consists of arbitrary constants. Hence, the general solution for nonhomogeneous matrix differential equation (3.8) is obtained by substituting (3.16) into (3.14) to have

$$\begin{aligned}H(x) &= \Gamma(x)C(x)\\ &= \Gamma(x)\left(C_0 - \int \Gamma^{-1}(x)s(x)dx\right).\end{aligned} \tag{3.17}$$

Using (3.1), the exact solutions for (1.3) are obtained, where $r_i(x)$ $(i = 1, 2)$ in (1.3) are determined by applying (3.3).

Example 3.4. This illustration is given by considering the case where $p_1(x) = cos\ x, p_2(x) = sin\ x, q_1(x) = sin\ x, q_2(x) = cos\ x, s_1(x) = sin\ x$, and $s_2(x) = e^x$, where $x \in (0, \pi/2)$. According to (3.9),

$$M(x) = \begin{bmatrix} cos\ x/sin\ x & 0\\ 0 & sin\ x/cos\ x\end{bmatrix},$$

and the fundamental matrix is obtained according to (3.11) as

$$\Gamma(x) = \begin{bmatrix} e^{ln\ sin\ x} & 0\\ 0 & e^{-ln\ cos\ x}\end{bmatrix} = \begin{bmatrix} sin\ x & 0\\ 0 & 1/cos\ x\end{bmatrix},$$

with

$$\Gamma^{-1}(x) = \begin{bmatrix} 1/sin\ x & 0 \\ 0 & cos\ x \end{bmatrix}.$$

Using (3.17), we obtain

$$H(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} C_1 sin\ x + x \\ C_2/cos\ x + e^x\,(cos\ x + sin\ x)/2 \end{bmatrix}. \tag{3.18}$$

Setting $C_1 = C_2 = 0$ in (3.18), we have

$$\begin{aligned} h_1(x) &= x, \\ h_2(x) &= e^x\,(cos\ x + sin\ x)/2. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.1) gives

$$\begin{aligned} u &= U(y), \quad y = t + x^2/2, \\ v &= V(z), \quad z = t + e^x sin\ x/2, \end{aligned} \tag{3.20}$$

and into (3.3) results in

$$\begin{aligned} r_1(x) &= x^2 cos\ x, \\ r_2(x) &= e^{2x} sin\ x\,(sin\ 2x + 1)/4, \end{aligned}$$

where we have taken $a_i \equiv 1, i = 1, 2$. Hence for arbitrary functions $H_1(u, \bar{u}, v, \bar{v})$ and $H_2(u, \bar{u}, v, \bar{v})$, the nonlinear ADR system

$$\begin{aligned} sin\ x\ u_t &= cos\ x\ u_{xx} + sin\ x\ u_x + x^2 cos\ x\ H_1(u, \bar{u}, v, \bar{v}), \\ e^x v_t &= sin\ x\ v_{xx} + cos\ x\ v_x + \frac{e^{2x} sin\ x\,(sin\ 2x + 1)}{4} H_2(u, \bar{u}, v, \bar{v}), \end{aligned}$$

admits the generalized traveling-wave solutions (3.20), where $U(z)$ and $V(z)$ are determined by the system of ODEs

$$\begin{aligned} U''_{yy} + H_1(U, \bar{U}, V, \bar{V}) = 0,\ \bar{U} = U(Y - \tau), \\ V''_{zz} + H_2(U, \bar{U}, V, \bar{V}) = 0,\ \bar{V} = V(Z - \tau). \end{aligned} \tag{3.21}$$

3.1.2 Nondegenerate case

For $b_1 \neq 0$ and $b_2 \neq 0$, the substitution

$$\begin{aligned} h_1 &= \frac{1}{b_1}\frac{\omega'}{\omega}, \\ h_2 &= \frac{1}{b_2}\frac{\xi'}{\xi} \end{aligned} \tag{3.22}$$

into (3.7) gives

$$\begin{aligned} &\frac{p_1(x)}{b_1}\left(\frac{\omega''}{\omega} - \frac{(\omega')^2}{\omega^2}\right) + b_1 p_1(x)\left(\frac{1}{b_1}\frac{\omega'_x}{\omega}\right)^2 + \frac{q_1(x)}{b_1}\frac{\omega'}{\omega} - s_1(x) = 0, \\ &\frac{p_2(x)}{b_2}\left(\frac{\xi''}{\xi} - \frac{(\xi')^2}{\xi^2}\right) + b_2 p_2(x)\left(\frac{1}{b_2}\frac{\xi'_x}{\xi}\right)^2 + \frac{q_2(x)}{b_2}\frac{\xi'}{\xi} - s_2(x) = 0. \end{aligned} \tag{3.23}$$

Using vector-matrix notation, simplification of (3.23) yields

$$\Omega''(x) + M(x)\Omega'(x) - m(x)\Omega(x) = 0, \tag{3.24}$$

where

$$\Omega(x) = \begin{bmatrix} \omega(x) \\ \xi(x) \end{bmatrix}, M(x) = \begin{bmatrix} q_1(x)/p_1(x) & 0 \\ 0 & q_2(x)/p_2(x) \end{bmatrix} \ \&\ m(x) = \begin{bmatrix} b_1 s_1(x)/p_1(x) \\ b_2 s_2(x)/p_2(x) \end{bmatrix}.$$

The readers who are interested in exact solutions of second-order ODEs, such as (3.24), when associated with different values of $p(x), q(x)$ and $s(x)$ can consult [20, 19].

Example 3.5. An illustration will be given by considering $p_i(x) = b_i s_i(x)$ in (3.24), for $i = 1, 2$. Let $W = \begin{bmatrix} \omega' \\ \xi' \\ \omega \\ \xi \end{bmatrix}$, then $W' = \begin{bmatrix} \omega'' \\ \xi'' \\ \omega' \\ \xi' \end{bmatrix} = \begin{bmatrix} -q_1(x)/b_1 s_1(x)\omega' + \omega \\ -q_2(x)/b_2 s_2(x)\xi' + \xi \\ \omega' \\ \xi' \end{bmatrix}$. Thus,

$$W'(x) = \begin{bmatrix} -q_1(x)/b_1 s_1(x) & 0 & 1 & 0 \\ 0 & -q_2(x)/b_2 s_2(x) & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} W(x). \tag{3.25}$$

The matrix differential equation (3.24), which is a system of second ODEs, has been transformed to the matrix differential equation (3.25), which is a homogeneous system of first ODEs. The matrix in (3.25) is observed to be symmetric. For $x \in (0, \pi/2)$, let $q_1(x) = -b_1 s_1(x), q_2(x) = b_2 s_2(x), s_1(x) = sin\ x$ and $s_2(x) = cos\ x$ in (3.25). The fundamental matrix for the matrix differential equation is given by

$$\Gamma(x) = e^{\int_0^x M(\psi) d\psi},$$

where $M(\psi) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Therefore, the fundamental matrix for matrix differential equation (3.25) is

$$\Gamma(x) = e^{M(x)} \text{ where } M(x) = \begin{bmatrix} x & 0 & x & 0 \\ 0 & -x & 0 & x \\ x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \end{bmatrix}. \tag{3.26}$$

Notice that the matrix $M(x)$ commutes with itself over any two arguments. A fast route for computing the matrix exponential is through diagonalization. The matrix exponential will be computed by converting the matrix to Jordan form. Using Matlab 2015a, the eigenvalues for the matrix $M(x)$ are obtained as $\mp\frac{1}{2}\left(\sqrt{5}-1\right)x, \mp\frac{1}{2}\left(\sqrt{5}+1\right)x$ and the corresponding eigenvectors are $[0, -\frac{1}{2}\left(\sqrt{5}+1\right), 0, 1]^T$, $[0, \frac{1}{2}\left(\sqrt{5}-1\right), 0, 1]^T$, $[\frac{1}{2}\left(1-\sqrt{5}\right), 0, 1, 0]^T$ and $[\frac{1}{2}\left(\sqrt{5}+1\right), 0, 1, 0]^T$, respectively. Thus, the matrix of eigenvector which is the transition matrix is given by

$$G = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left(1-\sqrt{5}\right) & \frac{1}{2}\left(\sqrt{5}+1\right) \\ -\frac{1}{2}\left(\sqrt{5}+1\right) & \frac{1}{2}\left(\sqrt{5}-1\right) & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

To compute the inverse matrix G^{-1}, firstly the determinant of G is evaluated as

$$|G| = -5,$$

and

$$G^{-1} = \begin{bmatrix} 0 & -\frac{\sqrt{5}}{5} & 0 & \frac{1}{10}\sqrt{5}(\sqrt{5}-1) \\ 0 & \frac{\sqrt{5}}{5} & 0 & \frac{1}{10}\sqrt{5}(\sqrt{5}+1) \\ -\frac{\sqrt{5}}{5} & 0 & \frac{1}{2}(\frac{\sqrt{5}}{5}+1) & 0 \\ \frac{\sqrt{5}}{5} & 0 & \frac{1}{10}\sqrt{5}(\sqrt{5}-1) & 0 \end{bmatrix}.$$

Applying the identity $exp\left(G\ M(x)\ G^{-1}\right) = G\ exp(M(x))G^{-1}$ gives:

$$
\begin{aligned}
\Gamma(x) &= exp\begin{bmatrix} x & 0 & x & 0 \\ 0 & -x & 0 & x \\ x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \end{bmatrix} = exp\left(G \begin{bmatrix} -\frac{1}{2}\left(\sqrt{5}+1\right)x & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(\sqrt{5}-1\right)x & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\left(\sqrt{5}-1\right)x & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(\sqrt{5}+1\right)x \end{bmatrix} G^{-1} \right) \\
&= G \begin{bmatrix} e^{-\frac{1}{2}\left(\sqrt{5}+1\right)x} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}\left(\sqrt{5}-1\right)x} & 0 & 0 \\ 0 & 0 & e^{-\frac{1}{2}\left(\sqrt{5}-1\right)x} & 0 \\ 0 & 0 & 0 & e^{\frac{1}{2}\left(\sqrt{5}+1\right)x} \end{bmatrix} G^{-1}
\end{aligned}
$$

$$
= \begin{bmatrix} \frac{1}{10}\sqrt{5}(\sqrt{5}+1)e^{\frac{1}{2}(\sqrt{5}-1)x}(4e^x+1) & \frac{1}{10}\sqrt{5}(\sqrt{5}-1)e^{\frac{1}{2}(\sqrt{5}-1)x}(e^x-1) & 0 & 0 \\ \frac{1}{10}\sqrt{5}(\sqrt{5}+1)e^{\frac{1}{2}(\sqrt{5}-1)x}(e^x-1) & \frac{1}{10}\sqrt{5}(\sqrt{5}-1)e^{\frac{1}{2}(\sqrt{5}-1)x}(4e^x+1) & 0 & 0 \\ 0 & 0 & \frac{1}{10}\sqrt{5}(\sqrt{5}-1)e^{-\frac{1}{2}(\sqrt{5}+1)x}(4e^x+1) & \frac{1}{10}\sqrt{5}(\sqrt{5}+1)e^{x-\frac{1}{2}(\sqrt{5}+1)x}(e^x-1) \\ 0 & 0 & \frac{1}{10}(\sqrt{5}-1)\sqrt{5}e^{-\frac{1}{2}(\sqrt{5}+1)x}(e^x-1) & \frac{1}{10}\sqrt{5}(\sqrt{5}+1)e^{-\frac{1}{2}(\sqrt{5}+1)x}(4e^x+1) \end{bmatrix}.
$$

According to (3.13), the general solution for (3.25) takes the form

$$W(x) = \Gamma(x)C,$$

where $\Gamma(x)$ is the fundamental matrix and $C = [C_1, C_2, C_3, C_4]^T$. This gives the system of solutions for (3.24) as follows:

$$
\begin{aligned}
\omega(x) &= \frac{C_1}{10}\sqrt{5}(\sqrt{5}+1)e^{\frac{1}{2}(\sqrt{5}-1)x}(4e^x+1) + \frac{C_2}{10}\sqrt{5}(\sqrt{5}-1)e^{\frac{1}{2}(\sqrt{5}-1)x}(e^x-1), \\
\xi(x) &= \frac{C_3}{10}\sqrt{5}(\sqrt{5}+1)e^{\frac{1}{2}(\sqrt{5}-1)x}(e^x-1) + \frac{C_4}{10}\sqrt{5}(\sqrt{5}-1)e^{\frac{1}{2}(\sqrt{5}-1)x}(4e^x+1).
\end{aligned} \tag{3.27}
$$

Setting $C_2 = C_4 = 0$ in (3.27) leads to

$$
\begin{aligned}
h_1(x) &= \frac{1}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{4e^x}{4e^x+1}\right), \\
h_2(x) &= \frac{1}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{e^x}{e^x-1}\right).
\end{aligned} \tag{3.28}
$$

Apply (3.1) to (3.28) to get

$$
\begin{aligned}
u = U(y), \quad y &= t + \frac{1}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right)x + \ln(4e^x+1)\right), \\
v = V(z), \quad z &= t + \frac{1}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right)x + \ln(e^x-1)\right),
\end{aligned} \tag{3.29}
$$

and substitute (3.28) into (3.3) to obtain

$$
\begin{aligned}
r_1(x) &= -\frac{\sin\ x}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{4e^x}{4e^x+1}\right), \\
r_2(x) &= \frac{\cos\ x}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{e^x}{e^x-1}\right),
\end{aligned}
$$

where we have taken $a_i \equiv 1, i = 1, 2$. Hence for arbitrary functions $H_1(u, \bar{u}, v, \bar{v})$ and $H_2(u, \bar{u}, v, \bar{v})$, the nonlinear ADR system

$$
\begin{aligned}
sin\ x\ u_t &= b_1 \sin\ x\ u_{xx} - b_1 \sin\ x\ u_x - \frac{\sin\ x}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{4e^x}{4e^x+1}\right) H_1(u, \bar{u}, v, \bar{v}), \\
cos\ xv_t &= b_2 \cos\ x\ v_{xx} + b_2 \cos\ x\ v_x + \frac{\cos\ x}{b_1}\left(\frac{1}{2}\left(\sqrt{5}-1\right) + \frac{e^x}{e^x-1}\right) H_2(u, \bar{u}, v, \bar{v}),
\end{aligned}
$$

admits the generalized traveling-wave solutions (3.29), where $U(z)$ and $V(z)$ are determined by the system of ODEs (3.5).

Example 3.6. This illustration will be given by using Remark 3.3. The fundamental matrix is given as $\Gamma(x) = \begin{bmatrix} e^x & 0 \\ 0 & x+2 \end{bmatrix}$, while $s_1(x) = x$ and $s_2(x) = 1$.

The derivative and inverse of Γ are respectively obtained as $\Gamma'(x) = \begin{bmatrix} e^x & 0 \\ 0 & 1 \end{bmatrix}$ and $\Gamma^{-1}(x) = \begin{bmatrix} e^{-x} & 0 \\ 0 & 1/(x+2) \end{bmatrix}$. These yield the product

$$\Gamma'(x)\Gamma^{-1}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1/(x+2) \end{bmatrix} = M(x),$$

according to Remark 3.3. Therefore, it can be deduced that $p_1 = q_1 = q_2 = 1$ and $p_2(x) = x + 2$. Let $H(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}$ and $s(x) = \begin{bmatrix} s_1(x)/p_1(x) \\ s_2(x)/p_2(x) \end{bmatrix}$, apply (3.17) to obtain

$$\begin{aligned} H(x) &= \Gamma(x)\left(C_0 - \int \Gamma^{-1}(x)s(x)dx\right) \\ &= \begin{bmatrix} e^x\left(C_1 + (1-x)e^x\right) \\ (x+2)\left(C_2 - \ln(x+2)\right) \end{bmatrix}. \end{aligned} \tag{3.30}$$

Setting $C_1 = C_2 = 0$ in (3.30) gives

$$\begin{aligned} h_1(x) &= (1-x)e^{2x}, \\ h_2(x) &= -(x+2)\ln(x+2). \end{aligned} \tag{3.31}$$

Apply (3.1) to get

$$\begin{aligned} u &= U(y), \quad y = t - (x-2)e^x, \\ v &= V(z), \quad z = t - ((x+2)^2(\ln(x+2) - 1/2))/2, \end{aligned} \tag{3.32}$$

and (3.3) to obtain

$$\begin{aligned} r_1(x) &= (1-x)^2 e^{4x}, \\ r_2(x) &= (x+2)^3\left(\ln(x+2)\right)^2, \end{aligned}$$

where we have taken $a_i \equiv 1, i = 1,2$. Hence for arbitrary functions $H_1(u,\bar{u},v,\bar{v})$ and $H_2(u,\bar{u},v,\bar{v})$, the nonlinear ADR system

$$\begin{aligned} xu_t &= u_{xx} + u_x + (1-x)^2 e^{4x} H_1(u,\bar{u},v,\bar{v}), \\ v_t &= (x+2)v_{xx} + v_x + (x+2)^3\left(\ln(x+2)\right)^2 H_2(u,\bar{u},v,\bar{v}), \end{aligned}$$

admits the generalized traveling-wave solutions (3.32), where $U(z)$ and $V(z)$ are determined by the system of ODEs (3.21).

3.2 Finding exact solutions when h_1 and h_2 are given

Given h_1 and h_2, a pair of algebraic equations (3.7) is required to be solved simultaneously for the pair $p_i(x), q_i(x)$ and $r_i(x)$, to obtain the exact solutions of (1.3), where $i = 1,2$. Apart from h_1 and h_2 which are given in (3.7), two pairs of functions are assumed given from $p_i(x), q_i(x)$, and $r_i(x)$. Using (3.3) and (3.7), the pairs of unknown functions can be derived. Apply (3.1) to deduce the exact solution of (1.3).

Example 3.7. Given that $h_1(x) = \ln x, h_2(x) = sin\ x, p_1 = e^x, p_2 = e^{-x}, q_1 = x, q_2 = 1$ and b_i are arbitrary constants with $i = 1,2$. The task is to find $r_i(x), s_i(x)$ and consequently derive the exact solution of the system of nonlinear ADR equations with delay and variable coefficients.

(I) Degenerate case: $b_i = 0, i = 1,2$.

It can be obtained from (3.7) that

$$s_1(x) = e^x \ln x + 1,$$
$$s_2(x) = cos\ x \left(e^{-x} + tan\ x\right),$$

and from (3.3) that

$$\begin{aligned} r_1(x) &= e^x \left(\ln x\right)^2, \\ r_2(x) &= e^{-x} sin^2 x, \end{aligned} \tag{3.33}$$

where without loss of generality, $a_i \equiv 1$. Thus, the system of nonlinear ADR equations with delay and variable coefficients

$$\begin{aligned} \left(e^x \ln x + 1\right) u_t &= e^x u_{xx} + x u_x + e^x \left(\ln x\right) H_1(u, \bar{u}, v, \bar{v}), \\ e^{-x} sin^2 x\ v_t &= e^{-x} v_{xx} + v_x + e^{-x} sin^2 x H_2(u, \bar{u}, v, \bar{v}), \end{aligned}$$

for arbitrary functions $H_1(u, \bar{u}, v, \bar{v})$ and $H_2(u, \bar{u}, v, \bar{v})$, admit the generalized traveling-wave solutions

$$\begin{aligned} u &= U(y), \quad y = t + x\left(\ln x - 1\right), \\ v &= V(z), \quad z = t + sin\ x, \end{aligned} \tag{3.34}$$

where $U(z)$ and $V(z)$ are determined by the system of ODEs (3.21).

(II) Nondegenerate case: $b_i \neq 0,\ i = 1, 2.$

It can be obtained from (3.7) that

$$\begin{aligned} s_1(x) &= e^x \left(1/x + b_1 \left(\ln x\right)^2\right) + 1, \\ s_2(x) &= e^{-x} \left(\cos\ x + b_2 \sin^2 x\right) + \sin\ x. \end{aligned} \tag{3.35}$$

and from (1.3) that

$$\begin{aligned} \left[e^x \left(1/x + b_1 \left(\ln x\right)^2\right) + 1\right] u_t &= e^x u_{xx} + x u_x + e^x \left(\ln x\right) H_1(u, \bar{u}, v, \bar{v}), \\ \left[e^{-x} \left(\cos\ x + b_2 \sin^2 x\right) + \sin\ x\right] v_t &= e^{-x} v_{xx} + v_x + e^{-x} \sin^2 x H_2(u, \bar{u}, v, \bar{v}). \end{aligned} \tag{3.36}$$

Hence, for arbitrary functions $H_1(u, \bar{u}, v, \bar{v})$ and $H_2(u, \bar{u}, v, \bar{v})$, the system of nonlinear ADR equations with delay and variable coefficients (3.36) are solved by (3.34), where $U(z)$ and $V(z)$ are determined by the system of ODEs (3.5).

4 Modification for exact solutions of more sophisticated ADR systems of equations

The results which have been presented are applicable to any form of ADR systems of equations with delay and variable coefficients. The required set of modifications for application of the results to obtain the exact solutions of more sophisticated ADR systems of equations are presented in this section.

4.1 Nonlinear multifaceted PDEs with delay and variable coefficients

Let $u = u(x,t), v = v(x,t)$, $\bar{u} = u(x, t-\tau)$ and $\bar{v} = v(x, t-\tau)$, where τ denotes the delay in time. A nonlinear ADR system of the form

$$\begin{aligned} s_1(x) u_t &= p_1(x) u_{xx} + q_1(x) u_x + r_1(x) h_1(x) H_1\left(h_1, u, \bar{u}, v, \bar{v}, u_x/h_1\right), \\ s_2(x) v_t &= p_2(x) v_{xx} + q_2(x) v_x + r_2(x) h_2(x) H_2\left(h_2, u, \bar{u}, v, \bar{v}, v_x/h_2\right), \end{aligned} \tag{4.1}$$

where each of the arbitrary functions $H_1\left(h_1, u, \bar{u}, v, \bar{v}, \theta_1\right)$ and $H_2\left(h_2, u, \bar{u}, v, \bar{v}, \theta_2\right)$ take six arguments, which include the derivatives u_x and v_x. The exact solutions for the system (4.1) are given by

$$\begin{aligned} u &= U(y), \quad y = t + \int h_1(x) dx, \\ v &= V(z), \quad z = t + \int h_2(x) dx, \end{aligned} \tag{4.2}$$

where $U(z)$ and $V(z)$ are determined by the system

$$\begin{aligned} U''_{yy} - b_1 U'_y + a_1 H_1(h_1, U, \bar{U}, V, \bar{V}, U'_y) = 0, \\ V''_{zz} - b_2 V'_z + a_2 H_2(h_2, U, \bar{U}, V, \bar{V}, V'_z) = 0, \end{aligned}$$

with $\bar{U} = U(Y - \tau)$ and $\bar{V} = V(Z - \tau)$. This is straight forward by substituting (4.2) into (4.1) and taking (3.3) and (3.4) into consideration.

4.2 Nonlinear PDEs without delay

Sometimes consideration is given to systems of equations that take place in the media with local equilibrium where reference is not made to inertial properties. Such systems of equations can be described by $H(u, v)$, which is a special case of $H(u, \bar{u}, v, \bar{v})$ where the time delay $\tau = 0$. The response of such systems of equations to action at any time t is spontaneous.

Consider

$$\begin{aligned} s_1(x)u_t = p_1(x)u_{xx} + q_1(x)u_x + r_1(x)H_1(u, v), \\ s_2(x)v_t = p_2(x)v_{xx} + q_2(x)v_x + r_2(x)H_2(u, v), \end{aligned} \tag{4.3}$$

where $H_1(u, v)$ and $H_2(u, v)$ are arbitrary functions with $u = u(x, t)$ and $v = v(x, t)$. The exact solutions for the system (4.3)are given by

$$\begin{aligned} u = U(y), \quad & y = t + \int h_1(x)dx, \\ v = V(z), \quad & z = t + \int h_2(x)dx, \end{aligned} \tag{4.4}$$

where $U(z)$ and $V(z)$ are determined by the system

$$\begin{aligned} U''_{yy} - b_1 U'_y + a_1 H_1(U, V) = 0, \\ V''_{zz} - b_2 V'_z + a_2 H_2(U, V) = 0, \end{aligned}$$

with $U = U(Y)$ and $V = V(Z)$. This is straight forward by substituting (4.4) into (4.3) and taking (3.3) and (3.4) into consideration.

Conclusion: A system of ADR equations allows any number of equations to be posed simultaneously to represent a set of interacting processes. The presence of delay in nonlinear PDEs generally makes them more difficult to solve and analyze. This study has considered nonlinear PDEs of ADR equations with delay and variable coefficients. Precepts for reducing such systems of equations to simpler systems of delayed ODEs by using modified methods of functional constraints are presented. Illustrations are given for clarification on how to apply the precepts. The required modifications to obtain the exact solutions of more sophisticated ADR systems of equations are elucidated.

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