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Further generalization of the cyclic contraction for the best proximity point problem

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Abstract

In this paper, we introduce a further generalization of the cyclic contraction mappings. Our main results generalize the recent related results proved by M. Jleli and B. Samet [8] and solve a best proximity point problem. In order to show the applicability of our main results, an example is presented.

Keywords: Best proximity point, Fixed point, Uniformly convex Banach space, Iterative sequence
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1 Introduction

Let X be a metric space and F and G nonempty subsets of X. Put

$$\begin{aligned} F^\circ &= \{x \in F : d(x,y) = dist(F,G) \ for\ some\ y \in G\}, \\ G^\circ &= \{x \in G : d(x,y) = dist(F,G) \ for\ some\ y \in F\}. \end{aligned}$$

If there is a pair $(x_0, y_0) \in F \times G$ for which $d(x_0, y_0) = dist(F,G)$, that $dist(F,G)$ is distance of F and G, then the pair (x_0, y_0) is called a best proximity pair for F and G.

We say that the point $x \in F \cup G$ is a best proximity point of the pair (F,G) for $T : F \cup G \to F \cup G$, if $d(x,Tx) = dist(F,G)$ and we denote the set of all best proximity points of (F,G) by $P_T(F,G)$, that is

$$P_T(F,G) = \{x \in F \cup G : d(x,Tx) = dist(F,G)\}.$$

Best proximity point also evolves as a expansion of the concept of fixed point of mappings, because if $F \cap G \neq \emptyset$ each best proximity point is a fixed point of T.

A best proximity point theorem for contractive mappings has been detailed in Sadiq Basha [10, 11]. Anthony Eldred et al.[3] have elicited a best proximity point theorem for relatively nonexpansive mappings, an alternative treatment to which has been focused in Sankar Raj and Veeramani [12]. Anuradha and Veeramani [1] have discussed best proximity point theorems for proximal pointwise contractions. Best proximity point theorems for various variants of contractions have been explored Eldred and Veeramani [4], Haddadi et al. [5, 6], Karpagam and Agrawal [9], and [2].

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Theorem 1.1. ([4]) Let (F, G) be a nonempty closed convex pair of disjoint subsets of a uniformly convex Banach space X. If $T : F \cup G \to F \cup G$ is a cyclic mapping such that

$$d(Tx, Ty) \leq kd(x, y) + (1-k)dist(F, G), \quad \forall x \in F,\ y \in G$$

and either F or G is boundedly compact, then T has a unique best proximity point. Further, if $x_0 \in F$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Consistent with [8], we denote by Θ_0 the family of functions $\theta : (0, +\infty) \to (1, +\infty)$ so that:
(θ_1) θF is increasing;
(θ_2) for each sequence $\{\rho_n\} \subseteq (0, +\infty)$, $\lim\limits_{n \to +\infty} \theta(\rho_n) = 1$ iff $\lim\limits_{n \to +\infty} \rho_n = 0$;
(θ_3) there are $\kappa \in (0, 1)$ and $\lambda \in (0, +\infty)$ so that $\lim\limits_{\rho \to 0^+} \frac{\theta(\rho)-1}{\rho^\kappa} = \lambda$.

Theorem 1.2. [8, Corollary 2.1] Let T be a self-mapping on a complete metric space (X, d) so that

$$x, \omega \in X, \quad d(Tx, T\omega) \neq 0 \ \Rightarrow\ \theta(d(Tx, T\omega)) \leq \theta(d(x, \omega))^\alpha.$$

where $\theta \in \Theta_0$ and $\alpha \in (0, 1)$. Then T has a unique fixed point.

Note that the Banach contraction principle is a particular case of Theorem 1.2.

Denote by ΘF the set of strictly increasing continuous functions $\theta : (0, +\infty) \to (1, +\infty)$. Here, we have a wider range of functions than those introduced in [8].

Remark 1.1. [7] It is clear that $f(t) = e^t$ is not an element of Θ_0, but it belongs to ΘF. Other examples are $f(t) = \cosh t$, $f(t) = \frac{2\cosh t}{1+\cosh t}$, $f(t) = 1 + \ln(1+t)$, $f(t) = \frac{2+2\ln(1+t)}{2+\ln(1+t)}$, $f(t) = e^{te^t}$ and $f(t) = \frac{2e^{te^t}}{1+e^{te^t}}$, for all $t > 0$.

Let Φ be the class of functions $\phi : (1, +\infty) \to (0, +\infty)$ so that:
(ϕ_1) ϕ is continuous;
(ϕ_2) $\phi(t) = 0$ iff $t = 1$;
(ϕ_3) for each sequence $\{t_n\} \subseteq (1, +\infty)$; $\lim\limits_{n \to +\infty} \phi(t_n) = 0$ iff $\lim\limits_{n \to +\infty} t_n = 1$.

The following functions $\phi(t) = \sinh(t-1)$, $\phi(t) = \cosh(t-1)-1$, $\phi(t) = \tanh(t-1)$, $\phi(t) = \arccos ht$, $\phi(t) = t - \sqrt{t}$, $\phi(t) = \sqrt{t} - \sqrt[3]{t}$ are in Φ.

We denote by Ξ_0 the family of functions $\theta : (0, +\infty) \to (1, +\infty)$ so that:
(θ_1) θF is increasing;
(θ_2) for each sequence $\{\rho_n\} \subseteq (0, +\infty)$, $\lim\limits_{n \to +\infty} \theta(\rho_n) = \theta(\rho)$ iff $\lim\limits_{n \to +\infty} \rho_n = \rho$;
(θ_3) there are $\kappa \in (0, 1)$ and $\lambda \in (0, +\infty]$ so that $\lim\limits_{\rho \to 0^+} \frac{\theta(\rho)-1}{\rho^\kappa} = \lambda$;
(θ_4) is continuous.

2 Main Results

In the following we provide a strong convergence theorem for a generalization of cyclic contraction for the best proximity point problem in a complete metric space.

Theorem 2.1. Let F and G be closed disjoint subsets of complete metric space X and $T : F \cup G \to F \cup G$ be a cyclic mapping so that for every $x, \omega \in F$, or $x, \omega \in G$,

$$d(Tx, T\omega) \neq 0 \ \Rightarrow\ \theta(d(Tx, T\omega)) \leq \theta(d(x, \omega))^\alpha. \tag{2.1}$$

and for every $x \in F,\ \omega \in G$,

$$d(Tx, T\omega) \neq dist(F, G) \ \Rightarrow\ \theta(d(Tx, T\omega)) \leq \theta(d(x, \omega))^\alpha \theta(dist(F, G))^{1-\alpha}. \tag{2.2}$$

where $\theta \in \Xi_0$ and $\alpha \in (0, 1)$. Then $P_T(F, G) \neq \emptyset$. Further, if $x_0 \in F$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Proof . Fix $x \in F \cup G$ and define a sequence $\{x_n\}$ in $F \cup G$ by $x_n = T^n x$, $n \in \mathbb{N}_0$. We divide the proof into 4 steps:

Step 1. $\lim_{n\to+\infty} d(x_n, x_{n+1}) = dist(F, G)$.

So, without restriction of the generality, we can suppose that $d(T^n x, T^{n+1}x) > dist(F, G)$ for all $n \in \mathbb{N}$. Now, from (2.2), for all $n \in \mathbb{N}$, we have Note

$$\begin{aligned}
\theta(dist(F,G)) \leq \theta(d(x_{n+1}, x_{n+2})) &= \theta(d(Tx_n, Tx_{n+1})) \\
&\leq \theta(d(x_n, x_{n+1}))^{\alpha}\theta(dist(F,G))^{1-\alpha} \\
&\leq \theta(d(x_{n-1}, x_n))^{\alpha^2}\theta(dist(F,G))^{1-\alpha^2} \\
&\dots \\
&\leq \theta(d(x_1, x_2))^{\alpha^n}\theta(dist(F,G))^{1-\alpha^n}.
\end{aligned}$$

Hence $\{\theta(d(x_n, x_{n+1}))\}$ is monotonic decreasing and bounded below. Therefore $\lim_{n\to+\infty} \theta(d(x_n, x_{n+1}))$ exists and so $\lim_{n\to+\infty} d(x_n, x_{n+1})$. Let $\lim_{n\to+\infty} d(x_n, x_{n+1}) = \rho \geq dist(F, G)$. Assume that $\rho > dist(F, G)$. By the right continuity of θ,

$$\theta(\rho) = \lim_{n\to+\infty} \theta(d(x_{n+1}, x_{n+2})) \leq \lim_{n\to+\infty} \theta(d(x_n, x_{n+1}))^{\alpha}\theta(dist(F,G))^{1-\alpha} < \theta(\rho),$$

so $\rho = dist(F, G)$.

Step 2. $\lim_{n\to+\infty} d(x_{n+1}, x_{n-1}) = 0$.

Now, from (2.1), for all $n \in \mathbb{N}$, we have Note

$$\begin{aligned}
1 \leq \theta(d(x_{n+1}, x_{n-1})) &= \theta(d(Tx_n, Tx_{n-2})) \\
&\leq \theta(d(x_n, x_{n-2}))^{\alpha} \\
&\leq \theta(d(x_{n-1}, x_{n-3}))^{\alpha^2} \\
&\dots \\
&\leq \theta(d(x_2, x_0))^{\alpha^{n-1}}. \qquad (2.3)
\end{aligned}$$

Hence $\{\theta(d(x_{n+1}, x_{n-1}))\}$ is monotonic decreasing and bounded below. Hence

$$\lim_{n\to+\infty} \theta(d(x_{n+1}, x_{n-1})) = 1.$$

and so

$$\lim_{n\to+\infty} d(x_{n+1}, x_{n-1}) = 0.$$

Step 3. $\{x_{2n}\}$ is Cauchy sequence.

From condition (θ_3), there exist $r \in (0, 1)$ and $\ell(0, +\infty]$ such that

$$\lim_{n\to+\infty} \frac{\theta(d(x_{n+1}, x_{n-1})) - 1}{[d(x_{n+1}, x_{n-1})]^r} = \ell.$$

Suppose that $\ell < +\infty$. In this case, let $L = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_{n+1}, x_{n-1})) - 1}{[d(x_{n+1}, x_{n-1})]^r} - \ell\right| \leq L, \quad \forall\ n \geq n_0.$$

This implies that

$$\frac{\theta(d(x_{n+1}, x_{n-1})) - 1}{[d(x_{n+1}, x_{n-1})]^r} \geq \ell - L = L, \quad \forall\ n \geq n_0.$$

Then

$$n[d(x_{n+1},x_{n-1})]^r \leq \frac{1}{L}n[\theta(d(x_{n+1},x_{n-1}))-1], \quad \forall\, n \geq n_0.$$

Using (2.3), we obtain

$$n[d(x_{n+1},x_{n-1})]^r \leq \frac{1}{L}n[\theta(d(T^2x_0,x_0))^{\alpha^{n-1}}-1], \quad \forall\, n \geq n_0.$$

Letting $\lim_{n\to+\infty}$ in the above inequality, we obtain

$$\lim_{n\to+\infty} n[d(x_{n+1},x_{n-1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_{n+1},x_{n-1}) \leq \frac{1}{n^{1/r}}, \quad \forall\, n \geq n_1.$$

Now, let $m = 2k$

$$\begin{aligned} d(x_n,x_{n+m}) &\leq d(x_n,x_{n+2}) + d(x_{n+2},x_{n+4}) + \dots + d(x_{n+m-2},x_{n+m}) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \dots + \frac{1}{(n+m)^{1/r}}. \\ &\leq \sum_{i=n}^{+\infty} \frac{1}{i^{1/r}}, \quad \forall\, n \geq n_1. \end{aligned}$$

From the convergence of the series $\sum_i \frac{1}{i^{1/r}}$, we deduce that $\{x_{2n}\}$ is a Cauchy sequence.

Step 4. Existence of best proximity pair.

Because $\{x_{2n}\}$ is Cauchy, X is complete and F is closed, $\lim_{n\to+\infty} x_{2n} = x \in F$. Now

$$dist(F,G) \leq d(x,x_{2n-1}) \leq d(x,x_{2n}) + d(x_{2n},x_{2n-1}).$$

Thus, by step 1 we have $d(x_{2n},x_{2n-1}) \to dist(F,G)$ and so $d(x,x_{2n-1})$ converges to $dist(F,G)$. Since

$$\theta(dist(F,G)) \leq \theta(d(x_{2n},Tx)) \leq \theta(d(x_{2n-1},x))^{\alpha}\theta(dist(F,G))^{1-\alpha},$$

therefore by upper semicontinuity of θ we have

$$\theta(dist(F,G)) \leq \lim_{n\to+\infty} \theta(d(x_{2n},Tx)) \leq \lim_{n\to+\infty} \theta(d(x_{2n-1},x))^{\alpha}\theta(dist(F,G))^{1-\alpha} = \theta(dist(F,G)).$$

Hence

$$\theta(d(x,Tx)) = \lim_{n\to+\infty} \theta(d(x_{2n},Tx)) = \theta(dist(F,G))$$

and so $d(x,Tx) = dist(F,G)$. □

It is notable that if in Theorem 2.1 we have $F \cap G \neq \emptyset$, then (2.1) and (2.2) coincide and so we conclude Theorem 1.2. In the following we provide a strong convergence theorem for a generalization of cyclic contraction for the best proximity point problem in the uniformly convex Banach space.

Theorem 2.2. Let F and G be two nonempty closed and convex disjoint subsets of a uniformly convex Banach space X. Suppose the mapping $T : F \cup G \to F \cup G$ satisfied in (2.1) and (2.2). Then there is a unique $p \in F$ such that $\|p - Tp\| = dist(F,G)$. Also, if $p_0 \in F$ and $p_{n+1} = Tp_n$, then $\{p_{2n}\}$ converges to the best proximity point.

Proof . By Theorem 2.1 $P_T(F,G) \neq \emptyset$. Suppose $p,q \in P_T(F,G)$ such that $p \neq q$. Hence $\|p - Tp\| = dist(F,G)$ and $\|q - Tq\| = dist(F,G)$ where necessarily uniformly convexity of X, $T^2p = p$ and $T^2q = q$. Since $p \neq q$, by (2.2) we have $\theta(dist(F,G)) < \theta(\|Tp - q\|)$ and $\theta(dist(F,G)) < \theta(\|p - Tq\|)$. Therefore

$$\theta(\|p - Tq\|) = \theta(\|T^2p - Tq\|) < \theta(\|Tp - q|)$$

and

$$\theta(\|Tp - q\|) = \theta(\|Tp - T^2q\|) < \theta(\|p - Tq\|)$$

that it is a contradiction and so $p = q$. □

Example 2.1. Let F and G be subsets of $\mathbb{R}^2$ defined by

$$F=\{(x,0):x\geq 1\},\ G=\{(0,y):y\geq 1\}.$$

Suppose $T(x,y)=(\sqrt{y},\sqrt{x})$ and

$$\theta(\varsigma)=\begin{cases}\sqrt{\varsigma} & \varsigma<dist(F,G)\\ \sqrt{dist(F,G)\varsigma} & \varsigma\geq dist(F,G).\end{cases}$$

Then T is a cyclic mapping on $F\cup G$ that satisfied in (2.1) and (2.2). Also we have $\|(0,1)-T((1,0))\|=dist(F,G)$.

Proof . Here $dist(F,G)=\sqrt{2}$. For $(x,0),(y,0)\in F$ we have

$$\begin{aligned}\theta(\|T(x,0)-T(y,0)\|) &= \theta(\|(0,\sqrt{x})-(0,\sqrt{y})\|=\sqrt{\|(0,\sqrt{x}-\sqrt{y})\|}=\sqrt{|\sqrt{x}-\sqrt{y}|}\\ &\leq \sqrt{|x-y|}=\theta(|x-y|)\\ &= \theta(\|(x,0)-(y,0)\|).\end{aligned}$$

Hence we have (2.1). Also, for $(x,0)\in F$ and $(0,y)\in G$ we have

$$\begin{aligned}\theta(\|T(x,0)-T(0,y)\|) &= \theta(\|(0,\sqrt{x})-(\sqrt{y},0))\|=\sqrt{dist(F,G)\|(\sqrt{y},\sqrt{x})\|}=\sqrt{dist(F,G)\sqrt{y+x}}\\ &\leq \sqrt{x+y}\leq\sqrt{\sqrt{2}\sqrt{x^2+y^2}}\\ &\leq \sqrt{dist(F,G)\|(x,0)-(0,y)\|}\\ &= \theta(\|(x,0)-(0,y)\|).\end{aligned}$$

Therefore we have (2.2). Also we have $\|(0,1)-T((0,1))\|=\|(0,1)-(1,0)\|=\sqrt{2}=dist(F,G)$ that it is calculated in $n=21$ iteration from Table 1 and Figure 1. □

Table 1: Rate of convergence of the Picard iteration of Example 2.1

	$x_{2n}\in F$		$x_{2n+1}\in G$
0	(3.000000,0)	1	(0,1.732051)
2	(1.316074,0)	3	(0,1.147203)
4	(1.071075,0)	5	(0,1.034928)
6	(1.017314,0)	7	(0,1.008620)
8	(1.004301,0)	9	(0,1.002148)
10	(1.001073,0)	11	(0,1.000537)
12	(1.000268,0)	13	(0,1.000134)
14	(1.000067,0)	15	(0,1.000034)
16	(1.000017,0)	17	(0,1.000008)
18	(1.000004,0)	19	(0,1.000002)
20	(1.000001,0)	21	(0,1.000001)
22	(1,0)	23	(0,1)

If in the Theorem 2.2 put $\theta(t)=e^t$ then we have the following corollary.

Corollary 2.3. Let (F,G) be a nonempty closed convex pair of disjoint subsets of a uniformly convex Banach space X. If $T:F\cup G\to F\cup G$ is a cyclic mapping such that

$$d(Tx,Ty)\leq kd(x,y),\quad \forall x,y\in F,\ nor\ x,y\in G,$$

$$d(Tx,Ty)\leq kd(x,y)+(1-k)dist(F,G),\quad \forall x\in F,\ y\in G.$$

Then T has a unique best proximity point. Further, if $x_0\in F$ and $x_{n+1}=Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Remark 2.1. In Corollary 2.3 boundedly compact F or G is omitted with respect to Theorem 1.1.

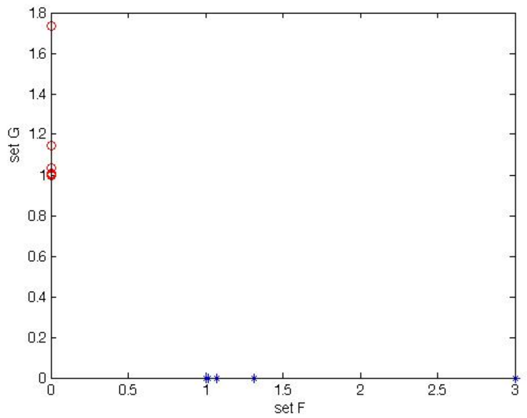

Figure 1: Rate of convergence of the Picard iteration of Example 2.1

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