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Existence of solutions to a certain type of non-linear difference-differential equations

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Abstract

The purpose of this paper is to investigate the finite-order transcendental entire solutions to specific types of non-linear differential-difference equations. Moreover, our results generalize some of the previous results. Some examples are provided to show that our results are best in certain sense.

Keywords: Entire function, Nevanlinna theory,Non-linear difference equations and Meromorphic function
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1 Introduction

Throughout the paper, we assume that the reader has prior knowledge of the fundamental results and standard notations of Nevanlinna theory. The terms $T(r,f)$, $N(r,f)$ and $m(r,f)$ represents the characteristic function, counting and proximity functions of f. Whenever $S(r,f)$ is defined, it has the property that $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of any set E of finite logarithmic measure. We say that function $h(z)$ is a small meromorphic with respect to $f(z)$ if and only if $T(r,h) = o(S(r,f))$. Specifically, Nevanlinna's theory plays an extremely important role to analyze the existence and solvability of non-linear differential, difference and differential-difference equations.

In 1964, Hayman [8] investigated the following non-linear differential equation

$$f^n + H_d(f) = F(z), \tag{1.1}$$

where d is the degree of the differential polynomial H_d and the result is:

Theorem 1.1. [8] If f and $F(z)$ be non-constant meromorphic functions and $n \geq d+1$ in (1.1). If $N(r,f)+N\left(r,\frac{1}{F}\right) = S(r,f)$, then $F = (f+\nu)^n$, where ν is small meromorphic function of f.

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An extension of Tumura–Clunie theory is *Theorem* 1.1 it is based on a theorem suggested by Tumura[18]. However, the proof of which was completed by Clunie[5]. Consequently, many studies have been done on the non-linear differential equation (1.1) by considering various forms of $F(z)$. One can refer [10, 13, 12, 16] for more details about non-linear differential equations.

In recent times, several authors have been interested in investigating the solution of the following type of equation

$$f^n + H_{d_*}(z, f) = h_1(z)e^{v_1(z)} + h_2(z)e^{v_2(z)}, \tag{1.2}$$

where d is the degree of the differential polynomial H_{d_*} and $v_1(z)$, $v_2(z)$, h_1 and h_2 are polynomials. There are a few works that are relevant to the topics that can be seen in [9, 21, 13, 17, 7, 1]. For instance, *Liu et al.* [15] studied the existence of meromorphic solution of (1.2) and the result is:

Theorem 1.2. [15] Let $n \geq 3$ be an integer and $d_* \leq n-2$ be the degree of differential polynomial H_{d_*}. Consider the polynomials v_1, v_2 of degree $k(\geq 1)$ and h_1, h_2 be two small non-zero meromorphic functions of e^{z^k}. If $\frac{v_1^{(k)}}{v_2^{(k)}} \notin \{\frac{n}{n-1}, \frac{n-1}{n}, -1, 1\}$,

and any one of the these occur:

1. $H_{d_*} \not\equiv 0$.

2. $H_{d_*} \equiv 0$, $\frac{v_1^{(k)}}{v_2^{(k)}} \notin \{\frac{n}{d_*}, \frac{d_*}{n}\}$, then (1.2)

does not have the meromorphic transcendental solution f with $N(r, f) = S(r, f)$.

L. W. Liao et al. [13] studied the differential equation of the form

$$f^n f' + H_{d_*}(z, f) = \xi(z)e^{p(z)}, \tag{1.3}$$

and obtained the result by taking $\xi(z)(\neq 0)$ as rational function and $p(z)$ as non-constant polynomial.

Theorem 1.3. [13] Let f be a meromorphic solution of (1.3) with finite number of poles, then

$$H_{d_*} \equiv 0, \quad f(z) = s(z)e^{\frac{p(z)}{n+1}}$$

for $d_* \leq n-1$ and the rational function $s(z)$ satisfies $s^n\left[(n+1)s' + p's\right] = (n+1)\xi$.

In 2012, Z. T. Wen et al. [19] classified certain non-linear difference equation of the form

$$f^n + h(z)e^{H(z)}f(z+c) = Q(z), \tag{1.4}$$

examined the entire solution of finite order. Later, 2017 *M. F. Chen et al.*[2] studied the existence of finite-order entire solutions of following non-linear difference equations

$$f^n + q(z)\Delta_c f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},\ n \geq 2$$

and

$$f^n + q(z)e^{Q(z)}f(z+c) = p_1 e^{\lambda z} + p_2 e^{-\lambda z},\ n \geq 3$$

where q, Q are non-zero polynomials, c, λ, p_i, $\alpha_i(i = 1, 2)$ are non-zero constants.

In this paper we consider the following non-linear difference equation of the form:

$$f^n + \xi(z)f(z+c)e^{G(z)} = h_1(z)\, e^{v_1(z)} + h_2(z)\, e^{v_2(z)}, \tag{1.5}$$

where n be an integer, $c \in \mathbb{C}\setminus\{0\}$, $h_1(z)$, $h_2(z)$ be non-zero small functions of f and $\xi(z)$, $G(z)$ [$G(z)$ *is non constant*], $v_1(z)$ and $v_2(z)$ are non-zero polynomials, and the result is:

Theorem 1.4. If f is finite-order transcendental entire solution of (1.5) with $n \geq 3$ and $deg\ v_1 \neq deg\ v_2$, then the following holds:

1. Suppose $deg\ v_1 < deg\ v_2$ and $\rho(f) = deg\ v_1$, then every solution of f satisfies $\rho(f) < max\{deg\ v_1, deg\ v_2\} = deg\ G$ and $f = \beta_2 e^{\frac{v_1}{n}}$, where ${\beta_2}^n = p_1$.
2. Suppose $deg\ v_1 < deg\ v_2$ and $\rho(f) \geq deg\ v_2$, then every solution of f satisfies $\rho(f) = deg\ G \geq max\{deg\ v_1, deg\ v_2\}$. Similarly we can get for $deg\ v_2 < deg\ v_1$, $\rho(f) \geq deg\ v_1$

Following are two examples that illustrate the sharpness of our result.

Example 1.5. Let $f = ze^{\frac{z}{3}}$ be a finite-order transcendental entire solution of the difference equation

$$f^3 + ze^{z^2} f(z+1) = z^3 e^z + (z^2+z)e^{z^2+\frac{z+1}{3}},$$

Here $n = 3$, $\xi(z) = z$, $G(z) = z^2$, $c = 1(\neq 0)$, $h_1(z) = z^2$, $h_2(z) = z^2 + z$, $v_1(z) = 2z$ and $v_2(z) = z^2 + \frac{z+1}{3}$. Then clearly we can see that $deg\ v_1 = 1 < 2 = deg\ v_2$ and $\rho(f) = deg\ v_1 = 1$, $\rho(f) = 1 < max\{1,2\} = 2 = deg\ G$ and $f = ze^z$. Thus, the conclusion (i) of the *Theorem* (1.4) holds.

Example 1.6. Let $f = ze^{-z^2}$ is a transcendental entire solution of finite order of the difference equation

$$f^3 + ze^{z^2+1} f(z+1) = (z^2+z)e^{-2z} + z^3 e^{-3z^2},$$

Here $n = 3$, $\xi(z) = z$, $G(z) = z^2 + 1$, $c = 1(\neq 0)$, $h_1(z) = z^2 + z$, $h_2(z) = z^2$, $v_1(z) = -2z$ and $v_2(z) = -3z^2$. Clearly $deg\ v_1 = 1 < 2 = deg\ v_2$ and $\rho(f) = 2 = deg\ G \geq max\{1,2\}$ and $f = ze^{-z^2}$. Thus, the conclusion (ii) of the *Theorem* (1.4) holds.

Later in 2016, K. Liu[14] studied the transcendental finite-order entire solutions to the differential-difference equation

$$f^n + h(z)e^{H(z)} f^{(k)}(z+c) = Q(z), \tag{1.6}$$

where $n \geq 2$, $(k \geq 1)$ is an integer, $c \in \mathbb{C}\setminus\{0\}$ and $h(z)(\neq 0)$, $Q(z)$ are polynomials and $H(z)$ is a polynomial of degree ≥ 1. Eventually, Chen et. al[2] and Xu et al. [20] replaced $Q(z)$ in (1.4), (1.6) by $p_1 e^{\eta z} + p_2 e^{-\eta z}$ and $p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$, where p_1, p_2, η, α_1 and α_2 are constants, obtained the results. Later, in 2020, W. Chen et al.[3] investigated the following non-linear differential-difference equation

$$f^n + af^{n-1}f' + \xi(z)e^{H(z)} f(z+c) = q(z)e^{p(z)}, \tag{1.7}$$

where $n \in I^+$, q, H, r, p are polynomials of degree ≥ 1, $c \neq 0$ and a are constants, proved the following result.

Theorem 1.7. [3] Let $n \in I$, $n \geq 3$ when $a \neq 0$ and $n \geq 2$ when $a = 0$. Let f be a entire non-vanishing transcendental solution to (1.7) with finite order. Thus, each solution f yields any of the following

1. $\rho(f) < deg\ p = deg\ H$ and $f = Ce^{\frac{-z}{a}}$, where C is constant.
2. $\rho(f) = deg\ H \geq deg\ p$.

In the same paper, the author also proved the solutions of equation (1.7), where $q(z)e^{p(z)}$ replaced by $p_1 e^{\lambda z} + p_2 e^{-\lambda z}$, λ, p_1 and p_2 are non-zero constants. In 2021, *Nan Li et al.* obtained the result to the equation (1.7) for the case $n = 2$ and $a = 0$ and also replaced $q(z)e^{p(z)}$ by $p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$, where p_1, p_2, α_1 and α_2 are non-zero constants, and proved the existence of entire solutions.

Theorem 1.8. [11] Let c, $a \neq 0$ be constants, ξ, G , q , p be polynomials such that G, p are not constants and ξ, $q \neq 0$. Suppose that f is a transcendental entire solution with finite order of the equation

$$f^2 + aff' + \xi(z)e^{G(z)} f(z+c) = q(z)e^{p(z)}, \tag{1.8}$$

satisfying $\lambda(f) < \rho(f)$, then $deg\ G = deg\ p$, and one among the following relations holds:

1. $\rho(f) < deg\ G = deg\ p$, and $f = Ce^{\frac{-z}{a}}$
2. $\rho(f) = deg\ G = deg\ p$.

It is also fascinating to explore the finite-order entire solutions of the following differential-difference equation

$$f^n + \eta f^{n-1} f' + s(z) e^{G(z)} f^{(k)}(z+c) = h(z) e^{p(z)}, \tag{1.9}$$

where $n > 0$ be an integer, $\eta \neq 0$, $c \in \mathbb{C} \setminus \{0\}$ and $h(z)$, $s(z)$, $G(z)(\geq 1)$ and $p(z)$ are non-constant polynomials, and the result is:

Theorem 1.9. Let f be a non-vanishing finite-order transcendental entire solution of (1.9), $\eta \neq 0$ when $n \geq 3$ and $\eta = 0$ when $n \geq 2$. Then each solution f satisfies any one of the following:

1. $\rho(f) < deg\ p = deg\ G$ and $f = Ce^{\frac{-z}{\eta}}$.
2. $\rho(f) = deg\ G \geq deg\ p$.

Following are two examples that illustrate the sharpness of our result.

Example 1.10. Let $f = e^{-z}$ be a finite-order transcendental entire solution of the differential-difference equation

$$f^3 + f^2 f' + z f^{(2)}(z+1) e^{z^2+z+1} = z e^{z^2}.$$

Example 1.11. Let $f = e^{z^2}$ is a transcendental entire solution of finite order of the differential-difference equation

$$f^3 + f^2 f' + z f'(z+1) e^{2z^2-2z-1} = (2z^2 + 4z + 1) e^{3z^2}.$$

Thus, by above examples we can see that the conclusion (i) and (ii) holds.

2 Preliminaries

Lemma 2.1. [22] If $f_k(z)$, $1 \leq k \leq m$, and $g_k(z)$, $1 \leq k \leq m$, $m \geq 2$ are entire functions that meet conditions listed below

1. $\sum\limits_{k=1}^{m} f_k(z) e^{g_k(z)} \equiv 0$,
2. The orders of f_k are less than that of $e^{g_l(z) - g_n(z)}$ for $1 \leq k \leq m$, $1 \leq k \leq l < n \leq m$, then $f_k \equiv 0$ for $1 \leq k \leq m$.

Lemma 2.2 ([6] Clunie's lemma). Let f be a non-constant finite order meromorphic solution of

$$f^n(z) P(z, f) = Q(z, f)$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in f with small meromorphic function as coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z, f)$ is a polynomial in f and its shifts are at most n, then

$$m\left(r, P(z, f)\right) = o\left(\frac{T(r + \mid c \mid, f)}{r^\delta}\right) + o\left(T(r, f)\right)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.3. [10] Assume that $f(z)$ be a transcendental meromorphic function, p, q, r and s are small functions of f with $prs \not\equiv 0$. If $pf^2 + qff' + r(f')^2 = s$, then

$$r(q^2 - 4pr)\frac{s'}{s} + q(q^2 - 4pr) - r(q^2 - 4pr)' + (q^2 - 4pr) r' \equiv 0.$$

Lemma 2.4. [4] Let f be a non-constant meromorphic function and η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$. Let $f(z)$ be a meromorphic function with finite order σ, then each $\epsilon > 0$, then

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma - 1 + \epsilon}\right).$$

Lemma 2.5. [22] Let f be a non-zero meromorphic function. Then

$$m\left(r, \tfrac{f'}{f}\right) = O(logr) \text{ as } r \to \infty$$

if f is finite order, and

$$m\left(r, \tfrac{f'}{f}\right) = O(logr(T(r,f))) \text{ as } r \to \infty$$

possibly outside a set E of r with finite linear measure if f is of infinite order.

Lemma 2.6. [4] Let $f(z)$ be a meromorphic function with order $\rho(f) < \infty$, and let η be a fixed non-zero complex number, then for each $\epsilon > 0$, we have $T(r, f(z+c)) = T(r,f) + O(r^{\rho-1+\epsilon}) + O(log\, r)$.

Lemma 2.7. [22] Let f be a meromorphic function in the complex plane that is not constant and k is a positive integer. Then we have the following inequality

$$N\left(r, \tfrac{1}{f^{(k)}}\right) \leq N\left(r, \tfrac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Proof of Theorem 1.4:

Suppose f be a transcendental entire solution of finite order to (1.5) and now, in order to prove the theorem, we will look at the following cases:

Case 1. If $\rho(f) < max\{deg\ v_1, deg\ v_2\}$, then from (1.5) and lemma 2.4, it follows that

$$\begin{aligned}
T(r, e^G) &= m(r, e^G), \\
&= m\left(r, \frac{h_1 e^{v_1} + h_2 e^{v_2} - f^n}{\xi f(z+c)}\right), \\
&\leq m\left(r, \frac{f}{\xi f(z+c)}\right) + m\left(r, \frac{1}{f}\right) + m\left(r, h_1 e^{v_1} + h_2 e^{v_2}\right) + nm(r,f) + S(r,f), \\
&= (n+1)T(r,f) + T\left(r, h_1 e^{v_1} + h_2 e^{v_2}\right) + S(r,f).
\end{aligned}$$

i.e $T(r, e^G) \leq T\left(r, h_1 e^{v_1} + h_2 e^{v_2}\right) + S(r,f)$, which implies

$$deg\ G \leq \max\{deg\ v_1, deg; v_2\}. \tag{2.1}$$

Meanwhile, we have from (1.5) and Lemma 2.4 that

$$\begin{aligned}
T\left(r, h_1 e^{v_1} + h_2 e^{v_2}\right) &= m\left(r, f^n + \xi f(z+c) e^G\right) + S(r,f), \\
&\leq nm(r,f) + m(r, e^G) + m\left(r, \frac{f(z+c)}{f}\right) + m\left(r, \frac{1}{f}\right) + S(r,f), \\
&= (n+1)T(r,f) + T\left(r, e^G\right) + S(r,f), \\
&\leq T\left(r, e^G\right) + S(r,f),
\end{aligned}$$

which implies

$$\max\{deg\ v_1, deg\ v_2\} \leq deg\ G. \tag{2.2}$$

From (2.1) and (2.2), we have

$$deg\ G = \max\{deg\ v_1, deg\ v_2\} \quad and \quad \rho(f) < deg\ G.$$

For convenience, we write $f(z+c) = f_c,\ G(z) = G$, similarly for $h_1, h_2,\ v_1$ and v_2, then (1.5) take the form

$$f^n + \xi f_c e^G = h_1 e^{v_1} + h_2 e^{v_2}, \tag{2.3}$$

By differentiating (2.3),we get

$$nf^{n-1}f' + A\xi f_c e^G = h_1 A_1 e^{v_1} + h_2 A_2 e^{v_2}, \tag{2.4}$$

where $A = \frac{\xi'}{\xi} + \frac{f_c'}{f_c} + G'$, $A_1 = \frac{h_1'}{h_1} + v_1'$ and $A_2 = \frac{h_2'}{h_2} + v_2'$ are small functions of f. Eliminating e^{v_1} and e^{v_2} from (2.3) and (2.4), we get

$$A_1 f^n - nf^{n-1}f' + (A_1 - A)\xi f_c e^G = h_2(A_1 - A_2)e^{v_2}, \tag{2.5}$$

$$A_2 f^n - nf^{n-1}f' + (A_2 - A)\xi f_c e^G = -h_1(A_1 - A_2)e^{v_1}, \tag{2.6}$$

since $deg\ v_1 \neq deg\ v_2$, clearly $A_1 - A_2 \neq 0$. We have $deg\ v_1 < deg\ v_2$ and $deg\ v_1 = \rho(f)$, differentiating (2.5) and eliminating e^{v_2}, we get

$$B_3 e^G + B_4 = 0, \tag{2.7}$$

where

$$B_3 = \left[A_4 - \left(\frac{(A_1 - A)'}{A_1 - A} + \frac{\xi'}{r} + \frac{f_c'}{f_c} + G'\right)\right](A_1 - A)\,\xi f_c$$

$$B_4 = f^{n-2}\left[(A_1 A_4 - A_1')\, f^2 - n\,(A_1 + A_4)\, ff' + n(n-1)(f')^2 + nff''\right]$$

and

$$A_4 = \left(\frac{h_2'}{h_2} + v_2' + \frac{(A_1 - A)'}{A_1 - A}\right).$$

Since $\rho(f) < deg\ G$, by (2.7) and lemma 2.1, we get $B_3 \equiv B_4 \equiv 0$. From $B_3 \equiv 0$, we must have either $A_1 - A \equiv 0$ or $\left[A_4 - \left(\frac{(A_1-A)'}{A_1-A} + \frac{\xi'}{\xi} + \frac{f_c'}{f_c} + G'\right)\right] \equiv 0$.

Subcase 1.1. Suppose $A_1 - A \equiv 0$, then we have $\frac{\xi'}{\xi} + \frac{f_c'}{f_c} + G' = \frac{h_1'}{h_1} + v_1'$, on integrating, we get

$$\xi f_c e^G = c_3 h_1 e^{v_1}, c_3 \neq 0. \tag{2.8}$$

If $c_3 = 1$, then substituting (2.8) in (2.3), we get $f^n = h_2 e^{v_2}$. Since $\rho(f) < degv_2$, which is absurd. If $c_3 \neq 1$, then substituting (2.8) in (2.3), we get

$$f^n + \left(1 - \frac{1}{c_3}\right)\xi f_c e^G = h_2 e^{v_2}. \tag{2.9}$$

On differentiating (2.9) and eliminating e^{v_2}, we get

$$A_2 f^n - nf^{n-1}f' + \left(1 - \frac{1}{c_3}\right)(A_2 - A)\,\xi f_c e^G = 0. \tag{2.10}$$

Equation (2.10) can be written as: $B_5 e^G + B_6 = 0$, where $B_5 = \left(1 - \frac{1}{c_3}\right)(A_2 - A)\,\xi f_c$ and $B_6 = f^{n-1}(A_2 f - nf')$. Similar to subcase 1.1, we get $B_5 \equiv B_6 \equiv 0$, from $B_5 = 0$, we must have $A_2 - A = 0$. Since $c_3 \neq 1$ and $\xi f_c \neq 0$, $A_1 = A_2$. Hence, it is contradictory to $A_1 - A_2 \neq 0$.

Subcase 1.2. Suppose $A_4 - \left(\frac{(A_1-A)'}{A_1-A} + \frac{\xi'}{r} + \frac{f_c'}{f_c} + G'\right) = 0$, then integrating, we get

$$(A_1 - A)\xi f_c e^G = c_4(A_1 - A_2)h_2 e^{v_2}, \quad c_4 \neq 0. \tag{2.11}$$

We claim that $c_4 = 1$, otherwise from (2.11), we have

$$f(z) = H(z)e^{u(z)}, \quad where \ \ \rho(f) = deg(u), \tag{2.12}$$

and $H(z) = c_4\left[\frac{h_2(z-c)(A_1(z-c)-A_2(z-c))}{\xi(z-c)(A_1(z-c)-A(z-c))}\right]e^{u(z)}$, $u(z) = v_2(z-c) - G(z-c)$. Substituting (2.11) and (2.12) in (2.5), we get $H^{n-1}(A_1 H - n(h' + Hu'))\,e^{nu(z)} = (1 - c_4)h_2(A_1 - A_2)\,e^{v_2}$. Since $c_4 \neq 1$, we have $deg\ u = deg\ v_2$, which is contradiction. Therefore $c_4 = 1$, putting (2.11) in (2.5), we get $f^{n-1}(A_1 f - nf') = 0 \Rightarrow A_1 f - nf' = 0$, on integrating, we get

$$f^n = c_5 h_1 e^{v_1}, \quad c_5 \neq 0. \tag{2.13}$$

We claim $c_5 = 1$. Otherwise, substituting (2.13) in (2.3) and on simple calculation, we get

$$(c_5 - 1)h_1 e^{v_1} = h_2 e^{v_2} - \xi f_c e^G. \tag{2.14}$$

Since $deg\ v_2 = deg\ G > deg\ v_1$ and by lemma 2.1, we get $(c_5 - 1)h_1 = 0$, since $h_1 \neq 0$, therefore we must have $c_5 = 1$. Similarly, we can prove another case as well.

Case 2. If $\rho(f) > max\{deg\ v_1, deg\ v_2\}$, it follows from lemma 2.4 and 2.3

$$\begin{aligned} T\left(r, e^G\right) &= T\left(r, e^G\right) + S(r,f), \\ &= m\left(r, \frac{h_1 e^{v_1} + h_2 e^{v_2} - f^n}{\xi f_c}\right) + S(r,f), \\ &\leq m\left(r, e^{v_1}\right) + m\left(r, e^{v_2}\right) + (n+1)m(r,f) + S(r,f). \end{aligned}$$

$i.e$, $T\left(r, e^G\right) \leq (n+1)T(r,f) + S(r,f)$, which implies that

$$deg\ (G) \leqslant \rho(f).$$

We now prove $deg\ G = \rho(f)$. Otherwise, if $deg(G) < \rho(f)$, denoting $R(z) = \xi e^G$ and $P(z) = h_1 e^{v_1} + h_2 e^{v_2}$, then $T(r,P) = S(r,f)$ and $T(r,R) = S(r,f)$, substituting $R(z)$ and $P(z)$ in (2.3), we get $f^n = P - Rf_c$ and using lemma 2.2 we get $m(r,f) = S(r,f)$ and $N(r,f) = S(r,f)$, therefore $T(r,f) = S(r,f)$, which is absurd.

$$\therefore\ \ deg\ G = \rho(f) > max\{deg\ v_1, deg\ v_2\}.$$

Case 3: If $\rho(f) = max\{deg\ v_1, deg\ v_2\}$, it follows from lemma 2.4 and (2.3)that

$$\begin{aligned} T\left(r, e^G\right) &= m\left(r, e^G\right) + S(r,f), \\ &= m\left(r, \frac{h_1 e^{v_1} + h_2 e^{v_2} - f^n}{\xi f_c}\right) + S(r,f), \\ &\leq T\left(r, e^{v_1}\right) + T\left(r, e^{v_2}\right) + (n+1)T(r,f) + S(r,f). \end{aligned}$$

$i.e$, $T\left(r, e^G\right) \leq 2\rho(f) + S(r,f)$, which implies that

$$deg\ G \leq \rho(f).$$

We now prove $deg\ G = \rho(f)$. Otherwise, if $deg\ G < \rho(f)$, and denoting $L(z) = \xi e^G$, then $T(r,L) = S(r,f)$ and (2.3) becomes

$$f^n + Lf_c = h_1 e^{v_1} + h_2 e^{v_2}. \tag{2.15}$$

differentiating (2.15) and eliminate e^{v_1} and e^{v_2} by using (2.15), we get

$$A_1 f^n - nf^{n-1}f' + R_1(z,f) = h_2 A_3 e^{v_2}, \tag{2.16}$$

$$A_2 f^n - nf^{n-1}f' + G_2(z,f) = -h_1 A_3 e^{v_1}, \tag{2.17}$$

where $G_1(z,f) = A_1 L f_c - (Lf_c)'$, $G_2(z,f) = A_2 L f_c - (Lf_c)'$ and $A_3 = A_1 - A_2$. On differentiating (2.16) and eliminating e^{v_2}, we get

$$f^{n-2}\phi(z) = G_2(z,f), \tag{2.18}$$

where $G_2(z,f) = G_1' - A_4 G_1$ and $\phi(z) = \left(A_4 A_1 - A_1'\right) f^2 - n(A_4 + A_1)ff' + n(n-1)(f')^2 + nff^{''}$. Suppose $G_2 = 0$, then we have $G_1' - A_4 G_1 = 0$.

If $G_1 = 0$, on integrating we get $Lf_c = c_6 h_1 e^{v_1}$ $(c_6 \neq 0)$, from this $f(z) = H_1(z)e^{v_1(z-c)}$, where $H_1(z) = \frac{c_6}{L(z-c)} h_1(z-c)e^{v_1(z-c)}$ and $deg\ v_1 = \rho(f)$. Since $deg\ v_2 = \rho(f) > deg\ v_1$, it is a contradiction. Therefore, $G_1(z,f) \neq 0$, then we have $G_1' - A_4 G_1 = 0$, on integrating $G_1 = c_7 A_3 h_2 e^{v_2},\ c_7 \neq 0$, substituting in (2.16), we get $f^{n-1}\left(A_1 f - nf'\right) = \left(\frac{1}{c_7} - 1\right) G_1(z,f)$. Since $n \geq 3$, whether or not $c_7 = 1$, we get from lemma 2.2 that $A_1 f - nf' = 0$, on integrating we get $f^n = c_8 h_1 e^{v_1},\ c_8 \neq 0$ and $\rho(f) = deg\ v_1$, again which is contradiction. Therefore, $G_2(z,f) \neq 0$ and it follows that $\phi(z) \neq 0$. Consider

$$\phi(z) = m_1 f^2 + m_2 ff' + m_3 (f')^2 + m_4 ff^{''} \tag{2.19}$$

where $m_1 = A_4A_1 - A'_1, m_2 = -n(A_4 + A_1), m_3 = n(n-1), m_4 = n$ and m_1, m_2 be a meromorphic functions that are non-zero with $T(r, m_i) = S(r, f),\ i = 1, 2$. We now turn to the following cases:

Subcase 3.1. If f has finite number of zeros, then it possible to assume f is of the form $f(z) = R_1(z)e^{R_2(z)}$, where R_1 and R_2 are polynomials, $R_1 \neq 0$ and $deg\ R_2 = deg\ v_2$, $deg\ R_2 > deg\ G$. Substituting $f(z)$ in (2.16), we get

$$\left[A_1R_1 - nR_1^{n-1}\left(R'_1 + R_1R'_2\right)\right] e^{nR_2(z)} + \left[A_1LR_1(z+c) - L'R_1 - L(R'_1 + R_2R_1(z+c))\right] e^{R_2(z+c)} = h_2A_3e^{v_2}. \quad (2.20)$$

If $A_1R_1 - nR_1^{n-1}\left(R'_1 + R_1R'_2\right) e^{nR_2(z)} = 0$, then on integrating we get

$$c_8h_1e^{v_1} = R_1^ne^{R_2}, \quad c_8 \neq 0$$

and since $deg\ v_1 < deg\ R_2$, it follows from lemma 2.1 that $h_1 = 0$, which is absurd. Therefore

$$A_1R_1 - nR_1^{n-1}\left(R'_1 + R_1R'_2\right) e^{nR_2(z)} \neq 0$$

and suppose

$$\left.\begin{array}{l} R_2(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_0 \\ v_2(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots\cdots + b_0 \end{array}\right\} \begin{array}{l}\text{where } a_i,\ b_i\ 0 \leq i \leq n \\ \text{are constants and} \\ a_nb_n \neq 0\end{array}$$

$$\left[R_1^{n-1}\left(A_1R_1 - n(R'_1 + R_1R'_2)\right)\right] e^{(na_n - b_n)z^k} + \cdots\cdots + (na_0 - b_0) +$$
$$\left[A_1LR_1(z+c) - L'R_1 - L(R'_1 + R_2R_1(z+c))\right] e^{(a_k - b_k)z^k} + \cdots\cdots + (a_0 - b_0) = h_2A_3.$$

From (2.1), we get contradiction.

Subcase 3.2. Suppose f has infinitely many zeors, then proceeding similar to case 3.2 of [3], we get simple zeros of f are infinite. On differentiating (2.19), we get

$$\phi' = m'_1f^2 + (2m_1 + m'_2)ff' + m_2(f')^2 + m_2ff'' + (2m_3 + m_4)f'f'' + m_4ff'''. \quad (2.21)$$

From (2.19) and (2.21), we obtain

$$\begin{aligned} f'\left[(m_2\phi - m_3\phi')f' + (2m_3 + m_4)\phi f''\right] = f\,[(m_1\phi' - m'_1\phi)f + \\ (m_2\phi' - (2m_1 + m'_2))f' + (m_4\phi' - m_2\phi)f'' - m_4\phi f'''\Big]. \end{aligned} \quad (2.22)$$

If f has simple zero at z_0 and not the zero and pole of the coefficients of (2.22). Putting z_0 in (2.22), we observe that z_0 is zero of $(m_2\phi - m_3\phi')f' + (2m_3 + m_4)\phi f''$. Let

$$\gamma(z) := \frac{(m_2\phi - m_3\phi')f' + (2m_3 + m_4)\phi f''}{f}. \quad (2.23)$$

Clearly $T(r, \gamma) = O(log\ r)$ and we can conclude by lemma 2.5 that γ is rational function. It follows from (2.23)

$$f'' = \left[\frac{-m_2}{n(2n-1)} - \frac{n-1}{2n-1}\frac{\phi'}{\phi}\right] f + \frac{\gamma f}{n(2n-1)\phi}. \quad (2.24)$$

Substituting (2.24) in (2.19), we obtain

$$\phi(z) = u_1f^2 + u_2ff' + u_3(f')^2, \quad (2.25)$$

where $u_1 = m_1 + \frac{\gamma}{(2n-1)\phi}$, $u_2 = m_2(n-1)\left[\left(\frac{2}{2n-1}\right) - \frac{n}{2n-1}\frac{\phi'}{\phi}\right]$ and $u_3 = n(n-1)$, $u_j,\ j = 1, 2$ are rational functions, and

$$T(r, u_i) = S(r, f)\ \ i = 1, 2. \quad (2.26)$$

By the similar argument of [3][from the equation (3.19) to (3.20)], we get

$$u_3(u_2^2 - 4u_1u_3)\frac{\phi'}{\phi} + u_2u_2^2 - 4u_1u_3 + u'_3u_2^2 - 4u_1u_3 = u_3u_2^2 - 4u_1u_3. \quad (2.27)$$

Denoting $u_2^2 - 4u_1u_3 = \psi$, now we will discuss the following cases

Subcase 3.2.1. If $\psi \neq 0$, then we get $\frac{u_2}{u_3} = \frac{\psi'}{\psi} - \frac{\phi'}{\phi} - \frac{u_3'}{u_3}$, on substituting all the parameters and integrating, we get

$$e^{v_1+v_2} = \frac{k}{h_1h_2A_3}\psi^{\frac{-(2n-1)}{2}}\phi^{n-1} \in S(r,f),$$

possible only when $v_1 = -v_2$, which is contradiction, since $deg\ v_1 < deg\ v_2$.

Subcase 3.2.2. If $\psi = 0$, then (2.25) becomes

$$\phi = u_3\left(f' + \frac{u_2}{2u_3}f\right)^2. \tag{2.28}$$

Let $\Psi = f' + \frac{u_2}{2u_3}f$, $\Psi \neq 0$ and $T(r,\phi) = S(r,f)$, we have

$$T(r,\Psi) = S(r,f) \quad and \quad f' = \Psi - \frac{u_2}{2u_3}f. \tag{2.29}$$

Putting (2.29) in (2.16) and (2.17), we get

$$\begin{aligned}\left(A_1 + \frac{u_2}{2u_3}n\right)f^n - n\Psi f^{n-1} + G_1(z,f) = h_2A_3e^{v_2},\\ \left(A_2 + \frac{u_2}{2u_3}n\right)f^n - n\Psi f^{n-1} + G_2(z,f) = -h_1A_3e^{v_1}.\end{aligned} \tag{2.30}$$

If $A_1 + \frac{u_2}{2u_3}n \equiv 0$ and $A_2 + \frac{u_2}{2u_3}n \equiv 0$, then we get $A_3 = 0$ which is absurd. Consequently, we claim

$$\left(A_1 + \frac{u_2}{2u_3}n\right)\left(A_2 + \frac{u_2}{2u_3}n\right) \equiv 0.$$

Otherwise, since $A_3 \neq 0$ and $h_2 \neq 0$, from (2.30), we have

$$N\left(r, \frac{1}{G_l}\right) + N(r,f) = N\left(r, \frac{1}{A_3}\right) + N(r,f) = S(r,f), \qquad l = 1, 2.$$

From Theorem 1.1, equation (2.26) and (2.29) there exist two small functions ν_1, ν_2 of f such that

$$H_1 = \left(A_1 + \frac{u_2}{2u_3}n\right)(f - \nu_1)^n = h_2A_3e^{v_2}, \tag{2.31}$$

and

$$H_2 = \left(A_2 + \frac{u_2}{2u_3}n\right)(f - \nu_2)^n = -h_1A_3e^{v_1}. \tag{2.32}$$

Based on Nevanlinna's second fundamental theorem concerning to small functions that $\nu_1 = \nu_2$, then from (2.31) and (2.32), we get

$$e^{v_1-v_2} = -\frac{\left(A_2 + \frac{nu_2}{2u_3}\right)}{\left(A_1 + \frac{nu_2}{2u_3}\right)} \in S(r,f),$$

possible only when $v_1 = v_2$, which is contradiction to $deg\ v_1 < deg\ v_2$. Therefore,

$$\rho(f) = deg\ G = \max\{deg\ v_1, deg\ v_2\}.$$

Proof of Theorem 1.9: Assuming $\eta \neq 0$. Suppose that f be a transcendental entire solution of finite order to (1.9) and now, in order to prove the theorem, we will look at the following cases:

Case 1. If $\rho(f) < deg\ p$, then from (1.9) and lemma (2.4) to (2.7), it follows that

$$
\begin{aligned}
T(r,e^G) &= m(r,e^G),\\
&= m\left(r, \frac{he^p - (f^n + \eta f^{n-1}f')}{s(z)f^{(k)}(z+c)}\right),\\
&\le m\left(r, he^p\right) + m\left(r, f^n + \eta f^{n-1}f'\right) + m\left(r, \frac{1}{sf^{(k)}(z+c)}\right) + O(1),\\
&\le T(r,e^p) + nT(r,f) + T\left(r, f^{(k)}(z+c)\right) - N\left(r, \frac{1}{f^{(k)}(z+c)}\right) + S(r,f),\\
&\le T(r,e^p) + nT(r,f) + T\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) + T(r,f(z+c)) - N\left(r, \frac{1}{f(z+c)}\right) - k\overline{N}\left(r, f(z+c)\right) + S(r,f),\\
&\le T(r,e^p) + (n+1)T(r,f) + m\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) + N\left(r, \frac{f^{(k)}(z+c)}{f(z+c)}\right) - N\left(r, \frac{1}{f(z+c)}\right) + S(r,f),\\
&\le T(r,e^p) + (n+1)T(r,f) + N\left(r, f^{(k)}(z+c)\right) + S(r,f),\\
&\le T(r,e^p) + (n+1)T(r,f) + S(r,f),
\end{aligned}
$$

i.e $T(r,e^G) \le T(r,e^p) + S(r,f)$, which implies

$$deg\ G \le deg\ p. \tag{2.33}$$

Meanwhile, we have from (1.9) and Lemma 2.4 that

$$
\begin{aligned}
T(r,e^p) &= m(r,e^p),\\
&= m\left(r, \frac{f^n + \eta f^{n-1}f' + se^G f^{(k)}(z+c)}{h}\right),\\
&\le m\left(r, \frac{sf^{(k)}(z+c)}{f}\right) + m(r,f) + m(r,e^G) + m(r, f^n + \eta f^{n-1}f') + S(r,f),\\
&\le T(r,e^G) + (n+1)T(r,f) + S(r,f),\\
&\le T(r,e^G) + S(r,f),
\end{aligned}
$$

which implies

$$deg\ p \le deg\ G. \tag{2.34}$$

From (2.33) and (2.34), we have

$$deg\ G = deg\ p \quad and \quad \rho(f) < deg\ G.$$

For convenient, we write (1.9) as follows

$$f^n + \eta f^{n-1}f' + s(z)f_c^{(k)}e^G = he^p. \tag{2.35}$$

Differentiating (2.35) and eliminating e^p, we get

$$H_1e^G + H_2 = 0, \tag{2.36}$$

where

$$
\begin{aligned}
H_1 &= (A - A_1)sf^{(k)}(z+c),\\
H_2 &= Af^n + (aA - n)f^{n-1}f' - a(n-1)f^{n-2}(f')^2 - af^{n-1}f^{''}.
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \frac{h^{'}}{h} + p^{'},\\
A_1 &= \frac{s^{'}}{s} + \frac{f_c^{(k+1)}}{f_c^{(k)}} + G^{'}.
\end{aligned}
$$

Since $\rho(f) < deg\ p$, by lemma 2.1, we get $H_1 \equiv H_2 \equiv 0$. From $H_1 \equiv 0$, it follows that

$$\frac{s^{'}}{s}+\frac{f_c^{(k+1)}}{f_c^{(k)}}+G^{'}=\frac{h^{'}}{h}+p^{'},$$

on integrating above expression, we get

$$se^G f_c^{(k)}=C_1he^p,\ \ C_1\neq 0. \tag{2.37}$$

Substituting (2.37) in (2.35), we get

$$f^n+\eta f^{n-1}f^{'}=(1-C_1)he^p. \tag{2.38}$$

If $C_1=1$, then from (2.38), we have $f^n+\eta f^{n-1}f^{'}=0$. Then easily we get

$$f=Ce^{\frac{-z}{\eta}},$$

where $C(\neq 0)$ is a constant. Thus, conclusion (1) is true. If $C_1\neq 1$, then it follows from (2.38) and lemma 2.5 that

$$\begin{aligned}
T(r,e^p)&=m(r,e^p),\\
&=m\left(r,\frac{f^n+\eta f^{n-1}f^{'}}{(1-C_1)h}\right),\\
&\leq m(r,f^n+\eta f^{n-1}f^{'})+S(r,f),\\
&\leq nm(r,f)+S(r,f),\\
&\leq nT(r,f)+S(r,f),
\end{aligned}$$

which implies $deg\ p\leq\rho(f)$. Which is contradiction to $\rho(f)<deg\ p$. Therefore $C_1\neq 0$.

Case 2. If $\rho(f)>deg\ p$. By lemma 2.4 to 2.7 and (2.35), it follows that

$$\begin{aligned}
T(r,e^G)&=m(r,e^G),\\
&=m\left(r,\frac{he^p-(f^n+\eta f^{n-1}f^{'})}{sf_c^{(k)}}\right),\\
&\leq m\left(r,\frac{1}{sf_c^{(k)}}\right)+m(r,e^p)+m(r,f^n+\eta f^{n-1}f^{'}+S(r,f),\\
&\leq T\left(r,f_c^{(k)}\right)-N\left(r,\frac{1}{f_c^{(k)}}\right)+T(r,e^p)+nm(r,f)+S(r,f),\\
&\leq T\left(r,\frac{f_c^{(k)}}{f(z+c)}\right)+T(r,f(z+c))-N\left(r,\frac{1}{f(z+c)}\right)-k\overline{N}\left(r,f(z+c)\right)+T(r,e^p)+nT(r,f)S(r,f),\\
&\leq(n+1)T(r,f)+T(r,e^p)+m\left(r,\frac{f_c^{(k)}}{f(z+c)}\right)+N\left(r,\frac{f_c^{(k)}}{f(z+c)}\right)-N\left(r,\frac{1}{f(z+c)}\right)+S(r,f),\\
&\leq(n+1)T(r,f)+T(r,e^p)+N(r,f_c^{(k)})+S(r,f),\\
&\leq(n+1)T(r,f)+S(r,f),
\end{aligned}$$

$$which\ \ implies\ \ deg\ G\leq\rho(f). \tag{2.39}$$

We will show now $deg\ G=\rho(f)$. Otherwise $deg\ G<\rho(f)$, let us denote $U_1(z)=he^p$, $U_2(z)=se^G$, clearly $T(r,U_1)=S(r,f)$, $T(r,U_2)=S(r,f)$. And substituting U_1, U_2 in (2.35), we get

$$f^{n-1}\left(f+\eta f^{'}\right)=U_1+U_2f_c^{(k)}. \tag{2.40}$$

Since $n\geq 3$, it follows from lemma 2.2 that

$$m(r,f+\eta f^{'})=S(r,f),\quad m(r,f(f+\eta f^{'}))=S(r,f). \tag{2.41}$$

Since f is an entire function, as a result, it's simple to infer,

$$T(r,f)=m(r,f)\leq m\left(r,\frac{1}{f+\eta f'}\right)+m(r,(f+\eta f^{'})f)$$
$$\leq T(r,f+\eta f^{'})+S(r,f)=m(r,f+\eta f^{'})+S(r,f)=S(r,f),$$

which is absurd. As a result, we have $deg\ p < deg\ G=\rho(f)$.

Case 3. If $\rho(f)=deg\ p$, in the same way as the proof in case 2, we can conclude that $deg\ G\leq\rho(f)=deg\ p$. We will now show that $deg\ G=\rho(f)$. Suppose $deg\ G<\rho(f)$ and let $D(z)=se^G$, then $T(r,D)=S(r,f)$. Therefore (2.35) becomes

$$f^n+\eta f^{n-1}f^{'}+Df_c^{(k)}=he^p. \tag{2.42}$$

On differentiating (2.42) and eliminating e^p, we get

$$f^{n-2}\left(Af^2+(\eta A-n)ff^{'}-\eta(n-1)(f^{'})^2-\eta ff^{''}\right)=\Psi(f), \tag{2.43}$$

where $\Psi(f)=Df_c^{(k+1)}+D^{'}f_c^{(k)}-ADf_c^{(k)}$ is a differential-difference polynomial in f, where the coefficients are small functions of f and degree at most 1. We will examine whether Ψ equivalent to zero or not. If $\Psi\equiv 0$, then we have

$$\frac{f_c^{(k+1)}}{f_c^{(k)}}=\frac{h^{'}}{h}+p^{'}-\frac{s^{'}}{s}-G^{'}, \tag{2.44}$$

which implies

$$se^Gf_c^{(k)}=C_2he^p, \tag{2.45}$$

where $C_2(\neq 0)$ constant. Substituting (2.45) in (2.42), we get $f^{n-1}\left(f+\eta f^{'}\right)=\left(\frac{1}{C_3}-1\right)Df_c^{(k)}$, whether or not $C_3=1$, we get $f+\eta f^{'}=0$, which impish $f=Ce^{-\frac{z}{\eta}}$, here $\rho(f)=1$. Since $deg\ G<\rho(f)=1$, thus G is constant, which is contradiction.

If $\Psi\not\equiv 0$, preceding similar to the *case 3* of *Theorem 1.4* [3], we can obtain a contradiction, the proof is skipped in this case.

If $\eta=0$, we can obtain the conclusion of *Theorem 4* by having a similar conversation as above, we skip the proof here.

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