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Multi-cubic functional equations in Lipschitz spaces

Mohammad Ali Abolfathi

Department of Mathematics, Faculty of Sciences, Urmia University, P. O. Box 165, Urmia, Iran

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Abstract

In this paper, we prove stability of multi-cubic functional equations in Lipschitz spaces by property multi-symmetric left invariant mean. Indeed, we prove under certain Lipschitz condition a family of Lipschitz mappings can be approximated by multi-cubic mappings.

Keywords: Lipschitz space, Multi-cubic functional equation, Stability
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1 Introduction

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?
The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940 and affirmatively solved by Hyers [14]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and by Th. M. Rassias [23] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\varepsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Th. M. Rassias' theorem was obtained by Găvruta [12], who replaced $\varepsilon(\|x\|^p+\|y\|^p)$ by ageneral control function $\varphi(x,y)$. During the last two decades, the subject has been established and developed by an increasing number of mathematicians in various spaces [1, 9, 22].

In 2002, Jun and Kim [15] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2(f(x+y)+f(x-y))+12f(x),$$

and established the general solution and the Hyers-Ulam stability for this functional equation (also see [21]). This functional equation is called cubic functional equation and every solution of cubic equation is said to be a cubic mapping. Obviously, the mapping $f(x)=x^3$ satisfies in the functional equation. Bodaghi [3] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(mx+ny)+f(mx-ny)=mn^2(f(x+y)+f(x-y))+2m(m^2-n^2)f(x)$$

where m,n are integer numbers with $m\geq 2$.

The stability of multi-quadratic mappings in Banach spaces has been studied for the first time in [20]. Then, X. Zhao et al., characterized them as an equation in [27]. Recently, some generalized forms of the multi-quadratic

Email address: m.abolfathi@urmia.ac.ir (Mohammad Ali Abolfathi)

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mappings and their Hyers-Ulam stability in Banach spaces by a known fixed point method have been studied in [4] and [5]. The Jensen-type multi-quadratic mappings and their characterization can be found in [24]. Ghaemi et al., in [13] introduced the multi-cubic mappings for the first time. Next, a special case of such mappings is studied in [6]. Park and Bodaghi [19] investigated multi-cubic functional equations and some results on the stability in modular spaces. Indeed, a mapping $f : \mathcal{W}^n \to \mathcal{V}$ is called multi-cubic if it is cubic in each variable, i.e., satisfies (2.1) in each variable. In [6], the authors unified the system of functional equations defining a multi-cubic mapping to a single equation, namely multi-cubic functional equation. The general system of cubic functional equations which was defined in [13], characterized as a single equation in [11].

The algebra of Lipschitz functions on a complete metric space plays a role in non-commutative metric theory similar to that played by the algebra of continuous functions on compact space in non-commutative topology. In 1997, Lipschitz stability-type problems for Cuachy and Jensen functional equations were studied by Tabor [25]. Czerwik and Dlutek [7] investigated the stability of the quadratic functional equations in Lipschitz spaces. The stability of cubic functional equations in Lipschitz spaces was proved by Ebadian et al. [10]. Nikoufar [16, 17, 18] verified Lipschitz stability of bi-quadratic functional equations, multi-quadratic functional equations and quartic functional equations, respectively. Moreover, the stability of generalized multi-quadratic mappings in Lipschitz spaces was investigated in [8].

In this paper, we prove Lipschitz stability of multi-cubic functional equations. Indeed, we prove under certain Lipschitz conditions a family of functions can be approximated by multi-cubic mappings.

2 Main results

Let $\mathcal{W}$ be an abelian group and $\mathcal{V}$ a real vector space. A mapping $\mathcal{C} : \mathcal{W}^n \to \mathcal{V}$ is called multi-cubic if $\mathcal{C}$ is cubic in each variable; that is, $\mathcal{C}$ satisfies the system of equations:

$$\begin{aligned} &2f(x_1, x_2, x_3, ..., x_i + y_i, ..., x_n) + 2f(x_1, x_2, x_3, ..., x_i - y_i, ..., x_n) \\ &\quad = f(x_1, x_2, x_3, ..., 2x_i + y_i, ..., x_n) + f(x_1, x_2, x_3, ..., 2x_i - y_i, ..., x_n) \\ &\quad - 12f(x_1, x_2, x_3, ..., x_i, ..., x_n) \end{aligned} \tag{2.1}$$

for all $x_i, y_i \in \mathcal{W}, i = 1, 2, 3, ..., n$. Let $\mathcal{L}(\mathcal{V})$ be a family of subsets of $\mathcal{V}$. We say that $\mathcal{L}(\mathcal{V})$ is linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, i.e., $x + \mathcal{A} \in \mathcal{L}(\mathcal{V})$ for every $\mathcal{A} \in \mathcal{L}(\mathcal{V})$ and every $x \in \mathcal{V}$ [7]. By $\mathcal{F}(\mathcal{W}, \mathcal{L}(\mathcal{V}))$ we denote the family of all mappings $f : \mathcal{W} \to \mathcal{V}$ such that $Imf \in \mathcal{B}$ for some $\mathcal{B} \in \mathcal{L}(\mathcal{V})$. A mapping $f : \mathcal{W}^n \to \mathcal{V}$ is said to be symmetric if $f(x_1, x_2, ..., x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$ for every permutation $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ of $\{1, 2, ..., n\}$.

Definition 2.1. Family mappings $\mathcal{F}(\mathcal{W}, \mathcal{L}(\mathcal{V}))$ is called multi-symmetric left invariant mean (briefly MSLIM), if the family $\mathcal{L}(\mathcal{V})$ is linearly invariant and there exists a linear operator
$\Lambda : \mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V})) \to \mathcal{V}$, such that
(i) if $Imf \in \mathcal{B}$ for some $\mathcal{B} \in \mathcal{L}(\mathcal{V})$, then $\Lambda[f] \in \mathcal{B}$,
(ii)if $f_{x_1,x_2,...,x_n} \in \mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V}))$ and $x_1, x_2, ..., x_n \in \mathcal{W}$, then

$$\Lambda[f_{x_1,x_2,...,x_n}] = \Lambda[f_{x_{\sigma(1)},x_{\sigma(2)},...,x_{\sigma(n)}}]$$

for every permutation $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ of $\{1, 2, ..., n\}$,
(iii) if $f \in \mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V}))$ and $a \in \mathcal{W}$, then $\Lambda[f^a] = \Lambda[f]$, where $f^a(x_1, x_2, ..., x_n) = f(x_1 + a, x_2, ..., x_n)$.

Definition 2.2. Let $\Delta : \mathcal{W}^n \times \mathcal{W}^n \longrightarrow \mathcal{L}(\mathcal{V})$ be a set-valued function such that

$$\begin{aligned} \Delta((x_1 + a_1, x_2 + a_2, ..., x_n + a_n), &(y_1 + a_1, y_2 + a_2, ..., y_n + a_n)) \\ &= \Delta((a_1 + x_1, a_2 + x_2, ..., a_n + x_n), (a_1 + y_1, a_2 + y_2, ..., a_n + y_n)) \\ &= \Delta((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n), \end{aligned}$$

for all $(a_1, a_2, ..., a_n), (x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n) \in \mathcal{W}^n$. A mapping $f : \mathcal{W}^n \to \mathcal{V}$ is said to be Δ-Lipschitz if

$$f(x_1, x_2, ..., x_n) - f(y_1, y_2, ..., y_n) \in \Delta((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))$$

for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in \mathcal{W}^n$.

For a given mapping $f : \mathcal{W}^n \to \mathcal{V}$ we consider its multi-cubic difference as follows:

$$\begin{aligned}\Theta_i f(x_1, x_2, ...x_i, y_i, x_{i+1}, x_n) : &= 2f(x_1, x_2, x_3, ..., x_i + y_i, ..., x_n) + 2f(x_1, x_2, x_3, ..., x_i - y_i, ..., x_n)\\ &- f(x_1, x_2, x_3, ..., 2x_i + y_i, ..., x_n) - f(x_1, x_2, x_3, ..., 2x_i - y_i, ..., x_n)\\ &+ 12f(x_1, x_2, x_3, ..., x_i, ..., x_n)\end{aligned}$$

for all $x_i, y_i \in \mathcal{W}$ and $i = 1, 2, ..., n$.

Theorem 2.3. Let $\mathcal{W}$ be an abelian group and $\mathcal{V}$ a vector space. Assume that the family $\mathcal{F}(\mathcal{W}, \mathcal{L}(\mathcal{V}))$ admits MSLIM. If $f : \mathcal{W}^n \to \mathcal{V}$ is an odd mapping and $\Theta_1 f(t, ., ..., .)$ is Δ-Lipschitz for every $t \in \mathcal{W}$, then there exists a multi-cubic mapping $\mathcal{C} : \mathcal{W}^n \to \mathcal{V}$ such that $f - \mathcal{C}$ is $\frac{1}{12}\Delta$-Lipschitz. Moreover, if $Im\Theta_1 f \subset \mathcal{A}$ for some $\mathcal{A} \in \mathcal{L}(\mathcal{V})$, then $Im(f - \mathcal{C}) \subset \frac{1}{12}\mathcal{A}$.

Proof . For every $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$, we define the mapping $\eta_{x_1}(\cdot, x_2, ..., x_n) : \mathcal{W} \to \mathcal{V}$ by the formula

$$\begin{aligned}\eta_{x_1}(\cdot, x_2, ..., x_n) &= \frac{1}{12} f(2x_1 + \cdot, x_2, ..., x_n)\\ &+ \frac{1}{12} f(2x_1 - \cdot, x_2, ..., x_n) - \frac{1}{6} f(x_1 + \cdot, x_2, ..., x_n) - \frac{1}{6} f(x_1 - \cdot, x_2, ..., x_n).\end{aligned}$$

We prove that $Im\eta_{x_1}(\cdot, x_2, ..., x_n) \subseteq \mathcal{A}$ for some $\mathcal{A} \in \mathcal{L}(\mathcal{V})$. For $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$, we get

$$\begin{aligned}\eta_{x_1}(\cdot, x_2, ..., x_n) &= f(0, x_2, ..., x_n) + \frac{1}{12} f(2x_1 + \cdot, x_2, ..., x_n)\\ &+ \frac{1}{12} f(2x_1 - \cdot, x_2, ..., x_n) - \frac{1}{6} f(x_1 + \cdot, x_2, ..., x_n) - \frac{1}{6} f(x_1 - \cdot, x_2, ..., x_n)\\ &- f(x_1, x_2, ..., x_n) + f(x_1, x_2, ..., x_n) - f(0, x_2, ..., x_n)\\ &= \frac{1}{12}\mathcal{D}_1 f(0, \cdot, x_2, ..., x_n) - \frac{1}{12}\mathcal{D}_1 f(x_1, \cdot, x_2, ..., x_n)\\ &+ f(x_1, x_2, ..., x_n) - f(0, x_2, ..., x_n).\end{aligned}$$

Hence

$$Im\eta_{x_1}(., x_2, ..., x_n) \subseteq \mathcal{A},$$

where

$$\mathcal{A} = \frac{1}{12}\Delta\big((0, x_2, ..., x_n), (x_1, x_2, ..., x_n)\big) + f(x_1, x_2, ..., x_n) - f(0, x_2, ..., x_n).$$

Since the family $\mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V}))$ is MSLIM, so there exists a linear operator $\Lambda : \mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V})) \to \mathcal{V}$, such that
(i) $\Lambda[\eta_{x_1}(\cdot, x_2, ..., x_n)] \in \mathcal{A}$ for some $\mathcal{A} \in \mathcal{L}(\mathcal{V})$ and every $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$,
(ii)$\Lambda[\eta_{x_1}(\cdot, x_2, ..., x_n)] = \Lambda[\eta_{x_{\sigma(1)}}(\cdot, x_{\sigma(2)}, ..., x_{\sigma(n)})]$ for every permutation $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ of $\{1, 2, ..., n\}$,
(iii) if $a \in \mathcal{W}$ and $\eta^a_{x_1}(\cdot, x_2, ..., x_n) : \mathcal{W} \to \mathcal{V}$ is defined by $\eta^a_{x_1}(\cdot, x_2, ..., x_n) = \eta_{x_1}(\cdot + a, x_2, ..., x_n)$ for every $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$, then $\eta^a_{x_1}(\cdot, x_2, ..., x_n) \in \mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V}))$ and $\Lambda[\eta^a_{x_1}(\cdot, x_2, ..., x_n)] = \Lambda[\eta_{x_1}(\cdot, x_2, ..., x_n)]$.

Define the mapping $\mathcal{C} : \mathcal{W}^n \to \mathcal{V}$ by

$$\mathcal{C}(x_1, x_2, ..., x_n) := \Lambda[\eta_{x_1}(\cdot, x_2, ..., x_n)].$$

We prove that $f - \mathcal{C}$ is $\frac{1}{12}\Delta$-Lipschitz. Since is Δ-Lipschitz for $t \in \mathcal{W}$,

$$\Theta_1 f(t, x_1, x_2, ..., x_n) - \Theta_1 f(t, y_1, y_2, ..., y_n) \in \Delta\big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\big)$$

for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in \mathcal{W}^n)$ and so

$$Im(\frac{1}{12}\Theta_1(\cdot, x_1, ..., x_n) - \frac{1}{12}\Theta_1(\cdot, y_1, ..., y_n)) \subseteq \frac{1}{12}\Delta\big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\big).$$

Hence

$$\Lambda\big[\frac{1}{12}\Theta_1 f(\cdot, x_1, ..., x_n) - \frac{1}{12}\Theta_1 f(\cdot, y_1, ..., y_n)] \in \frac{1}{12}\Delta\big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\big)$$

for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in \mathcal{W}^n$. Note that $\mathcal{F}(\mathcal{W}^n, \mathcal{L}(\mathcal{V}))$ contains constant mappings. By property (i) of Λ for constant mapping $\mathcal{K} : \mathcal{W}^n \to \mathcal{V}$ we have $\Lambda[\mathcal{K}] = \mathcal{K}$. We define the constant mapping $\mathcal{K}_{x_1,...,x_n} : \mathcal{W}^n \to \mathcal{V}$ by $\mathcal{K}_{x_1,...,x_n}(., ..., .) := f(x_1, ..., x_n)$. We see that

$$\begin{aligned}
\big(f(x_1, ..., x_n) - \mathcal{C}(x_1, ..., x_n)\big) &- \big(f(y_1, ..., y_n) - \mathcal{C}(y_1, ..., y_n)\big) \\
&= \big(\Lambda[\mathcal{K}_{x_1,...,x_n}(., ., ..., .)] - \Lambda[\Theta_1 f(\cdot, x_2, ..., x_n)]\big) \\
&- \big(\Lambda[\mathcal{K}_{y_1,...,y_n}(., ., ..., .)] - \Lambda[\Theta_1 f(\cdot, y_2, ..., y_n)]\big) \\
&= \big(\Lambda[\mathcal{K}_{x_1,...,x_n}(., ., ..., .) - \Theta_1 f(\cdot, x_2, ..., x_n)]\big) \\
&- \big(\Lambda[\mathcal{K}_{y_1,...,y_n}(., ., ..., .) - \Theta_1 f(\cdot, y_2, ..., y_n)]\big) \\
&= \Lambda[\frac{1}{12}\eta_1 f(\cdot, x_1, ..., x_n) - \frac{1}{12}\eta_1 f(\cdot, y_1, ..., y_n)]
\end{aligned}$$

for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in \mathcal{W}^n$. This shows that

$$\begin{aligned}
\big(f(x_1, ..., x_n) - \mathcal{C}(x_1, ..., x_n)\big) &- \big(f(y_1, ..., y_n) - \mathcal{C}(y_1, ..., y_n)\big) \\
&\in \frac{1}{12}\Delta\big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\big)
\end{aligned}$$

for all $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n) \in \mathcal{W}^n$, therefore $f - \mathcal{C}$ is a $\frac{1}{12}\Delta$-Lipschitz mapping. By property (iii) of Λ, we have

$$\begin{aligned}
&2\mathcal{C}(x_1 + y_1) + 2\mathcal{C}(x_1 - y_1) + 12\mathcal{C}(x_1) \\
&= 2\Lambda[\eta_{x_1+y_1}(\cdot, x_2, ..., x_n)] + 2\Lambda[\eta_{x_1-y_1}(\cdot, x_2, ..., x_n)] + 12\Lambda[\eta_{x_1}(\cdot, x_2, ..., x_n)] \\
&= \Lambda[\eta^{2x_1}_{x_1+y_1}(\cdot, x_2, ..., x_n)] + \Lambda[\eta^{-2x_1}_{x_1+y_1}(\cdot, x_2, ..., x_n)] + \Lambda[\eta^{2x_1}_{x_1-y_1}(\cdot, x_2, ..., x_n)] + \Lambda[\eta^{-2x_1}_{x_1-y_1}(\cdot, x_2, ..., x_n)] \\
&+ 2\Lambda[\eta^{x_1+y_1}_{x_1}(\cdot, x_2, ..., x_n)] + 2\Lambda[\eta^{x_1-y_1}_{x_1}(\cdot, x_2, ..., x_n)] + 2\Lambda[\eta^{-x_1-y_1}_{x_1}(\cdot, x_2, ..., x_n)] \\
&+ 2\Lambda[\eta^{-x_1+y_1}_{x_1}(\cdot, x_2, ..., x_n)] + 2\Lambda[\eta^{y_1}_{x_1}(\cdot, x_2, ..., x_n)] + 2\Lambda[\eta^{-y_1}_{x_1}(\cdot, x_2, ..., x_n)] \\
&= \Lambda[\frac{1}{12}f(4x_1 + 2y_1 + ., , x_2, ..., x_n) + \frac{1}{12}f(2y_1 - ., x_2, ..., x_n) - \frac{1}{6}f(3x_1 + y_1 + ., x_2, ..., x_n) \\
&- \frac{1}{6}f(y_1 - x_1 - \cdot, x_2, ..., x_n) + \frac{1}{12}f(2y_1 + \cdot, x_2, ..., x_n) + \frac{1}{12}f(4x_1 + 2y_1 - \cdot, x_2, ..., x_n) \\
&- \frac{1}{6}f(3x_1 + y_1 - \cdot, x_2, ..., x_n) - \frac{1}{6}f(-x_1 + y_1 + \cdot, x_2, ..., x_n) + \frac{1}{12}f(4x_1 - 2y_1 + \cdot, x_2, ..., x_n) \\
&+ \frac{1}{12}f(-2y_1 - \cdot, x_2, ..., x_n) - \frac{1}{6}f(3x_1 - y_1 + \cdot, x_2, ..., x_n) - \frac{1}{6}f(-x_1 - y_1 - \cdot, x_2, ..., x_n) \\
&+ \frac{1}{12}f(-2y_1 + \cdot, x_2, ..., x_n) + \frac{1}{12}f(4x_1 - 2y_1 - \cdot, x_2, ..., x_n) - \frac{1}{6}f(-x_1 - y_1 + \cdot, x_2, ..., x_n) \\
&- \frac{1}{6}f(3x_1 - y_1 - \cdot, x_2, ..., x_n) + \frac{1}{6}f(3x_1 + y_1 + \cdot, x_2, ..., x_n) + \frac{1}{6}f(x_1 - y_1 - \cdot, x_2, ..., x_n) \\
&- \frac{1}{3}f(2x_1 + y_1 + \cdot, x_2, ..., x_n) - \frac{1}{3}f(-y_1 - \cdot, x_2, ..., x_n) + \frac{1}{6}f(3x_1 - y_1 + \cdot, x_2, ..., x_n) \\
&+ \frac{1}{6}f(x_1 + y_1 - \cdot, x_2, ..., x_n) - \frac{1}{3}f(2x_1 - y_1 + \cdot, x_2, ..., x_n) - \frac{1}{3}f(y_1 - \cdot, x_2, ..., x_n) \\
&+ \frac{1}{6}f(x_1 - y_1 + \cdot, x_2, ..., x_n) + \frac{1}{6}f(3x_1 + y_1 - \cdot, x_2, ..., x_n) - \frac{1}{3}f(-y_1 + \cdot, x_2, ..., x_n) \\
&- \frac{1}{3}f(2x_1 + y_1 - \cdot, x_2, ..., x_n) + \frac{1}{6}f(x_1 + y_1 + \cdot, x_2, ..., x_n) + \frac{1}{6}f(3x_1 - y_1 - \cdot, x_2, ..., x_n) \\
&- \frac{1}{3}f(y_1 + \cdot, x_2, ..., x_n) - \frac{1}{3}f(2x_1 - y_1 - \cdot, x_2, ..., x_n) + \frac{1}{6}f(2x_1 + y_1 + \cdot, x_2, ..., x_n) \\
&+ \frac{1}{6}f(2x_1 - y_1 - \cdot, x_2, ..., x_n) - \frac{1}{3}f(x_1 + y_1 + \cdot, x_2, ..., x_n) - \frac{1}{3}f(x_1 - y_1 - \cdot, x_2, ..., x_n) \\
&+ \frac{1}{6}f(2x_1 - y_1 + \cdot, x_2, ..., x_n) + \frac{1}{6}f(2x_1 + y_1 - \cdot, x_2, ..., x_n) - \frac{1}{3}f(x_1 - y_1 + \cdot, x_2, ..., x_n) \\
&- \frac{1}{3}f(x_1 + y_1 - \cdot, x_2, ..., x_n)]
\end{aligned}$$

$$= \Lambda[\frac{1}{12}f(4x_1 + 2y_1 + \cdot, x_2, ..., x_n) + \frac{1}{12}f(4x_1 + 2y_1 - \cdot, x_2, ..., x_n) - \frac{1}{6}f(2x_1 + y_1 + \cdot, x_2, ..., x_n)$$
$$- \frac{1}{6}f(2x_1 + y_1 - ., x_2, ..., x_n)]$$
$$+ \Lambda[\frac{1}{12}f(4x_1 - 2y_1 + \cdot, x_2, ..., x_n) + \frac{1}{12}f(4x_1 - 2y_1 - \cdot, x_2, ..., x_n) - \frac{1}{6}f(2x_1 - y_1 + \cdot, x_2, ..., x_n)$$
$$- \frac{1}{6}f(2x_1 - y_1 - \cdot, x_2, ..., x_n)]$$
$$= \Lambda[\eta_{2x_1+y_1}(\cdot, x_2, ..., x_n)] + \Lambda[\eta_{2x_1-y_1}(\cdot, x_2, ..., x_n)]$$
$$= \mathcal{C}(2x_1 + y_1) + \mathcal{C}(2x_1 - y_1).$$

It follows that $\mathcal{C}$ is cubic on its first variable. On the other hand, by property (ii) of Λ, $\mathcal{C}$ is multi-symmetric and hence $\mathcal{C}$ is multi-cubic mapping. Moreover, if $Im\Theta_1 f \subset \mathcal{A}$, then

$$Im(\frac{1}{12}\Theta_1 f(\cdot, x_2, ..., x_n)) \subset \frac{1}{12}\mathcal{A}.$$

So

$$\frac{1}{12}\Theta_1 f(., x_2, ..., x_n) \in \mathcal{F}(\mathcal{W}^n, \mathcal{L}(V))$$

for all $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$. By property (i) of Λ, we get

$$f(x_1, x_2, ..., x_n) - \mathcal{C}(x_1, x_2, ..., x_n) = \Lambda[\frac{1}{12}\Theta_1 f(\cdot, x_2, ..., x_n)] \in \frac{1}{12}\mathcal{A}$$

for all $(x_1, x_2, ..., x_n) \in \mathcal{W}^n$. Therefore, $Im(f - \mathcal{C}) \subset \frac{1}{12}\mathcal{A}$. □

Definition 2.4. Consider an Abelian group $(\mathcal{W}^n, +)$ with a metric d invariant under translation, that is

$$d((x_1 + a_1, x_2 + a_2, ..., x_n + a_n), (y_1 + a_1, y_2 + a_2, ..., y_n + a_n)) = d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n), (a_1, a_2, ..., a_n) \in \mathcal{W}^n$. A metric D is called a metric pair on $\mathcal{W}^n \times \mathcal{W}$ if it is invariant under translation and the following condition holds:

$$\begin{aligned} D((x_1, x_2, ..., x_n, a), (y_1, y_2, ..., y_n, a)) &= D((x_1, x_2, ..., x_n, a), (y_1, y_2, ..., y_n, a)) \\ &= d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) \end{aligned} \tag{2.2}$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$ and $a \in \mathcal{W}$.

Definition 2.5. A mapping $f : \mathcal{W}^n \to \mathcal{V}$ is called Lipschitz mapping of order $\alpha > 0$ if there exists a constant $L > 0$ such that

$$\|f(x_1, x_2, ..., x_n) - f(y_1, y_2, ..., y_n)\| \leq Ld((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))^\alpha$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$.

Let $Lip_\alpha(\mathcal{W}^n, \mathcal{V})$ be the Lipschitz space consisting of all bounded Lipschitz of order $\alpha > 0$ with the norm:

$$\|f\|_{Lip} := \|f\|_{\sup} + lip_\alpha(f),$$

where $\|f\|_{\sup}$ is the supremum norm and

$$lip_\alpha(f) = \sup \frac{\|f(x_1, x_2, ..., x_n) - f(y_1, y_2, ..., y_n)\|}{d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))^\alpha}$$

such that $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$ and $(x_1, x_2, ..., x_n) \neq (y_1, y_2, ..., y_n)$.

Theorem 2.6. Let $(\mathcal{W}^n, +, d, D)$ be a metric pair, $(\mathcal{V}, \|.\|)$ a normed space. Assume that $\mathcal{S}(\mathcal{V})$ is a family of closed balls such that $(\mathcal{W}^n, \mathcal{S}(\mathcal{V}))$ admits MSLIM. Consider a mapping $f : \mathcal{W}^n \to \mathcal{V}$ and let $\gamma : R^+ \to R^+$ be the module of continuity of the mapping $\Theta_1 f$. Then there exists a multi-cubic mapping $\mathcal{C}$ such that $\gamma_{f-\mathcal{C}} = \frac{1}{12}\Theta_1 f$. Moreover, if $\Theta_1 f \in Lip_\alpha(\mathcal{W} \times \mathcal{W}^n, \mathcal{V})$, then

$$\|f - \mathcal{C}\|_{\sup} \leq \frac{1}{12}\|\Theta_1 f\|_{\sup}.$$

Proof . We consider the set-valued function $\Delta : \mathcal{W} \times \mathcal{W}^n \to \mathcal{S}(\mathcal{V})$ defined by

$$\Delta((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) := \inf_{d((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) \le \delta} \gamma_{\Theta_1 f}(\delta)\mathcal{B}(0, 1),$$

where $\mathcal{B}(0, 1)$ is the closed unit ball with center at zero. We have

$$\begin{aligned}\|\Theta_1 f(t, (x_1, x_2, ..., x_n) - \Theta_1 f(t, y_1, y_2, ..., y_n)\| &\le \inf_{D((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) \le \delta} \gamma_{\Theta_1 f}(\delta) \\ &= \inf_{d((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) \le \delta} \gamma_{\Theta_1 f}(\delta)\end{aligned}$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$ and so $\Theta_1 f(t, (., ..., .)$ is Δ-Lipschitz. Now, by Theorem 2.3, there exists a multi-cubic mapping $\mathcal{C}$ such that $f - \mathcal{C}$ is $\frac{1}{12}$-Lipschitz and consequently

$$\|(f - \mathcal{C})(x_1, x_2, ..., x_n) - (f - \mathcal{C})(y_1, y_2, ..., y_n)\| \le \inf_{d((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) \le \delta} \frac{1}{12}\gamma_{\Theta_1 f}(\delta). \tag{2.3}$$

This shows that $\gamma_{f-\mathcal{C}} = \frac{1}{12}\gamma_{\Theta_1 f}$. Moreover, $\|\Theta_1 f\|_{\sup} < \infty$ and clearly $Im\Theta_1 f \subset \|\Theta_1 f\|_{\sup}\mathcal{B}(0, 1)$. By Theorem 2.3, we have

$$\|f - \mathcal{C}\|_{\sup} \le \frac{1}{12}\|\Theta_1 f\|_{\sup}.$$

□

Theorem 2.7. Let $(\mathcal{W}^n, +, d, D)$ be a metric pair, $(\mathcal{V}, \|.\|)$ a normed space. Assume that $\mathcal{S}(\mathcal{V})$ is a family of closed balls such that $(\mathcal{W}^n, \mathcal{S}(\mathcal{V}))$ admits MSLIM. Consider a mapping $f : \mathcal{W}^n \to \mathcal{V}$.
If $\Theta_1 f \in Lip_\alpha(\mathcal{W} \times \mathcal{W}^n, \mathcal{V})$, then there exists a multi-cubic mapping $\mathcal{C}$ such that

$$\|f - \mathcal{C}\|_{Lip} \le \frac{1}{12}\|\Theta_1 f\|_{Lip}.$$

Proof . Define the function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ by the formula $\omega(t) := lip_\alpha(\Theta_1 f)t^\alpha$. Since
$\Theta_1 f \in Lip_\alpha(\mathcal{W} \times \mathcal{W}^n, \mathcal{V})$, we obtain

$$\|\Theta_1 f(t, x_1, x_2, ..., x_n) - \Theta_1 f(t, y_1, y_2, ..., y_n)\| \le \omega(D((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)))$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$ and $t \in \mathcal{W}$, which means that ω is the module of continuity of $\Theta_1 f$. In view of 2.6, there exists a multi-cubic mapping $\mathcal{C}$ such that $\gamma_{f-\mathcal{C}} = \frac{1}{12}\omega$. Then

$$\begin{aligned}\|(f - \mathcal{C})(x_1, x_2, ..., x_n) - (f - \mathcal{C})(y_1, y_2, ..., y_n)\| &\le \frac{1}{12}\omega(d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))) \\ &\frac{1}{12}lip_\alpha(\Theta_1 f)d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n))^\alpha,\end{aligned}$$

for all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathcal{W}^n$, which means that $f - \mathcal{C}$ is a Lipschitz mapping of order α and

$$lip_\alpha(f - \mathcal{C}) \le \frac{1}{12}lip_\alpha(\Theta_1 f).$$

Therefore by 2.6, we get

$$\begin{aligned}\|f - \mathcal{C}\|_{Lip} &= \|f - \mathcal{C}\|_{\sup} + lip_\alpha(f - \mathcal{C}) \\ &\le \frac{1}{12}\|\Theta_1 f\|_{\sup} + \frac{1}{12}lip_\alpha(\Theta_1 f) \\ &\le \frac{1}{12}\|\Theta_1 f\|_{Lip}.\end{aligned}$$

□

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