Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1557–1575
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.28570.3933

Existence of solutions to a periodic parabolic problem with Orlicz growth and L^1 data

Erriahi Elidrissi Ghita[a], Azroul Elhoussine[a,*], Lamrani Alaoui Abdelilah[b]

[a] *Sidi Mohamed Ben Abdellah University Faculty of Sciences Dhar el Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco*

[b] *CRMEF Fès, Morocco*

(Communicated by Abdolrahman Razani)

Abstract

In Â this paper, we are concerned with the Â existence Â of renormalized solution for a Â nonlinear periodic Â parabolic problem associated to the equation

$$\frac{\partial u}{\partial t} - A(u) + g(x,t,u,\nabla u) = f \in L^1 , \tag{0.1}$$

where $A(u)$ is the m-Laplacian operator defined on $W_0^{1,x} L_M(\mathcal{Q})$.

Keywords: Periodic solutions, Orlicz space, Renormalized solutions, L^1 data, Weak solutions
2020 MSC: 35B10, 35K55

1 Introduction

In the recent years, a great interest has been dedicated to mathematical studies of partial differential equations PDE which always has the benefit of participating in the development of several scientific fields: engineering, physics, chemistry, biology, biomedical, disease propagation etc. These fields offer new and exciting branches of research (see [27],[26],[29],[25]).

Periodic behavior of solutions of parabolic PDE intervenes in the mathematical modeling of a large variety of phenomena. The literature of time periodic solutions of ordinary differential equations have a great development. Most of the studies are devoted to the existence of global solutions, their periodic behavior and regularity properties. The periodicity of solutions for parabolic boundary value problems has attracted great interests of scientists, and a lots of results have been reported under either Dirichlet or Neumann boundary conditions.

These problems have many considerations, In the biomedical field, an example of the spread of early tumors along linear or tubular structures is mathematically modeled by a periodic partial differential equation (For more details see [24]). Further, in physics we present two mathematical models, The first one is based on the equation of thermal conduction with a variable temperature. In the second model, we consider the internal energy as a variable in the

*Corresponding author
Email addresses: ghita.idrissi.s6@gmail.com (Erriahi Elidrissi Ghita), elhoussine.azroul@gmail.com (Azroul Elhoussine), lamranii@gmail.com (Lamrani Alaoui Abdelilah)

Received: October 2022 *Accepted:* December 2022

problem. We obtain a nonlinear periodic parabolic equation with respect to the gradient of this energy (for more details see [21]).

Periodic solution of parabolic problem were studied by many authors in the setting of classical Sobolev space $L^p(0,T,W_0^{1,p})$. A. Deuel and Hess [7] has proved the existence of periodic solutions of the problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}+A(u)+F(u,\nabla u)=0 \text{ in } \Omega\times\mathbb{R}^+ \\ u=0 \quad \text{ on } \partial\Omega\times\mathbb{R}^+, \\ u(0)=u(T) \quad \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and open subset of $\mathbb{R}^N$, $N\geq 1$. For the usual Leray Lions operator, under the presence of well-ordered lower and upper-solutions for (P), those result has improved to the natural growth of $|\nabla u|$ in [10].

Alaa, N. and Iguernane in [3] has proved the existence of weak periodic solutions for some quasilinear parabolic equations with data measures and critical growth nonlinearity with respect to the gradient, these results has generalized to the p-Laplacian, measure data and natural growth of $|\nabla u|$ in [12] . Boldrini and Crema [5] was considered the case where $A(u)$ is the p-Laplacian operator, with $p\geq 2$. This results has generalized to the singular case $1<p<2$ in [11].

When trying to relax this restriction on a, we are led to replace the space $L^p(0,T,W_0^{1,p})$ with an inhomogeneous Sobolev space $W_0^{1,x}L_M(Q)$ built from an Orlicz space L_M instead of L^p where the N-function M is related to the actual growth of a. Many works has been done in this case, see Donaldson [8] where an existence result for equation (0.1) with $g\equiv 0$ and $u(0)=u_0$) was proved, Robert [28] for $g\equiv g(x,t,u)$ when A is monotone, $t^2\ll M(t)$ and $\overline{M}$ satisfies the Δ_2 condition. See also Elmahi [13] for $g=g(x,t,u,\nabla u)$ when M satisfies a Δ' condition and $M(t)\ll t^{N/(N-1)}$ and finally Elmahi-Meskine [14] for the case where f belongs to $L^1(Q)$.

The main purpose and novelty of this paper is to prove an existence of solution for the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}-A(u)+g(x,t,u,\nabla u)=f(x,t) & \text{in } Q \\ u=0 & \text{On } \partial\Omega\times(0,T) \\ u(x,0)=u(x,T) & \text{in } \Omega \end{cases} \tag{1.2}$$

where $A(u)$ is the m-Laplacian operator in the setting of Orlicz spaces, $f\in L^1$ and the initial data is replaced by the periodicity condition $u(0)=u(T)$, by using the concept of renormalized solution and a classical approximating method.

The paper is organized as follows. In Section 2, we recall some preliminaries concerning Orlicz-Sobolev spaces and some compactness results (see [20],[19]). Section 3 is devoted to the statement of our assumptions and the main result.

In the fourth section we prove the existence theorem by following these steps:

- We give the a priori estimates,
- We prove the almost everywhere convergence of the gradients,
- We demonstrate the modular convergence of the truncation,
- We pass to the limit.

2 Preliminaries

.

2.1 Orlicz-Sobolev Spaces-Notations and Properties

1. let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, i.e. continuous, convex, with $M(t)>0$ for $t>0, M(t)/t\to 0$ as $t\to 0$ and $M(t)/t\to\infty$ as $t\to\infty$. Equivalently, M admits the representation: $M(t)=\displaystyle\int_0^t m(\tau)d\tau$ where $m:\mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing, right continuous, with $m(0)=0$, $m(t)>0$ for $t>0$ and $m(t)\to\infty$ as $t\to\infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t)=\displaystyle\int_0^t \overline{m}(\tau)d\tau$ where $\overline{m}:\mathbb{R}^+\to\mathbb{R}^+$ is given by $m(t)=\sup\{s:m(s)\leq t\}$. The N-function M is said to satisfy a Δ_2 condition if, for some $k>0$:

$$M(2t)\leq kM(t)\quad \forall t\geq 0.$$

When this inequality holds only for $t\geq t_0>0$, M is said to satisfy the Δ_2-condition near infinity.

2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$, for some $\lambda > 0$).
   $L_M(\Omega)$ is a Banach space under the norm:
$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$$
   and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
   The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.
   The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition (near infinity only if Ω has finite measure).
3. We now turn to the Orlicz-Sobolev spaces. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:
$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$
   Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspace of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
   The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
4. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$
$$\int_\Omega M\left(\left(D^\alpha u_n - D^\alpha u\right)/\lambda\right)dx \to 0 \text{ for all } |\alpha| \leq 1.$$
   This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Note that, if $u_n \to u$ in $L_M(\Omega)$ for the modular convergence and $v_n \to v$ in $L_M(\Omega)$ for the modular convergence, we have
$$\int_\Omega u_n v_n dx \to \int_\Omega uv dx \quad \text{ as } n \to \infty.$$

2.2 The homogeneous Orlicz-Sobolev

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times]0, T[$. Let M be an N-function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The homogeneous Orlicz-Sobolev spaces of order 1 are defined as follows
$$W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q), \forall |\alpha| \leq 1\}$$
and
$$W^{1,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q), \forall |\alpha| \leq 1\}.$$
The latter space is a subspace of the former. Both are Banach spaces under the norm
$$\|u\| = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q}.$$

The space $W_0^{1,x}L_M(Q)$ is defined as the (norm) closure in $W^{1,x}L_M(Q)$ of $\mathcal{D}(Q)$ and we have .
$$W_0^{1,x}L_M(Q) = \overline{\mathcal{D}(Q)}^{\sigma\left(\Pi L_M, \Pi L_{\overline{M}}\right)}$$

.

Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M$.

Poincare's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has

$$\sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{M,Q}$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_M(Q)$ and it is shown that

$$W^{-1,x}L_M(Q) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha \ : \ f_\alpha \in L_M(Q) \right\}.$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{M,Q}$$

where the inf is taken over all possible decomposition $f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_M(Q)$. The space F_0 is then given by

$F_0 = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_M(Q) \right\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q)$.

2.3 Compactness results

Theorem 2.3.1. Let B be a Banach space and let $T > 0$ be a fixed real number. If $F \subset L^1(0,T;B)$ is such that

$$\left\{ \int_{t_1}^{t_2} f(t)dt \right\}_f \ \textit{is relatively compact in } B, \quad \textit{for all } 0 < t_1 < t_2 < T. \tag{2.1}$$

$$\|\tau_h f - f\|_{L^1(0,T;B)} \to 0 \text{ uniformly } \textit{in } f \in F, \text{ when } h \to 0. \tag{2.2}$$

Then F is relatively compact in $L^1(0,T;B)$.

Next, we have the following lemma, which it can be seen as a "Orlicz" version of the well known interpolation inequality related to the space $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$.

Lemma 2.3.1. (see [15]) Let M be an N-function. Let Y be a Banach space such that the following continuous embedding holds: $L^1(\Omega) \subset Y$. Then, for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$, with $|\nabla u|/\lambda \in \mathcal{L}_M(Q)$,

$$\|u\|_{L^2(Q)} \leq \varepsilon\lambda \left(\int_Q M\left(\frac{|\nabla u|}{\lambda}\right) dxdt + T \right) + C_\varepsilon \|u\|_{L^2(0,T;Y)}. \tag{2.3}$$

We have also the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 2.3.2. (see [14]) Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous embedding. If F is bounded in $W_0^{1,x}L_M(Q)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q)$.

Theorem 2.3.2. (see [14]) Let M be an N-function. If F is bounded in $W_0^{1,x}L_M(Q)$ and $\left\{ \dfrac{\partial f}{\partial t} : f \in F \right\}$ is bounded in $W^{-1,x}L_M(Q)$ then F is relatively compact in $L^1(Q)$.

Theorem 2.3.3. (see [14]) If $u \in W^{1,x}L_M(Q) \cap L^1(Q)$ (resp. $W_0^{1,x}L_M(Q) \cap L^1(Q)\Big)$ and $\partial u/\partial t \in W^{-1,x}L_M(Q) + L^1(Q)$ then there exists a sequence (v_j) $in\mathcal{D}(\overline{Q})$ such that

$$v_j \to u \text{ in } W^{1,x}L_M(Q) \quad \text{ and } \quad \frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_M(Q) + L^1(Q)$$

for the modular convergence.

Corollary 2.3.1. Let M be an N-function and u_n be a sequence of $W^{1,x}L_M(Q)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,x}L_M(Q) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}})$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \quad \text{in} \quad \mathcal{D}'(Q)$$

with h_n bounded in $W^{-1,x}L_{\overline{M}}(Q)$ and k_n bounded in the space $\mathcal{M}(Q)$ of measures on Q then :

1. $u_n \to u$ strongly in $L^1_{Loc}(Q)$
2. If further $u_n \in W_0^{1,x}L_M(Q)$ then $u_n \to u$ strongly in $L^1(Q)$

Corollary 2.3.2. Let $u \in L_M(Q)$, we define for all $\mu > 0$ and all $(x,t) \in Q$ a time mollification function u_μ such that

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s) \exp(\mu(s-t))dx$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of u. We have

-If $u_n \to u$ in $L_M(Q)$ strongly (resp. for the modular convergence) then $(u_n)_\mu \to u_\mu$ in $L_M(Q)$ strongly (resp. for the modular convergence).

-If $u_n \to u$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence) then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_M(Q)$ strongly (resp. for the modular convergence).

We will use the following technical Lemmas.

Lemma 2.3.3. (see [14]) Let Ω be a bounded open subset of $\mathbb{R}^N$. Then,

$$\left\{u \in W_0^{1,x}L_{\overline{M}}(Q) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)\right\} \subset C\left([0,T], L^1(\Omega)\right).$$

Lemma 2.3.4. Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \le p, q \le \infty$, let $W = \{u \in L^p([0,T]; X_0) \mid \dot{u} \in L^q([0,T]; X_1)\}$.
(i) If $p < \infty$ then the embedding of W into $L^p([0,T]; X)$ is compact.
(ii) If $p = \infty$ and $q > 1$ then the embedding of W into $C([0,T]; X)$ is compact.

3 Assumptions and statement of main results

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \ge 2)$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Let M be an N-function. Consider the $m-$Laplacien operator

$$\Delta_m u = \operatorname{div}\left(\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\right).$$

We set $a(x,t,\xi) = \dfrac{m(|\nabla\xi|)}{|\nabla\xi|}\nabla\xi$ with $\xi \in \mathbb{R}^{\mathbb{N}}$, where $a : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $(x,t) \in \Omega \times [0,T]$ and all $\xi \neq \xi^* \in \mathbb{R}^N$:

$$|a(x,t,\xi)| \le \overline{M}^{-1} M(\delta|\xi|) \; ; \tag{3.1}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)](\xi - \xi^*) > 0 \text{ if } \xi \neq \xi^* \; ; \tag{3.2}$$

$$a(x,t,\xi)\xi \geq \alpha M(|\xi|) \; ; \tag{3.3}$$

where $\delta, \alpha > 0$. Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying for a.e. $(x,t) \in \Omega \times (0,T)$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \quad \leq \quad b(|s|)\,(c(x,t) + M(|\xi|)) \; ; \tag{3.4}$$

$$g(x,t,s,\xi)s \quad \geq \quad 0 \, , \tag{3.5}$$

where $c(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function. Furthermore let

$$f \quad \in \quad L^1(Q). \tag{3.6}$$

Throughout this paper $< .,. >$ means for either the pairing between $W_0^{1,x}L_M(Q) \cap L^\infty(Q)$ and $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ or between $W_0^{1,x}L_M(Q)$ and $W^{-1,x}L_{\overline{M}}(Q)$. Consider, then, the following parabolic initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_m u + g(x,t,u,\nabla u) = f & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u(x,T) & \text{in } \Omega \end{cases} \tag{3.7}$$

Let us now precise in which sense the problem will be solved.

3.1 Definition of a renormalized solution

The definition of a renormalized solution for problem (3.7) can be stated as follows.

Definition 1. A measurable function u defined on Q is a renormalized solution of Problem (3.7) if

$$T_k(u) \in W_0^{1,x}L_M\,(Q) \quad \forall k \geqslant 0 \text{ and } u \in L^\infty\left(0,T; L^1(\Omega)\right) \tag{3.8}$$

$$\int_{\{(x,t)\in Q; h \leq |u(x,t)| \leq h+1\}} a(x,t,\nabla u)\nabla u dx dt \to 0 \text{ as } h \to +\infty, \tag{3.9}$$

and for every function S in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support and $S(0) = 0$, we have

$$\begin{cases} \dfrac{\partial S(u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,\nabla u)) + S''(u)a(x,t,\nabla u)\nabla u + g(x,t,u,\nabla u)S'(u) = fS'(u) & \text{in } D'\,(Q) \\ S(u(x,0)) = S(u(x,T)) & \text{in } \Omega. \end{cases} \tag{3.10}$$

The following remarks are concerned with a few comments on definition (1)

Remark 3.1.1. The first equation of (3.10) is formally obtained through pointwise multiplication of (3.7) by $S'(u)$. Note that due to (3.8) each term in the first equation of (3.10) has a meaning in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$.

3.2 Statement of the main result

The main result of the paper is the following existence theorem :

Theorem 3.2.1. Assume that (3.1)-(3.6) hold true, then the problem (3.7) admits at least one renormalized solution $u \in C(0,T,L^1(\Omega))$ satisfying $u(x,0) = u(x,T)$ a.e $x \in \Omega$.

4 Proof of the main result

Proof. We divide the proof in four steps.
step 1 : A priori e stimates
Let (f_n) be a sequence of smooth functions such that $f_n \to f$ in $L^1(Q)$ and let $(u_n(0), u_n(T))$ be a sequences in $L^2(\Omega)$ such that $u_n(0) \to u(0)$ and $u_n(T) \to u(T)$ in $L^1(\Omega)$. Consider the sequence of approximate problems:

$$\begin{cases} \partial u_n/\partial t - \operatorname{div} a(x,t,\nabla u_n) + g_n(x,t,u_n,\nabla u_n) = f_n. \\ u_n(x,0) = u_n(x,T) \\ u_n \in W_0^{1,x}L_M(Q) \cap C([0,T], L^2(\Omega)) \end{cases} \tag{4.1}$$

where $g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))$ and for $k > 0$, T_k means truncation operator such that

$$T_k(s) = \max\{-k, \min(k,s)\} \ .$$

Note that $g_n(x,t,s,\xi)s \geq 0$, $|g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$. The prove of the existence of solution for problem (4.1) is in progress. Although we can see [23], since we follow almost the same steps. Now we use in (4.1) the test function $T_k(u_n)$, $k > 0$ we get

$$\int_\Omega \Big[\frac{\partial S_k(u_n)}{\partial t}\Big]_0^T dxdt + \int_Q a(x,t,\nabla T_k(u_n))\nabla T_k(u_n)dxdt + \int_Q g(x,t,T_k(u_n),\nabla T_k(u_n))T_k(u_n) = \int_Q f_n T_k(u_n),$$

where $S_k(s) = \displaystyle\int_0^s T_k(r)dr$, then thanks to (3.5) and periodicity condition we have

$$\int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt \leq C_1 k, \tag{4.2}$$

where here C_1 denote positive constants not depending on n and k. On the other hand, thanks to Lemma 5.7 of [17], there exists two positive constants δ, λ such that

$$\int_Q M(v)dxdt \leq \delta \int_Q M(\lambda|\nabla v|)dxdt \quad \text{for all} \quad v \in W_0^{1,x}L_M(Q) \tag{4.3}$$

Taking $v = T_k\left(u_n\right)/\lambda$ in (4.3) and using (4.2) with (3.3), give

$$\alpha \int_Q M\left(\frac{T_k\left(u_n\right)}{\lambda}\right)dxdt \leq C_2 k \ ,$$

which implies that

$$\operatorname{meas}\left\{(x,t) \in Q : |u_n| > k\right\} \leq \frac{C_3 k}{M(k/\lambda)}.$$

So that

$$\lim_{k\to\infty}\left(\operatorname{meas}\left\{(x,t) \in Q : |u_n| > k\right\}\right) = 0 \quad \text{uniformly with respect to n.} \tag{4.4}$$

Consider now for $\theta, \varepsilon > 0$ a function $\rho_\theta^\varepsilon \in C^1(\mathbb{R})$ such that

$$\begin{array}{ll} \rho_\theta^\varepsilon(s) = 0 & \text{if } |s| \leq \theta \ ; \\ \rho_\theta^\varepsilon(s) = \operatorname{sign}(s) & \text{if } |s| \geq \theta + \varepsilon \ ; \\ \left(\rho_\theta^\varepsilon\right)'(s) \geq 0 & \forall s \in \mathbb{R} \ ; \end{array}$$

then, by using $\rho_\theta^\varepsilon\left(u_n\right)$ as a test function in (4.1) and following [28] and periodicity condition, we can see that

$$\int_{\{|u_n|>\theta\}} \left|g_n\left(x,t,u_n,\nabla u_n\right)\right| dxdt \leq \int_{\{|u_n|>\theta\}} |f_n|\, dxdt \tag{4.5}$$

and so by letting $\theta \to 0$ and using Fatou's lemma, we deduce that $g_n\left(x,t,u_n,\nabla u_n\right)$ is a bounded sequence in $L^1(Q)$. Moreover, we have from (4.2) that $T_k\left(u_n\right)$ is bounded in $W_0^{1,x}L_M(Q)$, for every $k > 0$.

Take a $C^2(\mathbb{R})$, and nondecreasing function ζ_k such that $\zeta_k(s)=s$ for $|s|\le k/2$ and $\zeta_k(s)=k\operatorname{sign}(s)$, for $|s|\ge k$. Multiplying the approximating equation by $\zeta_k'(u_n)$, we get

$$\frac{\partial}{\partial t}\left(\zeta_k\left(u_n\right)\right)-\operatorname{div}\left(a\left(x,t,\nabla u_n\right)\zeta_k'\left(u_n\right)\right)+a\left(x,t,\nabla u_n\right)\zeta_k''\left(u_n\right) + g_n\left(x,t,u_n,\nabla u_n\right)\zeta_k'\left(u_n\right)=f_n\zeta_k'\left(u_n\right)$$

in the sense of distributions. This implies, thanks to (4.2) and the fact that ζ_k' has compact support, that $\zeta_k\left(u_n\right)$ is bounded in $W_0^{1,x}L_M(Q)$ while its time derivative $\dfrac{\partial}{\partial t}\left(\zeta_k\left(u_n\right)\right)$ is bounded in $W^{-1,x}L_{\overline{M}}(Q)+L^1(Q)$, hence corollary (2.3.1) allows us to conclude that $\zeta_k\left(u_n\right)$ is compact in $L^1(Q)$. Therefore, following [6], we can see that there exists a measurable function u in $L^\infty\left(0,T;L^1(\Omega)\right)$ such that for every $k>0$ and a sub sequence, not relabeled,

$$\begin{gathered}T_k\left(u_n\right)\rightarrow T_k(u) \text{ weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma\left(\Pi L_M,\Pi E_{\overline{M}}\right),\\ \text{strongly in } L^1(Q) \text{ and a.e. in } Q.\end{gathered}\tag{4.6}$$

To prove that $a\left(x,t,\nabla T_k\left(u_n\right)\right)$ is a bounded sequence in $\left(L_{\overline{M}}(Q)\right)^N$. Let $\varphi\in\left(E_M(Q)\right)^N$ with $\|\varphi\|_{M,Q}=1$. In view of (3.2), we have

$$\int_Q\left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\varphi\right)\right]\left[\nabla T_k\left(u_n\right)-\varphi\right]dxdt\ge 0$$

which gives

$$\int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\varphi dxdt\le\int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt - \int_Q a\left(x,t,\varphi\right)\left[\nabla T_k\left(u_n\right)-\varphi\right]dxdt.$$

On the other hand, using (3.1), we see that

$$\overline{M}(|a\left(x,t,\varphi\right)|)\le M(\delta|\varphi|)$$

and hence $a\left(x,t,\varphi\right)$ is bounded in $\left(L_{\overline{M}}(Q)\right)^N$ and $T_k\left(u_n\right)$ is bounded in $W_0^{1,x}L_M(Q)$ then by using Holder inequality we get

$$\left|\int_Q a\left(x,t,\varphi\right)\left[\nabla T_k\left(u_n\right)-\varphi\right]dxdt\right|\le C$$

and so, by using (4.2) and the fact that $\|\varphi\|_{L_{\overline{M}}}=1$ we can deduce that $a\left(x,t,\nabla T_k\left(u_n\right)\right)$ is a bounded sequence in $\left(L_{\overline{M}}(Q)\right)^N$. Thus, up to a subsequences

$$a\left(x,t,\nabla T_k\left(u_n\right)\right)\rightarrow h_k \text{ in}\left(L_{\overline{M}}(Q)\right)^N \text{ for}\sigma\left(\Pi L_{\overline{M}},\Pi E_M\right)\tag{4.7}$$

for some $h_k\in\left(L_{\overline{M}}(Q)\right)^N$. Now we show (3.9). Using $\mathcal{V}_h=T_{h+1}(u_n)-T_h(u_n)$ as a test function in (4.1), then we have

$$\int_\Omega B_h(u_n(T))-B_h(u_n(0)dx+\int_Q a(x,t,\nabla u_n)\nabla\mathcal{V}_h(u_n)dxdt+\int_Q g_n(x,t,u_n,\nabla u_n)\mathcal{V}_h(u_n)dxdt=\int_Q f_n\mathcal{V}_h(u_n)dxdt$$

with $B_h(s)=\displaystyle\int_0^s\frac{\partial u_n}{\partial t}\mathcal{V}_h(\sigma)d\sigma$. By using the periodicity condition and the fact that $\displaystyle\int_Q g_n(x,t,u_n,\nabla u_n)\mathcal{V}_h(u_n)dxdt\ge 0$ we get

$$\int_Q a(x,t,\nabla u_n)\nabla\mathcal{V}_h(u_n)dxdt\le\int_Q f_n\mathcal{V}_h(u_n)dxdt.$$

Let remark that $\nabla\mathcal{V}_h(u_n)=\mathcal{V}_h(u_n)\chi_{\{h\le|u_n|\le h+1\}}$, then we can write that

$$\int_{\{h\le|u_n|\le h+1\}}a(x,t,\nabla u_n)\mathcal{V}_h(u_n)dxdt\le\int_Q f_n\mathcal{V}_h(u_n)dxdt.\tag{4.8}$$

From (3.6) and using Lebesgue theorem we see that

$$\lim_{h\rightarrow+\infty}\lim_{n\rightarrow+\infty}\int_Q f_n\mathcal{V}_h(u_n)dxdt=0.$$

Finally passing to the limit in (4.8) as $n \to +\infty$ and $h \to +\infty$ we deduce that

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \int_{\{h\le|u_n|\le h+1\}} a(x,t,\nabla u_n)\mathcal{V}_h(u_n)dxdt = 0 .$$

step 2 : Almost everywhere convergence of the gradients.

Fix $k > 0$ and let $\varphi(s) = se^{\delta s^2}, \delta > 0$. It is well known that when $\delta \ge \left(\frac{b(k)}{2\alpha}\right)^2$ one has

$$\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \ge \frac{1}{2} \quad \text{for all } s \in \mathbb{R} . \tag{4.9}$$

We have $T_k(u) \in C([0,T], L^1(\Omega))$ for all $k \ge 0$, then $T_k(u)(T) \in L^1(\Omega)$. Let $\mu_j \in \mathcal{D}$ and $(z_\nu)_\nu$ be two sequences such that

$$\mu_j \to u \text{ in } W_0^{1,x}L_M(Q) \text{ for modular convergence} \tag{4.10}$$

$$z_\nu \in W_0^{1,x} \cap L_M(Q)\ L^\infty(Q)\ \ z_\nu \to T_k(u)(T) \ \text{ a.e in } \Omega \text{ as } \nu \to \infty \tag{4.11}$$

$$\lim_{\nu\to\infty} \frac{1}{\nu}\|z_\nu\|_{W_0^{1,x}L_M(Q)} = 0. \tag{4.12}$$

We denote by $T_k(\mu_j)_\nu$ the unique solution of this problem :

$$\begin{cases} \partial_t T_k(\mu_j)_\nu = \nu(T_k(\mu_j) - T_k(\mu_j)_\nu) \\ T_k(\mu_j)_\nu(0) = z_\nu \end{cases}$$

such that

$$\begin{cases} |T_k(\mu_j)_\nu| \le k \\ T_k(\mu_j)_\nu \to T_k(u)_\nu \quad \text{in} \quad W_0^{1,x}L_M(Q) \text{ for the modular convergence as } j \to \infty \, ; \\ T_k(u)_\nu \to T_k(u) \quad \text{in} \quad W_0^{1,x}L_M(Q) \quad \text{for the modular convergence as } \mu \to \infty \end{cases}$$

$T_k(u)_\nu$ is defined as follows

$$T_k(u)_\nu(t) = \int_0^t \nu \exp^{\nu(s-t)} T_k(u)(s)ds + z_\nu \exp^{-\nu t} . \tag{4.13}$$

Let now the function ρ_m defined on $\mathbb{R}$ by

$$\rho_m(s) = \begin{cases} 1 & \text{if } |s| \le m, \\ m+1-|s| & \text{if } m \le |s| \le m+1, \\ 0 & \text{if } |s| \ge m+1, \end{cases}$$

where $m > k$. Using $\omega^{i,\nu}_{j,n,m} = \varphi(T_k(u_n) - T_k(\mu_j)_\nu)\rho_m(u_n)$ as test function in (4.1) we get

$$\begin{aligned} < u'_n, \omega^{i,\nu}_{j,n,m}) > &+ \int_Q a(x,t,\nabla u_n)[\nabla T_k(u_n) - T_k(\mu_j)_\nu)]\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\rho_m(u_n)dxdt \\ &+ \int_Q a(x,t,\nabla u_n)\varphi(T_k(u_n) - T_k(\mu_j)_\nu)\rho'_m(u_n)\nabla u_n dxdt \\ &+ \int_Q g_n\left(x,t,u_n,\nabla u_n\right)\omega^{i,\nu}_{j,n,m}dxdt \\ &= \int_Q f_n \omega^{i,\nu}_{j,n,m}dxdt \end{aligned}$$

which implies, since $g_n(x,t,u_n,\nabla u_n)\,\omega^{i,\nu}_{j,n,m} \geq 0$ on $\{|u_n| > k\}$:

$$\begin{aligned} < u_n', \omega^{i,\nu}_{j,n,m} > &+ \int_Q a(x,t,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(\mu_j)_\nu]\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\rho_m(u_n)dxdt \\ &+ \int_Q a(x,t,\nabla u_n)\varphi(T_k(u_n) - T_k(\mu_j)_\nu)\rho_m'(u_n)\nabla u_n dxdt \\ &+ \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\,\omega^{i,\nu}_{j,n,m} dxdt \\ &\leq \int_Q f_n\omega^{i,\nu}_{j,n,m} dxdt \end{aligned} \tag{4.14}$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then j,ν,i,s and finally m. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,j),\dots$ to mean that the limits are made only on the specified parameters. First all, let us prove that

$$\int_Q f_n\omega^{i,\nu}_{j,n,m} = \varepsilon(n,j,\nu). \tag{4.15}$$

Proof we have $\varphi(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n) \to \varphi(T_k(u) - T_k(\mu_j)_\nu)\,\rho_m(u))$ weakly * in $L^\infty(Q)$ as $n \to \infty$, then by letting $j \to \infty$ we get $\varphi(T_k(u) - T_k(\mu_j)_\nu)\,\rho_m(u_n) \to \varphi(T_k(u) - T_k(u)_\nu)\,\rho_m(u)$ weakly in $L^\infty(Q)$ and finally $\varphi(T_k(u) - T_k(u)_\nu)\,\rho_m(u) \to 0$ weakly * in $L^\infty(Q)$ as $\nu \to \infty$.
□

On one hand, from (4.1) one deduces that $u_n \in W_0^{1,x}L_M(Q)$ and $\partial u_n/\partial t \in W^{-1,x}(Q)+L^1(Q)$ and then by theorem (2.3.3) there exists a smooth function $u_{n\sigma}$ such that, as $\sigma \to 0^+, u_{n\sigma} \to u_n$ in $W_0^{1,x}L_M(Q)$ and $\partial u_{na}/\partial t \to \partial u_n/\partial t$ in $W^{-1,x}(Q)+ L^1(Q)$ for the modular convergence, so that, $\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,\rho_m(u_{n\sigma}) \to \omega^{i,\nu}_{j,n,m}$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and weakly * in $L^\infty(Q)$. This implies

$$\begin{aligned} \left\langle u_n', \omega^{i,\nu}_{j,n,m}\right\rangle &= \lim_{\sigma\to 0^+}\int_Q u_{n\sigma}'\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,\rho_m(u_{n\sigma})\,dxdt \\ &= \lim_{\sigma\to 0^+}\int_Q \left[(R_m(u_{n\sigma}))'\right]\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,dxdt \end{aligned}$$

where $R_m(s) = \displaystyle\int_0^s \rho_m(\eta)d\eta$. Hence

$$\begin{aligned} \left\langle u_n', \omega^{i,\nu}_{j,n,m}\right\rangle &= \lim_{\sigma\to 0^+}\int_Q \left[(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))'\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu) + \int_Q (T_k(u_{n\sigma}))'\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\right] \\ &= \lim_{\sigma\to 0^+}\left\{\int_\Omega \left[(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\right]_0^T \right. \\ &\quad - \int_Q (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)'\,dxdt \\ &\quad \left. + \int_Q (T_k(u_{n\sigma})'\varphi((T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)]\right\} \\ &= \lim_{\sigma\to 0^+}\{I_1(\sigma) + I_2(\sigma) + I_3(\sigma)\}. \end{aligned}$$

Observe that for $|s| \leq k$ we have $R_m(s) = T_k(s) = s$ and for $|s| > k$ we have $|R_m(s)| \geq |T_k(s)|$ and, since both

$R_m(s)$ and $T_k(s)$ have the same sign of s, we start by $I_1(\sigma)$, we can observe that

$$\begin{aligned} I_1(\sigma) &= \int_{\{|u_{n\sigma}(T)|>k\}} \big[R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T)\big]\varphi(T_k(u_{n\sigma})(T) - T_k(\mu_j)_\nu(T))dx \\ &\quad - \int_{\{|u_{n\sigma}(0)|>k\}} \big[R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0)\big]\varphi(T_k(u_{n\sigma})(0) - T_k(\mu_j)_\nu(0))dx \\ &= \int_{\{|u_{n\sigma}(T)|>k\}} \big[R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T)\big]\varphi(T_k(u_{n\sigma})(T) - z_\nu)dx \\ &\quad - \int_{\{|u_{n\sigma}(0)|>k\}} \big[R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0)\big]\varphi(T_k(u_{n\sigma})(0) - T_k(\mu_j)_\nu(0))dx. \end{aligned}$$

Letting $\sigma \to 0^+$ we have

$$\begin{aligned} \lim_{\sigma\to 0^+} I_1(\sigma) &= \int_{\{|u_n(T)|>k\}} \big[R_m(u_n)(T) - T_k(u_n)(T)\big]\varphi(T_k(u_n)(T) - T_k(\mu_j)_\nu(T))dx \\ &\quad - \int_{\{|u_n(0)|>k\}} \big[R_m(u_n)(0) - T_k(u_n)(0)\big]\varphi(T_k(u_n)(0) - T_k(\mu_j)_\nu(0))dx \\ &= \int_{\{|u_n(T)|>k\}} \big[R_m(u_n)(T) - T_k(u_n)(T)\big]\varphi(T_k(u_n)(T) - z_\nu)dx \\ &\quad - \int_{\{|u_n(0)|>k\}} \big[R_m(u_n)(0) - T_k(u_n)(0)\big]\varphi(T_k(u_n)(0) - T_k(\mu_j)_\nu(0))dx. \end{aligned}$$

When $n \to \infty$ we get

$$\begin{aligned} I_1(\sigma) &= \int_{\{|u(T)|>k\}} \big[R_m(u)(T) - T_k(u)(T)\big]\varphi(T_k(u)(T) - T_k(\mu_j)_\nu(T))dx \\ &\quad - \int_{\{|u(0)|>k\}} \big[R_m(u)(0) - T_k(u)(0)\big]\varphi(T_k(u)(0) - T_k(\mu_j)_\nu(0))dx \\ &= \int_{\{|u(T)|>k\}} \big[R_m(u)(T) - T_k(u)(T)\big]\varphi(T_k(u)(T) - z_\nu)dx \\ &\quad - \int_{\{|u(0)|>k\}} \big[R_m(u)(0) - T_k(u)(0)\big]\varphi(T_k(u)(0) - T_k(\mu_j)_\nu(0))dx \end{aligned}$$

finally, letting $j \to \infty$ then $\mu \to \infty$ then using the fact that $\varphi(0) = 0$ and (4.11), we obtain

$$\lim_{\sigma\to 0^+} \sup I_1(\sigma) = \varepsilon(n, j, \mu). \tag{4.16}$$

About $I_2(\sigma)$, we have, since $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}))' = 0$

$$\begin{aligned} I_2(\sigma) &= \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu))\,(T_k(\mu_j)_\nu)'\,dxdt \\ &= \nu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,(T_k(\mu_j) - T_k(\mu_j)_\nu)\,dxdt \end{aligned}$$

adding and subtracting $T_k(u_{n\sigma})$ we get

$$\begin{aligned} I_2(\sigma) &= \nu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,(T_k(\mu_j) - T_k(u_{n\sigma})\,dxdt \\ &\quad +\nu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\,\varphi'(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\,(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu))\,dxdt \end{aligned}$$

by using the fact that $\varphi' \geq 0$ and the fact that $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)) \geq 0$ when $\{|u_{n\sigma}| > k\}$ and so, by letting $\sigma \to 0^+$ in the last integral we obtained

$$\lim_{\sigma\to 0^+} \sup I_2(\sigma) \geq \nu \int_{\{|u_n|>k\}} \left(R_m\left(u_n\right) - T_k\left(u_n\right)\right) \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \left(T_k\left(\mu_j\right) - T_k\left(u_n\right)\right) dxdt.$$

Finally by letting $n \to \infty$, $j \to \infty$ then $\mu \to \infty$ we conclude that

$$\lim_{\sigma\to 0^+} \sup I_2(\sigma) \geq \varepsilon(n, j, \mu). \tag{4.17}$$

For what concern $I_3(\sigma)$, one has

$$I_3(\sigma) = \int_Q (T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)'\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)dxdt + \int_Q (T_k(\mu_j)_\nu)'\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)dxdt$$

and then, by setting $\Phi(s) = \displaystyle\int_0^s \varphi(\eta)d\eta$ and integrating by parts

$$\begin{aligned}
I_3(\sigma) &= \int_\Omega \left[\Phi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\right]_0^T dx + \int_Q [T_k(\mu_j)_\nu]'\varphi\left(T_k\left(u_{n\sigma}\right) - T_k(\mu_j)_\nu\right) \\
&= \int_\Omega \Phi(T_k(u_{n\sigma})(T) - T_k(\mu_j)_\nu(T)) - \int_\Omega \Phi(T_k(u_{n\sigma})(0) - z_\nu)) \\
&\quad +\nu \int_Q ((T_k(\mu_j) - T_k(u_{n\sigma}))\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)dxdt \\
&\quad +\nu \int_Q ((T_k(\mu_j) - T_k(u_{n\sigma}))\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)dxdt
\end{aligned}$$

letting $\sigma \to 0^+$ then $n \to \infty$, since $\varphi \geq 0$ and $(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu)\varphi(T_k(u_{n\sigma}) - T_k(\mu_j)_\nu) \geq 0$ we get

$$\begin{aligned}
\lim_{\sigma\to 0^+} \sup I_3(\sigma) \geq & \int_\Omega \Phi\Big(T_k(u(T)) - T_k(\mu_j)_\nu(T)\Big) - \int_\Omega \Phi\Big(T_k(u(0)) - z_\nu)\Big) \\
& + \mu \int_Q (T_k(\mu_j) - T_k(u)\varphi(T_k(u) - T_k(\mu_j)_\nu)dxdt + \varepsilon(n)
\end{aligned}$$

using the periodicity condition and letting j then $\nu \to \infty$ we can deduce that

$$\limsup_{\sigma\to 0^+} I_3(\sigma) \geq \varepsilon(n, j, \mu). \tag{4.18}$$

Combining (4.16),(4.17) and (4.18), we conclude

$$\langle u_n', \omega_{j,n,m}^{i,\nu} \rangle \geq \varepsilon(n, j, \mu) \tag{4.19}$$

On the other hand, the second term of the left hand side of (4.14) reads as

$$\begin{aligned}
& \int_Q a(x,t,\nabla u_n) \left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt \\
= & \int_{\{|u_n|\leq k\}} a\left(x,t,\nabla u_n\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt \\
& + \int_{\{|u_n|>k\}} a\left(x,t,\nabla u_n\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt \\
= & \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) dxdt \\
& + \int_{\{|u_n|>k\}} a\left(x,t,\nabla u_n\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt
\end{aligned}$$

where we have used the fact that, since $m > k, \rho_m(u_n) = 1$ on $\{|u_n| \le k\}$.

Setting for $s > 0$, $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \le s\}$ and $Q_j^s = \{(x,t) \in Q : |\nabla T_k(v_j)| \le s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Q^s and Q_j^s respectively, we deduce that

$$\begin{aligned}
&\int_Q a(x,t,\nabla T_k(u_n))\left[\nabla T_k(u_n) - \nabla T_k(\mu_j)_\nu\right] \varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
=&\int_Q \left[a(x,t,\nabla T_k(u_n)) - a\left(x,t,\nabla T_k(\mu_j)\chi_j^s\right)\right]\left[\nabla T_k(u_n) - \nabla T_k(\mu_j)\chi_j^s\right]\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
&+\int_Q a\left(x,t,\nabla T_k(\mu_j)\chi_j^s\right)\left[\nabla T_k(u_n) - \nabla T_k(\mu_j)\chi_j^s\right]\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
&+\int_Q a(x,t,\nabla T_k(u_n))\,\nabla T_k(\mu_j)\,\chi_j^s\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
&-\int_Q a(x,t,\nabla u_n)\,(\nabla T_k(\mu_j)_\nu)\varphi'((T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
=&J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We shall go to the limit as n, j, μ and $s \to \infty$ in the last three integrals of the last side. Starting with J_2, we have by letting $n \to \infty$

$$J_2 \;=\; \int_Q a\left(x,t,\nabla T_k(\mu_j)\chi_j^s\right)\left[\nabla T_k(u) - \nabla T_k(\mu_j)\chi_j^s\right]\varphi'(T_k(u) - T_k(\mu_j)_\nu)\,\rho_m(u)dxdt + \varepsilon(n).$$

Letting $j \to \infty$ then $\mu \to \infty$, by (4.10) we have $a\left(x,t,\nabla T_k(\mu_j)\chi_j^s\right) \to a(x,t,\nabla T_k(u)\chi^s)$ strongly in $E_{\overline{M}}(Q)^N)$, then using (3.1) and Lebesgue theorem while $\nabla T_k(\mu_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $L_M(Q^N)$ we obtained

$$J_2 = \varepsilon(n,j,\mu). \tag{4.20}$$

About J_3, we have

$$\begin{aligned}
J_3 \;=\;& \int_Q a(x,t,\nabla T_k(u_n))\nabla T_k(\mu_j)\chi_j^s\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\varphi_m(u_n)dxdt\\
=\;& \int_{\{|u_n|\le k\}} a(x,t,\nabla u_n)\nabla T_k(\mu_j)\chi_j^s\varphi'(T_k(u_n - T_k(\mu_j)_\nu)dxdt\\
&+\int_{\{|u_n > k\}} a(x,t,0)\nabla T_k(\mu_j)\chi_j^s\varphi'(T_k(u_n - T_k(\mu_j)_\nu)\varphi_m(u_n)dxdt
\end{aligned}$$

gives by letting $n \to \infty$ and the fact that $a(x,t,0) = 0$

$$J_3 = \int_{\{|u|\le k\}} a(x,t,\nabla u)\nabla T_k(\mu_j)\chi_j^s\varphi'(T_k(u) - T_k(\mu_j)_\nu) + \varepsilon(n)$$

using (4.7) and letting $j \to \infty$ we get

$$J_3 = \int_{\{|u|\le k\}} h_k\nabla T_k(u)\chi^s\varphi'(T_k(u) - T_k(u)_\nu)dxdt + \varepsilon(n,j)$$

implying that by letting $\nu, s \to \infty$ and $\varphi'(0) = 1$

$$J_3 = \int_Q h_k\nabla T_k(u)dxdt + \varepsilon(n,j,\nu,s). \tag{4.21}$$

For what concerns J_4 we can write, since $\rho_m(u_n) = 0$ on $\{|u_n| > m+1\}$

$$\begin{aligned}
J_4 \;=\;& -\int_Q a(x,t,\nabla T_{m+1}(u_n))\,\nabla T_k(\mu_j)_\nu\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
=\;& -\int_{\{|u_n|\le k\}} a(x,t,\nabla T_k(u_n))\,\nabla T_k(\mu_j)_\nu\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt\\
&-\int_{\{k<|u_n|\le m+1\}} a(x,t,\nabla T_{m+1}(u_n))\,\nabla T_k(\mu_j)_\nu\varphi'(T_k(u_n) - T_k(\mu_j)_\nu)\,\rho_m(u_n)\,dxdt.
\end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} J_4 &= -\int_{\{|u|\le k\}} h_k \nabla T_k(\mu_j)_\nu \varphi'\left(T_k(u) - T_k(\mu_j)_\nu\right) dxdt \\ &\quad -\int_{\{k\le |u|\le m+1\}} h_{m+1} \nabla T_k(\mu_j)_\nu \varphi'\left(T_k(u) - T_k(\mu_j)_\nu\right) \rho_m(u) dxdt + \varepsilon(n) \end{aligned}$$

which implies that, by letting $j \to \infty$

$$\begin{aligned} J_4 &= -\int_{\{|u|\le k\}} h_k \nabla T_k(u)_\mu \varphi'\left(T_k(u) - T_k(u)_\nu\right) dxdt + \varepsilon(n,j) \\ &\quad - \int_{\{k\le |u|\le m+1\}} h_{m+1} \nabla T_k(u)_\nu \ \varphi'\left(T_k(u) - T_k(u)_\nu\right) \rho_m(u) dxdt \end{aligned}$$

so that, by letting $\nu \to \infty$

$$J_4 = -\int_Q h_k \nabla T_k(u) dxdt + \varepsilon(n,j,\nu). \tag{4.22}$$

We conclude then that

$$\int_Q a\left(x,t,\nabla u_n\right)\left[\nabla T_k\left(u_n\right) - \nabla T_k(\mu_j)_\nu\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt \tag{4.23}$$

$$= \int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right]\left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] \varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) dxdt + \varepsilon(n,j,\mu,s). \tag{4.24}$$

To deal with the third term of the left hand side of (4.14), since $|T_k(u_n)| \le k$, $|T_k(\mu_j)_\nu| \le k$ and by the definition of φ_m observe that

$$\left|\int_Q a\left(x,t,\nabla u_n\right) \varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu \le k\right) \rho_m'\left(u_n\right) dxdt\right| \le \varphi(2k) \int_{\{m\le |u_n|\le m+1\}} a\left(x,t,\nabla u_n\right) \nabla u_n dxdt$$

On the other hand, using $\theta_m\left(u_n\right)$ as a test function in (4.1) where $\theta_m(s) = T_1\left(s - T_m(s)\right)$, we get

$$\langle u_n', \theta_m\left(u_n\right)\rangle + \int_Q a\left(x,t,\nabla u_n\right) \nabla u_n \theta_m'\left(u_n\right) dxdt + \int_Q g\left(x,t,u_n,\nabla u_n\right) \theta_m\left(u_n\right) dxdt = \int_Q f_n \theta_m\left(u_n\right) dxdt$$

which gives, by setting $\Theta_m(s) = \displaystyle\int_0^s \theta_m(\eta) d\eta$ (observe that $\theta_m(s)s \ge 0$)

$$\left[\int_\Omega \Theta_m\left(u_n(t)\right) dx\right]_0^T + \int_{\{m\le |u_n|\le m+1\}} a\left(x,t,\nabla u_n\right) \nabla u_n dxdt \le \int_{\{|u_n|\ge m\}} |f_n| \, dxd$$

and since $\theta_m\left(u_n\right)(T) = \theta_m\left(u_n\right)(0)$, we deduce that

$$\int_{\{m\le |u_n|\le m+1\}} a\left(x,t,\nabla u_n\right) \nabla u_n dxdt \le \int_{\{|u_n|\ge m\}} |f_n| \, dxdt.$$

Since, as it can be easily seen, each integral of the right hand side is of the form $\varepsilon(n,m)$ we obtain

$$\left|\int_Q a\left(x,t,\nabla u_n\right) \varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m'\left(u_n\right) dxdt\right| \le \varepsilon(n,m). \tag{4.25}$$

We now turn to the fourth term of the left hand side of (4.14) using (3.3)and (3.4) , we can write

$$\begin{aligned} \left|\int_{\{|u_n|\le k\}} g_n\left(x,t,u_n,\nabla u_n\right) \varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt\right| &\le b(k) \int_Q c(x,t) \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right|\right) dxdt \\ &\quad + \frac{b(k)}{\alpha} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right| dxdt. \end{aligned} \tag{4.26}$$

Since $c(x,t)$ belongs to $L^1(Q)$, by letting $n, j,$ then $\nu \to \infty$ it is easy to see that

$$b(k) \int_Q c(x,t) \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right|\right) dxdt = \varepsilon(n,j,\nu).$$

On the other hand, the second term of the right hand side of (4.26) reads as

$$\begin{aligned}
&\frac{b(k)}{\alpha} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right|\right) dxdt \\
=&\frac{b(k)}{\alpha} \int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right| dxdt \\
&+ \frac{b(k)}{\alpha} \int_Q a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right| dxdt \\
&+ \frac{b(k)}{\alpha} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right) \nabla T_k\left(\mu_j\right)\chi_j^s \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right| dxdt
\end{aligned}$$

and, as above, by letting first n then j, ν and finally s to infinity, we can easily see that each one of last two integrals is of the form $\varepsilon(n,j,\nu)$. This implies that

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n\left(x,t,u_n,\nabla u_n\right) \varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) \rho_m\left(u_n\right) dxdt\right| \\
\le&\frac{b(k)}{\alpha} \int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] \left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right| dxdt + \varepsilon(n,j,\nu).
\end{aligned} \tag{4.27}$$

Combining (4.14), (4.15), (4.19), (4.23), (4.25) and (4.27), we get

$$\begin{aligned}
&\int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \\
&\times \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] \left[\varphi'\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right) - \frac{b(k)}{\alpha}\left|\varphi\left(T_k\left(u_n\right) - T_k(\mu_j)_\nu\right)\right|\right] dxdt \\
\le&\varepsilon(n,j,\nu,s,m),
\end{aligned}$$

then by the fact that $\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \ge \frac{1}{2}$, we conclude

$$\int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] dxdt \le 2\varepsilon(n,j,\nu,s,m). \tag{4.28}$$

On the other hand, we have

$$\begin{aligned}
&\int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k(u)\chi^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k(u)\chi^s\right] dxdt \\
&- \int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] dxdt \\
=&\int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right) \left[\nabla T_k\left(\mu_j\right)\chi_j^s - \nabla T_k(u)\chi^s\right] dxdt \\
&- \int_Q a\left(x,t,\nabla T_k(u)\chi^s\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k(u)\chi^s\right] dxdt \\
&+ \int_Q a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right) \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] dxdt
\end{aligned} \tag{4.29}$$

then by letting n, j and s to infinity, each integral of the right hand side is of the form $\varepsilon(n,j,s)$, which implying that

$$\begin{aligned}
&\int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k(u)\chi^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k(u)\chi^s\right] dxdt \\
=&\int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right) - a\left(x,t,\nabla T_k\left(\mu_j\right)\chi_j^s\right)\right] \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(\mu_j\right)\chi_j^s\right] dxdt + \varepsilon(n,j,s).
\end{aligned} \tag{4.30}$$

For $r \leq s$, we have

$$\begin{aligned} 0 &\leq \int_{Q^r} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt \\ &\leq \int_{Q_s} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt \\ &= \int_{Q_s} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\chi^s\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right]dxdt \\ &\leq \int_{Q} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\chi^s\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right]dxdt \end{aligned}$$

then by using (4.30), we can write

$$0 \leq \int_{Q^r} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt$$

$$\leq \int_Q \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k\left(v_j\right)\chi_j^s\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(\mu_j\right)\chi_j^s\right]dxdt+\varepsilon(n,j,s)$$

hence by passing to the limit sup over n and using (4.28) , we get

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \int_{Q^r} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt \\ &\leq \lim_{n\to\infty} \varepsilon(n,j,\nu,s,m) \end{aligned}$$

in which we can let successively j,ν,s and m go to infinity, to obtain

$$\int_{Q_r} \left[a\left(x,t,\nabla T_k\left(u_n\right)\right)-a\left(x,t,\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt \to 0 \text{ as } n\to\infty$$

and thus, as in the elliptic case (see [12]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in Q.} \tag{4.31}$$

We deduce then that,

$$a\left(x,t,\nabla T_k\left(u_n\right)\right) \to a\left(x,t,\nabla T_k(u)\right) \text{ weakly in } \left(L_{\overline{M}}(Q)\right)^N \text{ for } \sigma\left(\Pi L_{\overline{M}},\Pi E_M\right), \text{ for every } k>0. \tag{4.32}$$

Step 3: Modular convergence of the truncation and equi-integrability of the nonlinearities.
Thanks to (4.28)and (4.30) we have

$$\begin{aligned} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt \quad &\leq \quad \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k(u)\chi^s dxdt \\ &\quad + \int_Q a\left(x,t,\nabla T_k(u)\chi^s\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right]dxdt \\ &\quad +\varepsilon(n,j,\nu,s,m). \end{aligned}$$

by passing to the limit sup when $n\to\infty$ we obtain

$$\begin{aligned} \limsup_{n\to\infty} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt \quad &\leq \quad \int_Q a\left(x,t,\nabla T_k(u)\right)\nabla T_k(u)\chi^s dxdt \\ &\quad + \int_Q a\left(x,t,\nabla T_k(u)\chi^s\right)\left[\nabla T_k(u)-\nabla T_k(u)\chi^s\right]dxdt \\ &\quad + \lim_{n\to\infty} \varepsilon(n,j,\nu,s,m) \end{aligned}$$

in which we can pass to the limit as $j, \mu, s, m \to \infty$ to obtain

$$\limsup_{n\to\infty} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt \leq \int_Q a\left(x,t,\nabla T_k(u)\right)\nabla T_k(u)dxdt.$$

On the other hand, Fatou's lemma implies

$$\int_Q a\left(x,t,\nabla T_k(u)\right)\nabla T_k(u)dxdt \leq \lim\inf_{n\to\infty} \int_Q a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt.$$

Finally, we deduce as $n \to \infty$, that

$$a\left(x,t,\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right) \to a\left(x,t,\nabla T_k(u)\right)\nabla T_k(u) \quad \text{in } L^1(Q). \tag{4.33}$$

We have,

$$M(2|\nabla T_k(u_n) - \nabla T_k(u)| \leq \frac{1}{2}M(|\nabla T_k(u_n)|) + \frac{1}{2}M(|\nabla T_k(u)|).$$

Using (3.3) we can write

$$M(2|\nabla T_k(u_n) - \nabla T_k(u)| \leq \frac{1}{2\alpha}a\left(x,t,\nabla T_k(u)\right)\nabla T_k(u_n) + \frac{1}{2}M(|\nabla T_k(u)|)$$

applying Vitali's theorem , by the fact that $M(|\nabla T_k(u)|) \in L^1(Q)$ and (4.33) we deduce that

$$\nabla T_k\left(u_n\right) \to \nabla T_k(u) \quad in \quad \left(L_M(Q)\right)^N \quad \text{for the modular convergence.} \tag{4.34}$$

We shall now prove that $g_n\left(x,t,u_n,\nabla u_n\right) \to g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $g_n\left(x,t,u_n,\nabla u_n\right) \to g(x,t,u,\nabla u)$ a.e. in Q, thanks to (4.6) and (4.31), it suffices to prove that $g_n\left(x,t,u_n,\nabla u_n\right)$ are uniformly equi-integrable in Q. Let $E \subset Q$ be a measurable subset of Q. We have for any $m > 0$,

$$\begin{aligned}\int_E |\, g_n(\,x,t,u_n,\nabla u_n)\,|\,dxdt =& \int_{E\cap\{|u_n|\leq m\}} |g_n\left(x,t,u_n,\nabla u_n\right)|\,dxdt + \int_{\{|u_n|>m\}} |g_n\left(x,t,u_n,\nabla u_n\right)|\,dxdt\\ \leq& b(m)\int_E c(x,t) + M(|\nabla u_n|)dxdt + \int_{\{|u_n|>m\}} |f_n|\,dxdt\\ \leq& \frac{b(m)}{\alpha}\int_E a\left(x,t,\nabla T_m\left(u_n\right)\right)\nabla T_m\left(u_n\right)dxdt + b(m)\int_E c(x,t)dxdt + \int_{\{|u_n|>m\}} |f_n|\,dxdt\end{aligned}$$

where we have used (3.4) and (4.5). Therefore, it is easy to see that there exists $\eta > 0$ such that

$$|E| < \eta \Longrightarrow \int_E |g_n\left(x,t,u_n,\nabla u_n\right)|\,dxdt \leq \varepsilon,$$

which shows that $g_n\left(x,t,u_n,\nabla u_n\right)$ are uniformly equi-integrable in Q as required.

step 4 : Passage to the limit.

In this step, u is shown to satisfy (3.1) and (3.4). Let S be a function in $W^{2,\infty}(Q)$ such that S' has a compact support. Let k be a positive real number such that $supp(S') \subset [-k,k]$. Pointwise multiplication of the approximate equation (4.1) by $S'(u_n)$ leads to

$$\frac{\partial S(u_n)}{\partial t} - \operatorname{div}(S'(u_n)a(x,t,\nabla u_n)) + S''(u_n)a(x,t,\nabla u_n)\nabla u + g_n(x,t,u_n,\nabla u_n)S'(u_n) = f_nS'(u_n) \text{ in } D'\left(Q\right). \tag{4.35}$$

Starting by the limit of $-\operatorname{div}(S'(u_n)a(x,t,\nabla u_n))$, since $supp(S') \subset [-k,k]$ we have ,

$$S'(u_n)a(x,t,\nabla u_n) = S'(u_n)a(x,t,\nabla T_k(u_n)) \quad \text{a.e in } Q$$

(4.31) and (4.32) imply that $S'\left(u_n\right)a\left(x,t,\nabla T_k\left(u_n\right)\right) \to S(u)a\left(x,t,\nabla T_k(u)\right)$ weakly in $\left(L_{\overline{M}}\left(Q\right)\right)^N$, for $\sigma\left(\Pi L_{\overline{M}}, \Pi E_M\right)$ as n tends to $+\infty$, because $S'(u) = 0$ for $|u| \geq k$ a.e. in Q. And the term $S'(u)a\left(x,t,\nabla T_k(u)\right) = S'(u)a(x,t,\nabla u)$ a.e in Q, then

$$S'\left(u_n\right)a\left(x,t,\nabla u_n\right) \to S'(u)a(x,t,\nabla u) \quad \text{a.e. in} \quad Q. \tag{4.36}$$

Limit of $S''(u_n)\, a(x,t,\nabla u_n)\nabla u_n$, since supp $S'' \subset [-k,k]$, we have

$$S''(u_n)\, a(x,t,\nabla u_n)\nabla u_n = S''(u_n)\, a(x,t,\nabla T_k(u_n))\nabla T_k(u_n) \text{ a.e. in } Q.$$

The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'' , (4.32), (4.31) , and (4.33) allow to conclude that $S''(u_n)a(x,t,\nabla T_k(u_n))\nabla T_k(u_n) \to S''(u)a(x,t,\nabla T_k(u))\nabla T_k(u)$ weakly in $L^1(Q)$, as n tends to $+\infty$, and $S''(u)a(x,t,\nabla T_k(u))\nabla T_k(u) = S''(u)a(x,t,\nabla u)\nabla u$ a.e. in Q. Then

$$S''(u_n)\, a(x,t,\nabla u_n)\nabla u_n \to S''(u)a(x,t,\nabla u)\nabla u \quad \text{ a.e. in } \quad Q. \tag{4.37}$$

Similarly, for the Limit of $g_n(x,t,u_n,\nabla u_n)S'(u_n)$,using the fact that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ it is easy to see that

$$g_n(x,t,u_n,\nabla u_n)\, S'(u_n) \to g(x,t,u,\nabla u)\, S'(u). \tag{4.38}$$

Using the fact $f_n \to f$ in $L^1(Q)$, we deduce also that

$$f_n S'(u_n) \to f S'(u) \text{ strongly in } L^1(Q). \tag{4.39}$$

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in Eq. (4.35) and to conclude that u satisfies (3.10). It remains to show that $S(u(0)) = S(u(T))$.

Firstly, we have that $S(u_n)$ is bounded in $W_0^{1,x}L_M(Q) \cap L^\infty(Q)$, secondly (4.35) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$ As a consequence, lemma (2.3.3) and (2.3.4) implies that

$$S(u_n) \to S(u) \quad \text{strongly in} \quad C\left([0,T]; L^1(\Omega)\right)$$

Finaly using the fact that $S(u_n)(0) = S(u_n)(T)$ we deduce that

$$S(u)(0) = S(u)(T) \text{ in } \Omega$$

the proof of (3.2.1) is complete. □

References

[1] A. Aberqi, J. Bennouna, M. Elmassoudi and M. Hammoumi, *Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces*, Monatsh Math. **189** (2019), 195–219.

[2] R. Adams, *Sobolev Spaces*, Ac. Press, New York, 1975.

[3] N. Alaa and M. Iguernane, *Weak periodic solutions of some quasilinear parabolic equations with data measures*, J. Ineq. Pure Appl. Math. **3** (2002), no. 3.

[4] A. Benkirane, A. Elmahi, *Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application*, Nonlinear Anal. Theory Methods Appl. **28** (1997), 1769–1784.

[5] J.-L. Boldrini and J. Crema, *More on forced solutions of quasi-parabolic equations*, Cadernos Mate. **75** (2000), 71–88.

[6] H. Brezis, *Analyse Fonctionnelle, Theorie et Applications*, Masson, Paris, 1992.

[7] J. Deuel and P. Hess, *Nonlinear parabolic boundary value problems with upper and lower solutions*, Israel J. Math. **29** (1978), 92–104.

[8] T. Donaldson, *Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial-boundary value problems*, J. Differ. Equ. **16** (1974), 201–256.

[9] N.C. Eddine and M.A. Ragusa, *Generalized critical Kirchhoff-type potential systems with Neumann boundary conditions*, Appl. Anal. **101** (2022), no. 11, 3958–3988.

[10] A. El Hachimi and A. Lamrani Alaoui, *Existence of stable periodic solutions for quasilinear parabolic problems in the presence of well-ordered lower and upper-solutions*, Electronic J. Differ. Equ. **2002** (2002), 117–126.

[11] A. El Hachimi and A. Lamrani Alaoui, *Time periodic solutions to a nonhomogeneous Dirichlet periodic problem*, Appl. Math. E-Notes **8** (2008), 1–8.

[12] A.El Hachimi and A. Lamrani Alaoui, *Periodic solutions of nonlinear parabolic equations with measure data and polynomial growth in* $|\nabla u|$, Recent Dev. Nonlinear Analysis, 2010.

[13] A. Elmahi, *Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces*, Electron. J. Differ. Equ. **2002** (2002), 203–220.

[14] A. Elmahi and D. Meskine, *Strongly nonlinear parabolic equation with natural growth term and* L^1 *data in Orlicz space*, Port. Math. Nova **62** (2005), 143–183.

[15] A. Elmahi and D. Meskine, *Parabolic initial-boundary value problems in Orlicz spaces*, Ann. Polon. Math. **85** (2005), 99–119.

[16] A. Elmahi and D. Meskine, *Strongly nonlinear parabolic equations having natural growth terms in Orlicz spaces*, Nonlinear Anal. **60** (2005), 1–35.

[17] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974), 163–205.

[18] P. Gwiazda, I. Skrzypczak and A. Zatorska-Goldstein, *Existence ofrenormalized solutions to ellpitic equation in Muiselak-Orlicz space*, J. Differ. Equ. **2018** (2018), 341–377.

[19] S. Heidari and A. Razani, *Infinitely many solutions for nonlocal elliptic systems in Orlicz-Sobolev spaces*, Georgian Math. J. **29** (2021), no. 1, 45–54.

[20] S. Heidari and A. Razani, *Multiple solutions for a class of nonlocal quasilinear elliptic systems in Orlicz-Sobolev spaces*, Boundary Value Prob. **2021** (2021), no. 1, 1–15.

[21] G. Kouadri, *Etude de quelques problème inverses paraboliques dans des domaines non réguliers*, Doctoral thesis, Université Mohamed Khider-Biskra, 2019.

[22] R. Landes and V. Mustonen, *A strongly nonlinear parabolic initial-boundary value problem*, Ark. F. Mat. **25** (1987), 29–40.

[23] J.L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites Nonlineaires*, Dunod, Gauthier-Villars, 1969.

[24] A. Marciniak-Czochra and M. Kimmel, *Reaction-diffusion approach to modeling of the spread of early tumors along linear or tubular structures*, J. Theor. Bio. **244** (2007), no. 3, 375–387.

[25] J.J. Neito, *Periodic solutions of nonlinear parabolic equations*, J. Differ. Equ. **60** (1985), no. 1, 90–102.

[26] A. Razani, *Game-theoretic p-Laplace operator involving the gradient*, Miskolc Math. Notes **23** (2022), no. 2, 867–879.

[27] A. Razani, *Nonstandard competing anisotropic (p, q)-Laplacians with convolution*, Boundary Value Prob. **2022** (2022), no. 1.

[28] J. Robert, *Inéquations variationnelles paraboliques fortement non linéaires*, J. Math. Pures Appl. **53** (1974), 299–321.

[29] G. Tomás and K. Uriel, *Periodic parabolic problems with nonlinearities indefinite in sign*, Pub. Mate. **51** (2007), no. 1, 45–57.