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Solving system of first kind integral equations via the Chebyshev collocation approach

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Abstract

This paper discusses a numerical method for solving a first-kind Volterra integral equations system. Because of the ill-posedness of these equations, we need to apply an efficient computational method to discrete them to the system of algebraic equations. An expansion method known as the Chebyshev collocation method, based on the Chebyshev polynomials of the third kind, is employed to convert the system of integral equations to the linear algebraic system of equations. By solving the algebraic system, we conclude an approximate solution. Some numerical results support the accuracy and efficiency of the stated method.

Keywords: System of first-kind Volterra integral equations, Chebyshev polynomials of the third-kind, Collocation method, Absolute error
2020 MSC: 45D05, 45F05, 33C45, 65M70, 15A30

1 Introduction

System of linear and nonlinear integral equation of the first-kind is appeared in many branches of science and advanced technology, and since the theoretical solutions are not available for most of these types of problems, numerical methods are valuable and the focus of study [3, 14, 16, 17, 19]. Daily progress in different fields and modeling of relevant phenomena causes the creation of different equations, for which it is especially important to find a suitable and efficient numerical solution [7, 10, 12, 15]. System of first-kind Volterra integral equations is defined by

$$\mathbf{f}(x) = \int_a^x \mathbf{k}(x,t)\ \mathbf{u}(t)dt, \tag{1.1}$$

so that

$$\begin{aligned} \mathbf{f}(x) &= [f_i(x)], \\ \mathbf{u}(x) &= [u_i(x)], \\ \mathbf{k}(x,t) &= [k_{i,j}(x,t)], \quad i,j = 1,2,...,L, \end{aligned}$$

where $k_{i,j}(x,t)$ and $f_i(x)$ are known functions and $u_i(x)$ are unknown functions, $a \in \mathbb{R}$ and x is a variable.

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Integral equations appear both in the way of solving differential problems by inverting differential operators, and in describing phenomena by models that require summations on space or time or both. In the modeling of physics phenomena, especially quantum mechanics and statistical mechanics, the appearance of integral equations attracts more attention.

The rest of the manuscript is organized as follows. In Section 2, we explain some basic concepts and describe the process of implementing the method for approximating the solution of the system of first-kind Volterra integral equations. In Section 3, we present two examples of the equations studied in this article to test the possibility of implementing the presented method and the accuracy of the approximate solutions. Finally, we end the article by stating the conclusion.

2 Basic concepts and method implementation

We express the Chebyshev polynomials of the third-kind on the interval $[-1,1]$ based on the Chebyshev polynomials of the first-kind. The Chebyshev polynomial of the third-kind on $[-1,1]$ of degree n is denoted by V_n and is defined by [13, 18]

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad n = 1, 2, \dots, \tag{2.1}$$

so that $V_0(x) = 1$ and $V_1(x) = 2x - 1$. These polynomials are orthogonal with respect to the weight function $\omega(x) = \sqrt{\frac{1+x}{1-x}}$. We have the following relationship between the Chebyshev polynomials of the third-kind and the Chebyshev polynomials of the first-kind,

$$V_n(x) = \sqrt{\frac{2}{1+x}}T_{2n+1}\left(\sqrt{\frac{1+x}{2}}\right),$$

and we can obtain the properties and relations of the third-kind from the first-kind with minor changes, where $T_n(x)$ is the Chebyshev polynomial of the first-kind on $[-1,1]$ of degree n and these polynomials are given by the following recursive formula [1],

$$T_0(x) = 1, \qquad T_1(x) = x,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad n = 1, 2, \dots.$$

Using the Chebyshev polynomial of the third-kind, we apply the collocation method to convert Eq. (1.1) to an algebraic system of linear equations $AX = b$. We approximate $u_i(x)$'s, such that

$$u_i(x) \simeq \sum_{k=0}^{m} c_{ik}V_k(x), \tag{2.2}$$

where $V_k(x)$ is the kth Chebyshev polynomial of the third-kind and c_{ik}'s are unknown coefficients which are determined by solving an algebraic system. By substituting relation (2.2) in Eq. (1.1) we have

$$f_1(x) = \sum_{i=1}^{L}\int_a^x k_{1i}(x,t)\sum_{k=0}^{m} c_{ik}V_k(t)dt,$$
$$f_2(x) = \sum_{i=1}^{L}\int_a^x k_{2i}(x,t)\sum_{k=0}^{m} c_{ik}V_k(t)dt,$$
$$\vdots$$
$$f_L(x) = \sum_{i=1}^{L}\int_a^x k_{Li}(x,t)\sum_{k=0}^{m} c_{ik}V_k(t)dt.$$

Because the Chebyshev polynomials are orthogonal polynomials in $[-1,1]$, we select the following transformation

$$t = \frac{x-a}{2}\tau + \frac{x+a}{2},$$

and let

$$\bar{k}(x,\tau) = k(x, \frac{x-a}{2}\tau + \frac{x+a}{2}),$$
$$\bar{V}(x,\tau) = \frac{x-a}{2}V_k(\frac{x-a}{2}\tau + \frac{x+a}{2}),$$

so that

$$f_1(x) = \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{1i}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau,$$
$$f_2(x) = \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{2i}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau,$$
$$\vdots$$
$$f_L(x) = \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{Li}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau.$$

Now, we choose some collocation points such as

$$x_i = -1 + \frac{2i}{m} \quad \text{for} \quad i = 0,1,\ldots,m,$$

which are equidistant, also define system of residual equations by

$$R_1(x) = f_1(x) - \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{1i}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau,$$
$$R_2(x) = f_2(x) - \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{2i}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau,$$
$$\vdots$$
$$R_L(x) = f_L(x) - \sum_{i=1}^{L}\int_{-1}^{1}\bar{k}_{Li}(x,\tau)\sum_{k=0}^{m}c_{ik}\bar{V}_k(x,\tau)d\tau.$$

Then, by imposing the conditions

$$R_i(x_j) = 0 \quad \text{for} \quad i = 1,2,\ldots,L, \quad j = 0,1,\ldots,m;$$

we can conclude algebraic system of linear equations $AX = b$ [2, 8].

For example, for $L = 3$ we have;

$$\begin{cases} f_1(x) = \int_a^x k_{11}(x,t)u_1(t)dt + \int_a^x k_{12}(x,t)u_2(t)dt + \int_a^x k_{13}(x,t)u_3(t)dt, \\ f_2(x) = \int_a^x k_{21}(x,t)u_1(t)dt + \int_a^x k_{22}(x,t)u_2(t)dt + \int_a^x k_{23}(x,t)u_3(t)dt, \\ f_3(x) = \int_a^x k_{31}(x,t)u_1(t)dt + \int_a^x k_{32}(x,t)u_2(t)dt + \int_a^x k_{33}(x,t)u_3(t)dt, \end{cases} \tag{2.3}$$

after discretization, the algebraic system of linear equations $AX = b$ is concluded as follow;

$$A = (a_{ij}), \qquad i,j = 1,2,\ldots,3m+3,$$
$$b^T = [f_1(x_0), f_1(x_1),\ldots,f_1(x_m), f_2(x_0), f_2(x_1),\ldots,f_2(x_m), f_3(x_0), f_3(x_1),\ldots,f_3(x_m)],$$
$$X^T = [c_{10}, c_{11},\ldots,c_{1m}, c_{20}, c_{21},\ldots,c_{2m}, c_{30}, c_{31},\ldots,c_{3m}],$$

where

$$
a_{ij}=\begin{cases}
\int_{-1}^{1}\bar{k}_{11}(x_{i-1},\tau)\bar{V}_{j-1}(x_{i-1},\tau)d\tau, & \begin{cases} i=1,2,\dots,m+1\\ j=1,2,\dots,m+1\end{cases}\\
\int_{-1}^{1}\bar{k}_{12}(x_{i-1},\tau)\bar{V}_{j-m-2}(x_{i-1},\tau)d\tau, & \begin{cases} i=1,2,\dots,m+1\\ j=m+2,m+3,\dots,2m+2\end{cases}\\
\int_{-1}^{1}\bar{k}_{13}(x_{i-1},\tau)\bar{V}_{j-2m-3}(x_{i-1},\tau)d\tau, & \begin{cases} i=1,2,\dots,m+1\\ j=2m+3,2m+4,\dots,3m+3\end{cases}\\
\int_{-1}^{1}\bar{k}_{21}(x_{i-m-2},\tau)\bar{V}_{j-1}(x_{i-m-2},\tau)d\tau, & \begin{cases} i=m+2,m+3,\dots,2m+2\\ j=1,2,\dots,m+1\end{cases}\\
\int_{-1}^{1}\bar{k}_{22}(x_{i-m-2},\tau)\bar{V}_{j-m-2}(x_{i-m-2},\tau)d\tau, & \begin{cases} i=m+2,m+3,\dots,2m+2\\ j=m+2,m+3,\dots,2m+2\end{cases}\\
\int_{-1}^{1}\bar{k}_{23}(x_{i-m-2},\tau)\bar{V}_{j-2m-3}(x_{i-m-2},\tau)d\tau, & \begin{cases} i=m+2,m+3,\dots,2m+2\\ j=2m+3,2m+4,\dots,3m+3\end{cases}\\
\int_{-1}^{1}\bar{k}_{31}(x_{i-2m-3},\tau)\bar{V}_{j-1}(x_{i-2m-3},\tau)d\tau, & \begin{cases} i=2m+3,2m+4,\dots,3m+3\\ j=1,2,\dots,m+1\end{cases}\\
\int_{-1}^{1}\bar{k}_{32}(x_{i-2m-3},\tau)\bar{V}_{j-m-2}(x_{i-2m-3},\tau)d\tau, & \begin{cases} i=2m+3,2m+4,\dots,3m+3\\ j=m+2,m+3,\dots,2m+2\end{cases}\\
\int_{-1}^{1}\bar{k}_{33}(x_{i-2m-3},\tau)\bar{V}_{j-2m-3}(x_{i-2m-3},\tau)d\tau, & \begin{cases} i=2m+3,2m+4\dots,3m+3\\ j=2m+3,2m+4,\dots,3m+3.\end{cases}
\end{cases}
$$

3 Numerical Experiments

We use the method presented in Section 2 for two examples, one of which is a system of equations with two unknown functions and another system with three unknown functions. Our aim is to approximate the solution of Eq. (1.1) by employing the Chebyshev polynomial of the third-kind together with the collocation approach. We present some examples of first-kind system of Volterra integral equations which illustrate the accuracy and efficiency of stated method in comparison with other methods.

Example 3.1. In Eq. (1.1) with $a=0$, for $L=2$ if let

$$
\begin{cases}
k_{11}(x,t)=\sin(x-2t)\\
k_{12}(x,t)=t\cos(x-t)\\
k_{21}(x,t)=xt^2\\
k_{22}(x,t)=e^{t-x}\\
f_1(x)=\frac{4}{3}\sin^4\left(\frac{x}{2}\right)+\frac{1}{2}\left(xe^x-\sin(x)\right)\\
f_2(x)=x\left(\left(x^2-2\right)\sin(x)+2x\cos(x)\right)+\sinh(x)
\end{cases}
\tag{3.1}
$$

where the exact solutions are $u_1(x)=\cos(x)$ and $u_2(x)=e^x$, the numerical results for $m=9$ and $m=12$ are reported in Table 1. In this table, E_u represents the absolute error of the approximations. In Figures 1 and 2, the results for $m=10$ and $m=15$ are shown by using the Chebyshev polynomials of the third-kind basis functions and the collocation points $x=-1,-0.8,-0.6,\dots,1$ and $x=-1,-\dfrac{13}{15},-\dfrac{11}{15},\dots,1$, respectively.

Table 1: Numerical results for Example 3.1

	E_{u_1}		E_{u_2}	
x	$m = 9$	$m = 12$	$m = 9$	$m = 12$
-1	2.721×10^{-3}	1.393×10^{-4}	1.0961×10^{-4}	7.3153×10^{-5}
-0.8	3.2173×10^{-5}	6.6618×10^{-6}	7.1990×10^{-6}	5.2955×10^{-7}
-0.6	6.3122×10^{-5}	4.4286×10^{-6}	2.4554×10^{-6}	4.5564×10^{-7}
-0.4	4.7988×10^{-5}	7.1720×10^{-6}	4.8132×10^{-6}	8.1277×10^{-7}
-0.2	3.4338×10^{-5}	6.7711×10^{-6}	8.1532×10^{-7}	4.4437×10^{-7}
0	9.0221×10^{-7}	6.9005×10^{-7}	5.2992×10^{-6}	2.3222×10^{-7}
0.2	7.7710×10^{-6}	1.0255×10^{-6}	5.5002×10^{-6}	3.0475×10^{-8}
0.4	1.8036×10^{-5}	7.6324×10^{-7}	3.7760×10^{-6}	7.8826×10^{-7}
0.6	3.3946×10^{-5}	4.4418×10^{-6}	3.1449×10^{-6}	6.9553×10^{-7}
0.8	5.0989×10^{-5}	5.3372×10^{-6}	4.0979×10^{-5}	1.1105×10^{-5}
1	4.2933×10^{-4}	7.6441×10^{-5}	5.2170×10^{-5}	4.3385×10^{-5}

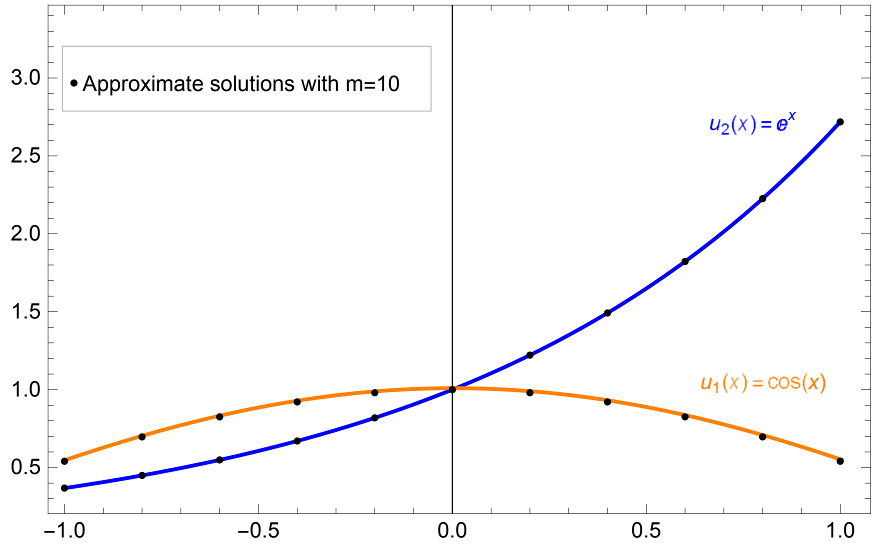

Figure 1: The exact solution and approximate solution related to system (3.1) with $m = 10$

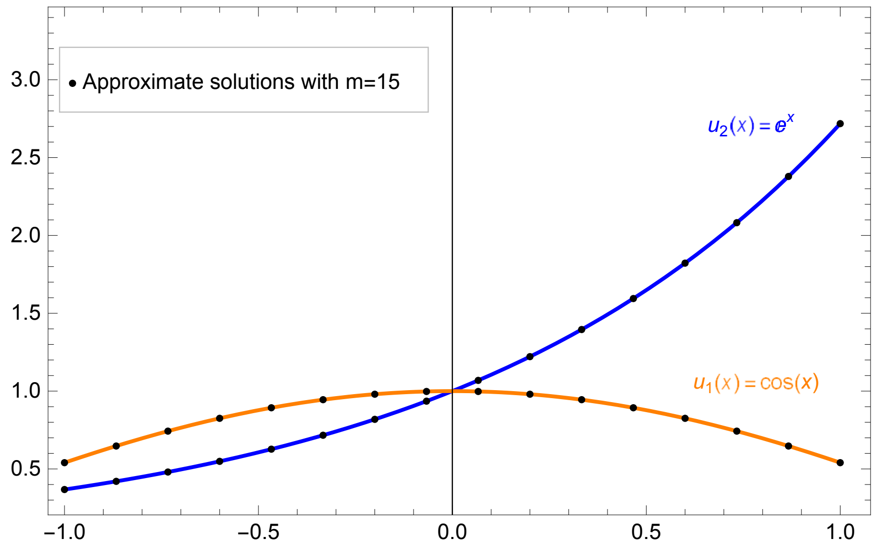

Figure 2: The exact solution and approximate solution related to system (3.1) with $m = 15$

Example 3.2. We try to solve the system of equations

$$\begin{cases} f_1(x) = \int_{-1}^{x}(x-t)u_1(t)\,dt + \int_{-1}^{x}(2t+x-1)u_2(t)\,dt + \int_{-1}^{x}\cos(x-t)u_3(t)\,dt, \\ f_2(x) = \int_{-1}^{x}\left(t-x^2\right)u_1(t)\,dt + \int_{-1}^{x}e^{t-2x}u_2(t)\,dt + \int_{-1}^{x}(t+x)u_3(t)\,dt, \\ f_3(x) = \int_{-1}^{x}e^{2t+x}u_1(t)\,dt + \int_{-1}^{x}\left(t^2-3x\right)u_2(t)\,dt + \int_{-1}^{x}\cos(t+2x)u_3(t)\,dt, \end{cases} \tag{3.2}$$

where

$$\begin{cases} f_1(x) = \frac{7x^3}{3} + x^2 + ex - x + e^{-x} + \frac{1}{30}(5\cos(3-x) - 8\cos(4x) + 3\cos(x+5)) + \frac{1}{3}, \\ f_2(x) = -e^{-x}\left(\left(e^{x+1}-1\right)x^2 + x + 1\right) + e^{-2x-1}\left(e^{x+1}(2x-1)+3\right) \\ \qquad + \frac{1}{16}(\sin(4x) + 4x\cos(4) - 8x\cos(4x) + \sin(4) - 4\cos(4)), \\ f_3(x) = -\frac{1}{6} + \frac{x^4}{2} - \frac{8x^3}{3} - 3x^2 + e^{x-1}\left(e^{x+1}-1\right) \\ \qquad + \frac{1}{30}(-5\cos(x) - 3\cos(7x) + 5\cos(2x+3) + 3\cos(5-2x)). \end{cases} \tag{3.3}$$

and the exact solutions are $u_1(x) = e^{-x}$, $u_2(x) = 2x+1$ and $u_3(x) = \sin(4x)$. Absolute errors of the Chebyshev collocation method with $m = 10$ and $m = 15$ are reported in Table 2. Figures 3 and 4 show the results for $m = 8$ and $m = 12$ by using the Chebyshev polynomials of the third-kind basis functions and the collocation points $x = -1, -\frac{3}{4}, -\frac{1}{2}, \ldots, 1$ and $x = -1, -\frac{5}{6}, -\frac{2}{3}, \ldots, 1$, respectively.

Table 2: Numerical results for Example 3.2

	E_{u_1}		E_{u_2}		E_{u_3}	
x	$m=10$	$m=15$	$m=10$	$m=15$	$m=10$	$m=15$
-1	2.0777×10^{-3}	3.8512×10^{-4}	8.5544×10^{-5}	2.3550×10^{-5}	4.3420×10^{-2}	2.4487×10^{-3}
-0.8	7.3445×10^{-4}	1.2008×10^{-4}	5.6090×10^{-5}	8.8772×10^{-6}	8.2884×10^{-3}	6.3555×10^{-3}
-0.6	5.3444×10^{-4}	5.3242×10^{-5}	8.8801×10^{-6}	4.4339×10^{-6}	3.0222×10^{-2}	2.3311×10^{-2}
-0.4	4.8804×10^{-4}	4.5662×10^{-5}	7.6605×10^{-5}	5.7766×10^{-6}	2.3304×10^{-2}	7.1648×10^{-3}
-0.2	7.7550×10^{-5}	4.6602×10^{-5}	7.0743×10^{-5}	2.0110×10^{-5}	8.5505×10^{-3}	5.1997×10^{-3}
0	2.3225×10^{-5}	8.6238×10^{-6}	9.4427×10^{-6}	3.3379×10^{-6}	4.4478×10^{-3}	2.2280×10^{-3}
0.2	3.3323×10^{-4}	9.7755×10^{-6}	8.8808×10^{-6}	5.7243×10^{-6}	6.4886×10^{-3}	6.7744×10^{-3}
0.4	1.7553×10^{-4}	5.5505×10^{-5}	1.0322×10^{-5}	4.2525×10^{-6}	3.3385×10^{-2}	9.3361×10^{-3}
0.6	4.6629×10^{-4}	3.7090×10^{-5}	5.1554×10^{-5}	6.2774×10^{-6}	3.0675×10^{-2}	1.6640×10^{-2}
0.8	4.1965×10^{-5}	7.6167×10^{-5}	6.5072×10^{-5}	4.2525×10^{-5}	5.9950×10^{-2}	6.4433×10^{-2}
1	3.5569×10^{-4}	1.1990×10^{-4}	5.3225×10^{-4}	4.6346×10^{-5}	7.0505×10^{-3}	8.3774×10^{-3}

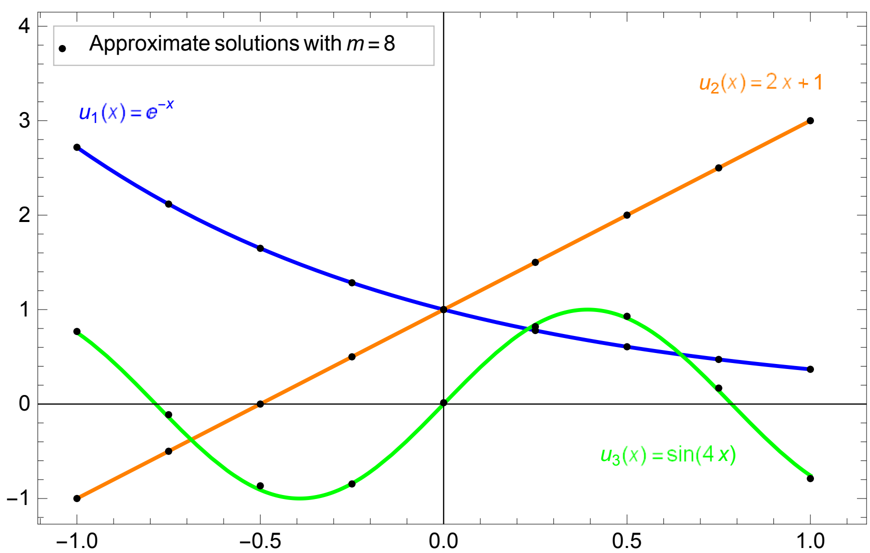

Figure 3: The exact solution and approximate solution related to system (3.2) and (3.3) with $m = 8$

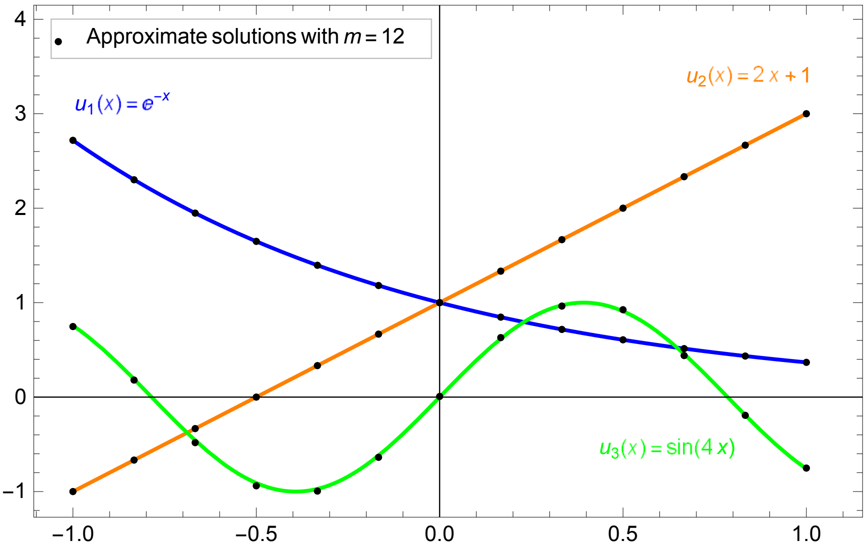

Figure 4: The exact solution and approximate solution related to system (3.2) and (3.3) with $m = 12$

Conclusion

In this paper, a projection method known as collocation method, based on the Chebyshev polynomials of the third-kind, are chosen to discrete and solve the system of the first-kind integral equations. The presented method has some advantages; this method is easy to apply, and we need less computations than other methods [4, 5, 6, 9, 11]. By using this method, we can get high accuracy, by solving an algebraic system of linear equations with rank less than 10×10, for many systems of integral equations.

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